# INVARIANTS OF WEIGHTED HOMOGENEOUS SINGULARITIES

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## 1.- Introduction :

In singularity theory one has first studied hypersurface singularities, where a holomorphic map germ $f : (\mathbb{C}^m,0) \to (\mathbb{C},0)$ is given. Especially simple is the example $f(z) = z_1^{a_1} + \ldots + z_m^{a_m}$ (Brieskorn polynomial). A more general class is given by the weighted homogeneous polynomials : Let $d, w_1, \ldots, w_m$ be positive integers and $f \in \mathbb{C}[z_1,\ldots,z_m]$ a polynomial.

Definition :

$f$ *is weighted homogeneous of degree* $d$ *with respect to the weights* $w_1, \ldots, w_m$ *if* $f$ *is a linear combination of monomials* $z_1^{j_1} \cdot \ldots \cdot z_m^{j_m}$, $j_1 w_1 + \ldots + j_m w_m = d$.

Let us assume that $f$ is such a weighted homogeneous polynomial which has an isolated singularity at $0$ and that $n := m-1 > 0$. Let $\varepsilon$ be a positive real number (in general, $\varepsilon$ should be small, but since $f$ is weighted homogeneous this is not necessary here), $t \in \mathbb{C}$, $0 < |t| \ll \varepsilon$, $Y_t = f^{-1}(\{t\})$, $Y_0 = f^{-1}(\{0\})$. Then $B_\varepsilon \cap Y_t$ is the Milnor fibre of $f$ and $\Sigma = \partial B_\varepsilon \cap Y_0$ the link of $Y_0$ at $0$. It is well-known that $B_\varepsilon \cap Y_t$ has the homotopy type of a bouquet of spheres of dimension $n$, their number is called the Milnor number $\mu$, and the singularity of $Y_0$ at $0$ is determined by $\Sigma$ since $B_\varepsilon \cap Y_0$ is homeomorphic to the cone over $\Sigma$. There is an endomorphism $h^*$ of $H^n(B_\varepsilon \cap Y_t;\mathbb{Z})$ - the (Picard-Lefschetz) monodromy - such that we have an exact sequence (with coefficients $\mathbb{Z}$) :

$$0 \to H_n(\Sigma) \to H^n(B_\varepsilon \cap Y_t) \xrightarrow{h^*-id} H^n(B_\varepsilon \cap Y_t) \to \widetilde{H}_{n-1}(\Sigma) \to 0$$

Cf. [6].

As we are looking at the weighted homogeneous case $B_\varepsilon \cap Y_t$ and $B_\varepsilon \cap Y_0$ are deformation retracts of $Y_t$ and $Y_0$, respectively. Therefore we will consider $Y_t$ instead of $B_\varepsilon \cap Y_t$, the condition $|t| \ll \varepsilon$ is no longer necessary then.

Let us list some invariants which have been calculated in the weighted homogeneous case already long ago :

$\mu = \mathrm{rk}\, H_n(Y_t;\mathbb{Z})$ : Milnor-Orlik [7]

characteristic polynomial $\chi(x)$ of $h^*$ : Milnor-Orlik [7]

$\mathrm{rk}\, \widetilde{H}_{n-1}(\Sigma;\mathbb{Z})$ : Orlik [8] (as a consequence of [7])

$\sigma$ = signature of $Y_t$ : Steenbrink [11]

Hodge numbers of $Y_t$ (with respect to the mixed Hodge structure) : Steenbrink [11].

In the next paragraph we will briefly recall the methods used in the hypersurface case and go over in the third paragraph to complete intersections. As a by-product we will get a further result for the hypersurface case, by determining the group $\widetilde{H}_{n-1}(\Sigma,\mathbb{Z})$ .

2.- Methods in the hypersurface case :

Let us shortly discuss how the computations of the invariants cited above have been performed :

a) The Milnor number $\mu$ can be calculated via the mapping degree of $(z_1,\dots,z_m) \to \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_m}\right)$ , because of the formula $\mu = \dim \mathcal{O}_{\mathbb{C}^m,0}/\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_m}\right)$ , cf. [7].

