WHITNEY (b)- REGULARITY FOR ANALYTICALLY CONSTRUCTIBLE SETS

Zbigniew Hajto
Kraków, Poland

In his La Rábida lectures [TE] Professor B. Teissier gave several equivalent conditions for Whitney (b)- regularity. One of these conditions characterizes (b)- regularity by a dimension of the fibres of certain analytic morphism. We shall give a similar description of (b)- regularity for analytically constructible leaves in the complex space $\mathbb{C}^n$. Such a description provides a direct answer to the Zariski's question C [Z]. For the pair of semi-algebraic leaves in the real space $\mathbb{R}^n$ we will show that the answer to the question C may be negative.

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1. Analytically constructible sets. Let us recall that a subset A of a complex manifold M is said to be analytically constructible if for each $x \in M$ the germ of A at x belongs to the smallest class $\mathcal{C}$ of germs at x of subsets of M satisfying the following condition

$$\alpha, \beta \in \mathcal{C} \Longrightarrow \alpha \cup \beta, \alpha \setminus \beta \in \mathcal{C}$$

and containing each analytic germ at x.

A nonempty connected analytic submanifold X of a complex manifold M is called an analytically constructible leaf if it is an analytically constructible subset of M. Basic facts about constructible sets are contained e.g. in the book [Ł] or in the lectures [TE]. Recall only that the family of constructible sets is closed by the following set-theoretical operations:

a) locally finite union and intersection

b) difference of any two sets.

An often used method to prove the analyticity of certain subset is to prove that it is a closure of an analytically constructible subset.

2. Whitney (b)- regularity. Let X and Y be two constructible leaves in an open subset $G \subset \mathbb{C}^n$. We say that X is (b)- regular over Y at $y_o \in Y \cap (\bar{X} \setminus X)$ if the following holds: For every se-

quence $\{x_n\}$ of X and $\{y_n\}$ of Y such that

$$x_n \longrightarrow y_o, \quad y_n \longrightarrow y_o, \quad \mathbb{C}(x_n - y_n) \longrightarrow 1 \quad \text{and} \quad T_{x_n}X \longrightarrow \tau$$

we have $\tau \supset 1$.

Consider now X and Y two constructible leaves of G such that $X \cap Y \neq \emptyset$ and $\overline{X} \neq G$. Let us denote

$$C(X,Y) = \left\{(x,y,\mathbb{C}(x-y),H) \in G \times G \times P^{n-1} \times \check{P}^{n-1} : (x,y) \in X \times Y, \ H \supset T_xX\right\},$$

$$\Omega = C(X,Y) \cap (\Delta_G \times P^{n-1} \times \check{P}^{n-1})$$

$$\Omega_{(y,y)} = \left\{(l,H) \in P^{n-1} \times \check{P}^{n-1} : (y,y,l,H) \in \Omega\right\}$$

where Δ_G is the diagonal subset of the $G \times G$ and $\check{P}^{n-1}$ is a dual projective space.

We can characterize Whitney (b)- regularity as follows:

Theorem 1. If $y_o \in Y \cap \overline{X}$ then $\dim \Omega_{(y_o,y_o)} \geqslant n-2$ and X is Whitney (b)- regular over Y at y_o if and only if

$$\dim \Omega_{(y_o,y_o)} = n-2 .$$

Using the upper semicontinuity of fiber dimension for analytic mappings we shall obtain

Corollary 1. The set $S_b(X,Y)$ of points $y \in Y$ where condition (b) fails is a closed subset of Y.

3. Proof of the Theorem 1. First we shall formulate the following elementary lemma proposed by S. Łojasiewicz ([Ł], [W2]).

Lemma 1. If $f: V \longrightarrow W$ is a holomorphic map of two local analytic varieties then for every $k \in \mathbb{N}$

(1) $\dim V \geqslant k + \dim f(V)$ if $\dim f^{-1}(w) \geqslant k$ for $w \in f(V)$

(2) $\dim V \leqslant k + \dim f(V)$ if $\dim f^{-1}(w) \leqslant k$ for $w \in f(V)$ and $V \neq \emptyset$.

Proof. It follows from the definition of the dimension of an analytic variety (see e.g. [W2] Lemma 6.7.).

Proposition 1. $C(X,Y)$ is a constructible leaf in $G \times G \times P^{n-1} \times \check{P}^{n-1}$ and $\dim C(X,Y) = \dim Y + n - 1$.