b) For the study of the endomorphism $h^*$ of $H^n(Y_t;\mathbb{Z})$ it is useful to note that $f : \mathbb{C}^m \to \mathbb{C}$ is equivariant with respect to $\mathbb{C}^*$ actions on $\mathbb{C}^m$ and $\mathbb{C}$ : $c \circ z := (c^{w_1} z_1,\dots,c^{w_m} z_m)$ , $c \circ t' := c^d t'$ for $c \in \mathbb{C}^*$ , $z \in \mathbb{C}^m$ , $t' \in \mathbb{C}$ . Then $h^*$ is induced by $h : Y_t \to Y_t : h(z) = e^{2\pi i/d} \circ z$ . From the Euler characteristics of the fixed point sets $\{z \in Y_t | h^\nu(z) = z\}$ , $\nu = 1, 2, \dots$ , one can compute the characteristic polynomial of $h^*$ , cf. [7].

c) The signature $\sigma$ of $Y := Y_t$ can be computed from the Hodge numbers, cf. [11]. In order to understand the mixed Hodge structure on $Y$ let us begin with the special case $w_1 = \dots = w_m = 1$ ($f$ homogeneous) : then $Y$ is the affine part of some smooth projective hypersurface $\bar{Y}$ of degree $d$ in $\mathbb{P}_m(C)$ In general one takes $\mathbb{P}_w :=$ Projan $\mathbb{C}\,[z_0,\dots,z_m]$ , where $\deg z_j = w_j$ , $j = 0, \dots, m$ , $w_0 = 1$ , instead of $\mathbb{P}_m(\mathbb{C})$ . The underlying topological space of $\mathbb{P}_w$ is the quotient of $\mathbb{C}^{m+1} - \{0\}$ by the $\mathbb{C}^*$ action which corresponds to the weights $w_0, \dots, w_m$. Now $\mathbb{P}_w$ is no longer smooth but still a V-manifold, hence a rational homology manifold. Let $\tilde{f} \in \mathbb{C}[z_0,\dots,z_m]$ be defined by $\tilde{f}(z_0,\dots,z_m) = f(z_1,\dots,z_m) - t\,z_0^d$. Then $\tilde{f}$ is weighted homogeneous of degree $d$ with respect to $w_0, \dots, w_m$. The equation $\tilde{f} = 0$ defines subvarieties $\overline{Y}$ and $Y_\infty$ of $\mathbb{P}_w$ and $\{[z] \in \mathbb{P}_w \mid z_0 = 0\}$ which are V-manifolds. Since $\overline{Y}$ and $Y_\infty$ are compact algebraic varieties they have canonical mixed Hodge structures according to Deligne [1], but in fact these are pure because we have V-manifolds, so we can speak of Hodge numbers $h^{pq}(\overline{Y}) = \dim_{\mathbb{C}} Gr_F^p\ H^{p+q}(\overline{Y};\mathbb{C})$. More important for our purpose are the numbers $h_0^{pq}(\overline{Y}) = \dim_{\mathbb{C}} Gr_F^p\ H^{p+q}(\overline{Y};\mathbb{C})_0$, where $H^j(\overline{Y};\mathbb{C})_0$ denotes the $j$-th primitive cohomology group. Then $h_0^{pq}(\overline{Y}) = h^{pq}(\overline{Y}) - \delta_{pq}$, $0 \le p \le \dim \overline{Y}$. From the Gysin sequence for $Y_\infty \subset \overline{Y}$ one gets :

$$\dim Gr_F^p\, Gr_n^W\ H^n(Y;\mathbb{C}) = h_0^{p,n-p}(\overline{Y})$$

$$\dim Gr_F^p\, Gr_{n+1}^W\ H^n(Y;\mathbb{C}) = h_0^{p-1,n-p}(Y_\infty)$$

$$Gr_i^W\, H^n(Y;\mathbb{C}) = 0 \quad , \quad i \neq n \quad , \quad n+1 \ .$$

See [11] for details. By generalizing Griffiths' description of the Hodge structure for a projective hypersurface [3] Steenbrink was able to describe the spaces $F^p\, Gr_i^W\ H^n(Y;\mathbb{C})$ explicitly, see [11].

## 3.- Complete intersections :

Let us leave the hypersurface case now and assume that $f_1, \dots, f_k$ are polynomials such that $f_j$ is weighted homogeneous of degree $d_j$ with respect to $w_1, \dots, w_m$ for $j = 1, \dots, k$. Here $w_1, \dots, w_m$, $d_1, \dots, d_k$ are positive integers. Let $f : \mathbb{C}^m \to \mathbb{C}^k$ be defined by $f = (f_1,\dots,f_k)$, let $t$ be a regular value of $f$, $Y_t = f^{-1}(\{t\})$, $Y_0 = f^{-1}(\{0\})$, $n = \dim Y_0$, $\varepsilon > 0$, $\Sigma = Y_0 \cap \partial B_\varepsilon$. In order to be able to compute invariants from the weights and degrees alone we assume that $n = m - k$ (i. e. $Y_0$ is a complete intersection) and that $Y_0 - \{0\}$ is non-singular, $n > 0$. Again $Y_t$ has the homotopy type of a bouquet of spheres of dimension $n$ [5]. Let us discuss the calculation of invariants now.

a) $\mu = \mathrm{rk}\, H_n(Y_t;\mathbb{Z})$ : This invariant has been calculated using differential forms in [2], the method is not just a generalization of the method described in the hypersurface case.

b) The monodromy is a more complicated object in the case $k \geq 2$ than for $k = 1$ : one has an action of $\pi_1(\mathbb{C}^k - D, t)$ on $H^n(Y_t;\mathbb{Z})$ , $D$ being the discriminant of $f$ . For $k = 1$ , $\pi_1(\mathbb{C}^k - D, t) \cong \mathbb{Z}$ , and the endomorphism $h^*$ of $H^n(Y_t;\mathbb{Z})$ introduced in this case corresponds to the action of the canonical generator. In the weighted homogeneous case, however, we have another possibility of generalizing the definition of $h^*$ . There is a $\mathbb{C}^*$ action on $\mathbb{C}^k$ defined by $c \circ t' = (c^{d_1} t'_1, \ldots, c^{d_k} t'_k)$ , $c \in \mathbb{C}^*$ , $t' \in \mathbb{C}^k$ , such that $f : \mathbb{C}^m \to \mathbb{C}^k$ is equivariant. Let $d$ be a positive integer such that $t = e^{2\pi i/d} \circ t$ , i. e. $d | d_j$ for all $j$ with $t_j \neq 0$ . Let $h^* : H^n(Y_t;\mathbb{Z}) \to H^n(Y_t;\mathbb{Z})$ be induced by $h(z) = e^{2\pi i/d} \circ z$ . Now the characteristic polynomial $\chi(x)$ of $h^*$ can be calculated by the results of [2].

c) The algebraic variety $Y = Y_t$ has a mixed Hodge structure which can be described just as in the hypersurface case : Let $\tilde{f}_j \in \mathbb{C}[z_0, \ldots, z_m]$ be defined by $\tilde{f}_j(z_0, \ldots, z_m) = f_j(z_1, \ldots, z_m) - t_j z_0^{d_j}$ , then $\tilde{f}_1 = \ldots = \tilde{f}_k = 0$ defines subvarieties $\overline{Y}$ and $Y_\infty$ of $\mathbb{P}_w$ and $\{[z] \in \mathbb{P}_w | z_0 = 0\}$ , and the results on $Gr_F^p\, Gr_i^W\, H^n(Y;\mathbb{C})$ are the same as in the hypersurface case. In order to state the formulae it is convenient to use the abbreviation

$$Q(x,y) = \frac{1}{1+y} \prod_{\nu=1}^{m} \frac{1 + y\, x^{w_\nu}}{1 - x^{w_\nu}} \prod_{\kappa=1}^{k} \frac{1 - x^{d_\kappa}}{1 + y\, x^{d_\kappa}} .$$

<u>Theorem 1</u> (cf. also formula (1) in [4]) :

$$h_0^{p,n-p}(\overline{Y}) = (-1)^{n-p} \operatorname{res}_{x=\infty} \operatorname{res}_{y=0}\, x^{-1} y^{-p-1} \frac{1+yx}{1-x} Q(x,y) ,$$

$$h_0^{p-1,n-p}(Y_\infty) = (-1)^{n-p} \operatorname{res}_{x=\infty} \operatorname{res}_{y=0}\, x^{-1} y^{-p} Q(x,y) .$$

The proof uses the description of the pure Hodge structure of projective varieties which are V-manifolds due to Steenbrink [11], it will be published elsewhere, as the proofs of the following theorems.

The same technique also yields more information about $h^*$ than that obtained in b). Note that $h^d = id$, therefore $h^* : H^n(\bar{Y};\mathbb{C})_0 \to H^n(\bar{Y};\mathbb{C})_0$ is diagonalizable, and the eigenvalues are of the form $e^{2\pi ir/d}$, $r \in \mathbb{Z}$. Let $h_0^{p,n-p}(e^{2\pi ir/d})$ be the dimension of the subspace of $Gr_F^p\, H^n(\bar{Y};\mathbb{C})_0$ on which $h^*$ operates as multiplication by $e^{2\pi ir/d}$.