Proof. Let us denote $C(X) = \left\{(x,H) \in X \times \check{P}^{n-1} : H \supset T_xX\right\}$ the conormal space of X [TE] and $\tilde{\sigma}$ the space $\mathbb{C}^n \times \mathbb{C}^n$ blown up at the diagonal i.e. the closure of the graph of σ :

$$\sigma : \mathbb{C}^n \times \mathbb{C}^n \ni (x,y) \longrightarrow \mathbb{C}(x-y) \in P^{n-1} .$$

$\tilde{\sigma}$ admits a structure of the complex analytic manifold and is a

modification of $\mathbb{C}^n \times \mathbb{C}^n$ (see e.g. [W1] p.260). Now

$$C(X,Y) = (\sigma_{X \times Y} \times \check{P}^{n-1}) \cap \alpha(C(X) \times Y \times P^{n-1})$$

where α is a suitable permutation of the Cartesian product $X \times \check{P}^{n-1} \times Y \times P^{n-1}$. Therefore $C(X,Y)$ is analytically constructible as an intersection of two constructible sets. On the other hand locally $C(X,Y)$ is biholomorphic to $\sigma_{X \times Y} \times S^{n-1}(\mathbb{C}^k)$ where $k = \dim X$ and $S^{n-1}(\mathbb{C}^k)$ denotes Schubert's cycle of all hyperplanes $H \supset \mathbb{C}^k$ in $\mathbb{C}^n$, so we have that $C(X,Y)$ is a connected manifold of dimension $\dim Y + n - 1$.

Corollary 2. The set $\Omega = \overline{C(X,Y)} \cap (\Delta_G \times P^{n-1} \times \check{P}^{n-1})$ is an analytic subset of $G \times G \times P^{n-1} \times \check{P}^{n-1}$ and $\dim \Omega = \dim Y + n-2$.

Proof. $\Delta_G \times P^{n-1} \times \check{P}^{n-1}$ is a hypersurface in $\tilde{\sigma}_{G \times G} \times \check{P}^{n-1}$ and $\overline{C(X,Y)}$ is the irreducible variety as a closure of a constructible leaf.

Now it is easy to prove that $\dim \Omega_{(y_o,y_o)} \geqslant n-2$, for if $q: P^{n-1} \times \check{P}^{n-1} \longrightarrow P^{n-1}$ is the natural projection then the projective tangent cone $C^*(X,y_o)$ is contained in $q\left(\Omega_{(y_o,y_o)}\right)$. $\dim C^*(X,y_o) = \dim X - 1$ and for any τ such that $T_{x_n}X \longrightarrow \tau$, $\mathbb{C}(x_n - y_n) \longrightarrow 1$, we have $\dim S^{n-1}(\tau) = n - \dim X - 1$, so by Lemma 1 $\dim \Omega_{(y_o,y_o)} \geqslant n - \dim X - 1 + \dim C^*(X,y_o) = n - 2$.

3.1. In this section we shall prove that (b)- regularity implies that $\dim \Omega_{(y_o,y_o)} = n - 2$.

First proof: The simplest way to prove it is to apply the theory of absolute polar varieties (see e.g. [TE], [H-M]) i.e. let us denote $p: P^{n-1} \times \check{P}^{n-1} \longrightarrow \check{P}^{n-1}$ the natural projection onto the second factor, $k = \dim X$, Z any irreducible component of the fiber $\Omega_{(y_o,y_o)}$, $r = \dim p(Z)$. Now, if E is a sufficiently general linear subspace of $\mathbb{C}^n$ of dimension r such that $S^{n-1}(E)$ is transverse to $p(Z)$ and E defines polar variety $P_{k+r-n+1}$ of codimension $k+r-n+1$ in X then $\dim[S^{n-1}(E) \cap p(Z)] = 0$ and for every $H \in S^{n-1}(E) \cap p(Z)$

$$\dim\left\{l \in P^{n-1} : (l,H) \in \Omega_{(y_o,y_o)}\right\} \leqslant \dim C^*(P_{k+r-n+1}, y_o) = n - r - 2.$$

From Lemma 1 we have that $\dim Z \leqslant n-r-2 + \dim p(Z) = n - 2$.

Second proof: Another proof that $\dim \Omega_{(y_o,y_o)} = n - 2$ is possible if we use the following theorem of V. Navarro Aznar (one disadvantage of this method is that we must use the equivalence between (b)- regularity and (w)- regularity of J. L. Verdier).

Lemma 2. (V. Navarro Aznar [N] Theorem 3.5) Let X be an analytically constructible leaf in $\mathbb{C}^r \times \mathbb{C}^s$, Y open subset of $\mathbb{C}^s$ such that $Y \cap (\overline{X} \setminus X) \neq \emptyset$. If X is (w)- regular over Y at the point $y_o = 0 \in Y$, then there exists stratification $\bigcup_l N_l$ of the fiber of the normal cone $C(\overline{X},Y)_{y_o}$ over y_o such that:

if $a(z) = (p(z),q(z)) \in \mathbb{C}^r \times \mathbb{C}^s$ is an analytic arc with $a(z) \longrightarrow y_o$ and $\frac{p(z)}{|p(z)|} \longrightarrow x \in C(\overline{X},Y)_{y_o}$ then:

$\tau = \lim_{z \to y_o} T_{a(z)}X$ containes tangent space to this stratum N_{l_o} which containes x.