Theorem 2 :

$h_0^{p,n-p}(e^{2\pi ir/d}) =$

$$= (-1)^{n-p}\, \mathrm{res}_{z=\infty}\, \mathrm{res}_{x=\infty}\, \mathrm{res}_{y=0}\; x^{-1}\, y^{-p-1}\, z^{r-1}\, \frac{z^d}{1-z^d}\, \frac{1+yzx}{1-zx}\, Q(x,y) .$$

Note that $h^*$ acts on $H^{n-1}(Y_\infty;\mathbb{C})_0$ as the identity. Now let $n$ be even, let $S$ be the intersection form on $H_c^n(Y;\mathbb{C})$ and $S^h$ the hermitian form on $H_c^n(Y;\mathbb{C})$ defined by $S^h(x,y) = S(x,\bar{y})$. Let $\mu^+(e^{2\pi ir/d})$ be the dimension of a maximal linear subspace of $H_c^n(Y;\mathbb{C})$ on which $S^h$ is positive definite and $h^*$ acts as multiplication by $e^{2\pi ir/d}$. Let $\mu^-(e^{2\pi ir/d})$ and $\mu_0(e^{2\pi ir/d})$ be defined in an analogous way, with "negative definite" resp. " identically zero" instead of "positive definite". From Theorem 2 and [11] one obtains

Theorem 3 :

If $n$ is even we have :

$$\mu_+(e^{2\pi ir/d}) = \mathrm{res}_{z=\infty}\, \mathrm{res}_{x=\infty}\, \mathrm{res}_{y=0}\; x^{-1}\, y^{-n-1}\, z^{r-1}\, \frac{1}{1-y^2}\, \frac{z^d}{1-z^d}\, \frac{1+yzx}{1-zx}\, Q(x,y) ,$$

$$\mu_-(e^{2\pi ir/d}) = \mathrm{res}_{z=\infty}\, \mathrm{res}_{x=\infty}\, \mathrm{res}_{y=0}\; x^{-1}\, y^{-n}\, z^{r-1}\, \frac{-1}{1-y^2}\, \frac{z^d}{1-z^d}\, \frac{1+yzx}{1-zx}\, Q(x,y) ,$$

$$\mu_0(e^{2\pi ir/d}) = 0 \quad \text{if } d \nmid r ,$$

$$\mu_0(1) = \dim H^{n-1}(Y_\infty;\mathbb{C})_0 = \mathrm{rk}\, \tilde{H}_{n-1}(\Sigma;\mathbb{Z}) \quad \text{(see Theorem 4).}$$

d) Calculation of $\widetilde{H}_{n-1}(\Sigma;\mathbb{Z})$ : As Steenbrink pointed out to me, $\widetilde{H}^{n-1}(Y_0-\{0\};\mathbb{C}) \cong H^{n-1}(Y_\infty;\mathbb{C})_0$ because of the existence of a rational Gysin sequence for the map $Y_0-\{0\} \to Y_\infty$ which is the canonical map to the orbit space with respect to the $\mathbb{C}^*$ action. Using this and the homotopy equivalence between $Y_0-\{0\}$ and $\Sigma$ we obtain

Theorem 4 :

$$\mathrm{rk}\ \widetilde{H}_{n-1}(\Sigma;\mathbb{Z}) = \mathrm{res}_{x=\infty}\ \mathrm{res}_{y=0}\ x^{-1}\ y^{-n}\ \frac{1}{1+y}\ Q(x,y)\ .$$

So it remains to calculate the torsion subgroup of $\widetilde{H}_{n-1}(\Sigma;\mathbb{Z})$ . There is an explicit way of doing this for $n$ odd by looking at certain fixed point sets similarly to section b) of the second paragraph.

In the special case where $f_1, \ldots, f_k$ are Brieskorn polynomials the torsion has been calculated for all $n$ by Randell [10].