Now, let us take $x \in N_l$, then we have analytic arc a(z) such that: $\tau = \lim_{z \to y_o} T_{a(z)}X \supset T_xN_l + Y$, this means that

$$\dim X = \dim \tau \geqslant \dim N_l + \dim Y = l + \dim Y .$$

If we denote by K_l the restriction of $\Omega_{(y_o,y_o)}$ to the projectivization of (N_l+Y) we obtain (using Lemma 1) that:

$$\dim K_l \leqslant \dim(N_l+Y)^* + \dim S^{n-1}(T_xN_l+Y) = n - 2 .$$

$\Omega_{(y_o,y_o)} = \bigcup_l K_l$, hence we have proved that

$$\dim \Omega_{(y_o,y_o)} = n - 2 .$$

Question 1.

Is it possible to obtain the stratification of Lemma 2 without use of equivalence $(w) \Longleftrightarrow (b)$?

3.2. We shall prove the following: $\dim \Omega_{(y_o,y_o)} = (n - 2)$ implies that X is (b)- regular over Y at y_o.

Lemma 3. The set $S_b(X,Y)$ is a nowhere dense analytically constructible subset of Y.

Proof. $S_b(X,Y)$ is constructible as a proper projection of a constructible set $\Omega \cap (\Delta_{G\times G} \times \{l \subset H\})$. Using Whitney´s whing´s lemma it is easy to prove that $S_b(X,Y)$ is nowhere dense in Y(see e.g. [W2]).

Let us take $\Omega = \bigcup_\iota V_\iota$ decomposition of Ω into irreducible components. From Corollary 2 it follows that $\dim V_\iota \equiv \dim X + n-2$ for every ι. If $\dim \Omega_{(y_o,y_o)} = n - 2$ then by upper semicontinuity of fiber dimension we have that $\dim \Omega_{(y,y)} = n - 2$ in a neighbourhood U of (y_o,y_o). If we take

$$W = \{(x,x,l,H) \in \Delta_G \times P^{n-1} \times \check{P}^{n-1} : l \subset H\}$$

and V_ι any irreducible component of Ω which intersects

$(y_o,y_o) \times \Omega_{(y_o,y_o)}$ then $\dim(V_\iota \cap W) = \dim V_\iota$ (because $S_b(X,Y)$ is nowhere dense). Now it is easily seen that $V_\iota \cap W = V_\iota$ because V_ι is irreducible and hence $V \subset W$. By free choice of V_ι,

$$\Omega_{(y_o,y_o)} \subset \{(l,H) \quad P^{n-1} \times \check{P}^{n-1} : l \subset H\},$$

so we have shown that X is (b)- regular over Y at y_o and this completes the proof of Theorem 1.

Corollary 3. If X, Y are two analytically constructible leaves in a complex manifold M such that $y_o \in Y \cap \overline{X}$ and X is (b)- regular over Y at y_o then $Y \subset \overline{X}$.

Proof. Let us suppose that X, Y are embeded in $\mathbb{C}^r \times \mathbb{C}^s$, Y is an open subset of $\mathbb{C}^s$ and $y_o = 0$. If $Y \not\subset \overline{X}$ then there exists a line $\mathbb{C}v \subset \mathbb{C}^s$ such that in a sufficiently small neighbourhood of 0 we have $\overline{X} \cap \mathbb{C}v = \{0\}$. Now, we can apply Theorem 1 and Corollary 2 to the pair $(X,\mathbb{C}v)$ and we have $\dim \Omega_{(y_o,y_o)} = \dim \Omega$,

$$\dim \Omega = \dim \mathbb{C}v + n - 2 = n - 1 .$$

This contradicts Theorem 1.

4. An example to the question of D. Trotman. In the paper about canonical Whitney stratifications ([T] p.17) D. Trotman asked whether the set $S_b(X,Y)$ is closed in the real case. Let us consider the following example of two semi-algebraic leaves:

$$X = \text{Regular points of } \{(u,v,x,y) \in \mathbb{R}^4 : y^2 = uv^2x^2 + x^4 , u > 0\}$$

$$Y = \{(u,v,x,y) \in \mathbb{R}^4 : x = y = 0\}.$$

Direct calculations show that

$$S_b(X,Y) = \{(u,v,x,y) \quad \mathbb{R}^4 : x = y = u = 0\} \setminus \{0\}.$$

Question 2.

Is it possible to obtain similar counterexample for two leaves of Łojasiewicz´s normal decomposition ?

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Z. Hajto
Inst. Mat. WSP
ul. Podchorazych 2
30-084 Kraków, Poland