The following object is related to $\widetilde{H}_{n-1}(\Sigma;\mathbb{Z})$ : Let us assume that $\widetilde{Y}_0 = \{z \in \mathbb{C}^m | f_1(z) = \ldots = f_{k-1}(z) = 0\}$ has also an isolated singularity at $0$ . Then let us consider $H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z})$ , where $\widetilde{\Sigma} = \partial B_\varepsilon \cap \widetilde{Y}_0$ .

e) Calculation of $H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z})$ : Because of the assumption just made we may choose $t = (0,\ldots,0,t_k)$ and $d = d_k$ . Then the exact sequence of the introduction has the following analogue (cf. [5], coefficients : $\mathbb{Z}$) :

$$0 \to H_{n+1}(\widetilde{\Sigma},\Sigma) \to H^n(Y) \xrightarrow{h^*-\mathrm{id}} H^n(Y) \to H_n(\widetilde{\Sigma},\Sigma) \to 0\ .$$

As we can compute the characteristic polynomial $\chi(x)$ of $h^*$ (cf. b)), we can deduce a formula for $\mathrm{rk}\ H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z})$ .

Note that there is an exact sequence

$$0 \to H_n(\widetilde{\Sigma};\mathbb{Q}) \to H_n(\widetilde{\Sigma},\Sigma;\mathbb{Q}) \to \widetilde{H}_{n-1}(\Sigma;\mathbb{Q}) \to 0$$

(since $\widetilde{H}_{n-1}(\Sigma;\mathbb{C}) \cong H^{n-1}(Y_\infty;\mathbb{C})_0$ etc...) so that $\mathrm{rk}\ H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z})$ can also be calculated from Theorem 4. On the other hand, one can prove Theorem 4 inductively using a suitable formula for $\mathrm{rk}\ H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z})$ .

But in fact, the whole group $H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z})$ can be calculated from $\chi(x)$ : Write $\chi = m_1 \cdot \ldots \cdot m_\mu$ where $m_1 | m_2, \ldots, m_{\mu-1} | m_\mu$ and $m_\mu$ is square-free, $m_1, \ldots, m_\mu \in \mathbb{Z}[x]$ .

## REFERENCES

[1] P. DELIGNE, Théorie de Hodge III.
Publ. Math. IHES 44, 5-77 (1975).

[2] G.M. GREUEL, H.A. HAMM, Invarianten quasihomogener vollständiger Durchschnitte. Invent. Math. 49, 67-86 (1978).

[3] Ph. GRIFFITHS, On the periods of certain rational integrals : I, II.
Ann. of Math. (2), 90, 460-541 (1969).

[4] H.A. HAMM, Genus $\chi_y$ of quasihomogeneous complete intersections Russian .
Funkcional. Anal. i Priložen. 11, n° 1, 87-88 (1977) = Functional Anal. Appl. 11, 78-79 (1977).

[5] H.A. HAMM, Lokale topologische Eigenschaften komplexer Räume.
Math. Ann. 191, 235-252 (1971).

[6] J. MILNOR, Singular points of complex hypersurfaces.
Ann. of Math. Studies 61 (1968).

[7] J. MILNOR, P. ORLIK, Isolated singularities defined by weighted homogeneous polynomials.
Topology 9, 385-393 (1970).

[8] P. ORLIK, On the homology of weighted homogeneous manifolds.
Proc. Second Conf. Transf. Groups I, Springer LN 298 (1972), 260-269.

[9] P. ORLIK, R. RANDELL, The monodromy of weighted homogeneous singularities.
Invent. Math. 39, 199-211 (1977).

[10] R.C. RANDELL, The homology of generalized Brieskorn manifolds.
Topology 14, 347-355 (1975).

[11] J.H.M. STEENBRINK, Intersection form for quasi-homogeneous singularities.
Compositio Math. 34, 211-223 (1977).

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Theorem 5 :

$$H_n(\widetilde{\Sigma},\Sigma;\mathbb{Z}) = (\mathbb{Z}/m_1(1)\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/m_\mu(1)\mathbb{Z}) .$$

In the case $k = 1$ we have $\widetilde{\Sigma} = \partial B_\varepsilon$ , so $H_n(\widetilde{\Sigma},\Sigma) \cong \widetilde{H}_{n-1}(\Sigma)$ :

Corollary :

$$\widetilde{H}_{n-1}(\Sigma;\mathbb{Z}) = (\mathbb{Z}/m_1(1)\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/m_\mu(1)\mathbb{Z}) \quad \text{if} \quad k = 1 .$$

The corollary has been conjectured by Orlik [8] and proved in special cases by Orlik and Randell (see [8], [9], [10]).