ON THE CLASSIFICATION OF SMOOTH PROJECTIVE SUBVARIETIES OF P^N

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§ 0. Introduction. There can be no question that the interplay between the computer and pure mathematics has added a new and excieting facet to our science.

The ways in which the computer has influenced pure mathematics are many, and not limited only to the various uses of the computer as a tool. But the variety of uses this tool has been put to is in itself facinating, ranging as it does from on the one hand large but relatively unsophisticated numerical computations via numerical experimentation to genuine computer aided proofs of mathematical theorems. Add to this features like the development and use of advanced systems for symbolic manipulation like MACSYMA or SCRACHPAD.

Beyons its role as a tool, the computer is starting to influence the way in which mathematicians think aboute mathematics. This manifests itself in the growing interest in constructive existence proofs, the efficiency of the corresponding algorithms etc. But also in some cases in a new perspective on the concept of a mathematical theorem, or better: fact or result. And of course also the concept of a mathematical proof, as exemplified by a continuing philosophical controversy over computer aided proofs.

The point of view we take on the calssification of smooth, projective subvarieties of P^N in this paper should be understood in the context above. It centers on the Classification Program formulated below, in §1. However, since this is obviously out of reach at the present time, we may at least propose to do the following, which is possible by definition: Collect all known variety generating algorithms and classify the output, (within some bounds).

To carry out such a program say for curves and surfaces would generate important new mathematical knowledge, but not necessary mean proving new theorems. That one might hope for - or fear - counterexamples to existing conjectures is another matter.

§ 1. The problems. Our aim here is to present some ideas on the following general classification problem in algebraic geometry. Let $P_k^N = P^N$ be projective N-space over some $k = \bar{k}$; where we may think of $k = \mathbb{C}$ to focus our ideas. Let $n < N$ be an integer.

<u>Problem 0.1</u> <u>Classify in some precise, definitive way, all smooth projective subvarieties of P^N.</u>

Part of the problem is of course to specify the type of classification one wants. If the approach of the Hilbert scheme is taken, then the solution should amount to understanding the structure of $\underline{\mathrm{Hilb}}_P P^N$, the Hilbert scheme parametrizing all smooth, projective subvarieties of P^N with Hilbert polynomial P, cf [Har 2]. In general this is a hard problem. Moreover, it is not even known exactly when $\underline{\mathrm{Hilb}}_P P^N \neq \emptyset$, i.e. when a given polynomial P can be realized as the Hilbert polynomial of some smooth X in P^N. Even for curves in P^3 this was open until recently, but solved in [G.P.].

So following [Har 2] one naturally divides the problem in two parts:

Let there be given a set of numerical invariants $e_0(X), \ldots, e_m(X)$ defined for all smooth, projective n-dimensional subvarieties of P^N. (Say, the coefficients of the Hilbert polynomial as above.)

<u>Problem 0.2</u> Determine the range of the invariants

$$R(n,N) = \left\{ (e_0, \ldots e_m) \in \mathbb{Q}^{m+1} \;\middle|\; \begin{array}{l} \exists \text{ smooth projective} \\ X \subset P^N \text{ with} \\ \underline{e}(X) = \underline{e} \end{array} \right\}$$

Here we put $\underline{e} = (e_0, \ldots, e_m)$. It turns out to be convenient to examine the closely related set $S(n,N)$ defined by those subvarieties X which are <u>strictly</u> contained in P^N, i.e. which are not contained in a hyperplane. Then of course

$$R(n,N) = \bigcup_{m=n}^{N} S(n,m)$$

We assume for the rest of this paragraph that $e_0(X), \ldots, e_m(X)$ are the coefficients of the Hilbert polynomial of X.

Clearly Problem 0.2 is simpler than Problem 0.1. The gap between them consists in the

Problem 0.3. Given $(e_0, \ldots, e_m) \in R(n,N)$. Describe the set of all smooth projective sabvarieties of P^N which have these numbers as invariants.

The most ambitious version of this last problem would be to carry out the following program:

Classification Program Specify a set of algorithms which produce all smooth projective subvarieties of P^N of a given dimension n and given set of invariants.

The output from these algorithms should be sets of explicitly described classes of subvarieties, say such as the set of all complete intersections, or varieties of the form

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^r}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^r}(a_s))$$

with the tantological embedding, etc.; or special varieties such as P^r blown up at some locus and embedded by a specified sheaf. For samples of such results, see [Ho 5], [Har 1], [Io], [Ok].

Complete sets of algorithms in the sense of the Classification Program above are not known except for the trivial case of $n = N - 1$: Even for smooth curves in P^3 it is open, but here at least one knows the domain $R(1,3)$, as finally settled in [GP]. Thus the question arises if one can specify curve generating algorithms which cover $R(1,3)$ completely. This does not seem to follow from the methods in [GP].

On the other hand, for smooth surfaces in P^4 not even $R(2,4)$ is known. A full solution to the classification program in this case is appearently out of reach for the time being, while there are some interesting results for small degrees, [Io], [Ok] (and Oknek's work refered to there) and [EP].

In the absence of complete lists of algorithms, one should do the following: Collect as many variety-generating algorithms as possible, and classify their output. A first coarse classsification obviously is to list the numerical invariants thus generated, in order to get a better understanding of the domains $R(n,N)$.

For curves in P^3 this turns out to be quite successfull.

§ 2. Curves in P^3. Let X be a smooth curve in P^3 of degree d and genus g . By Riemann Roch, the Hilbert polynomial is

$$HP_X(t) = dt + 1 - g ,$$

so that with the terminology of §1,

$$e_0(x) = d , \quad e_1(x) = 1 - g .$$

Thus to know R(1,3) is equivalent to knowing

$$G(3,1) = \left\{ (d,g) \,\middle|\, \begin{array}{l} \exists \text{ a smooth projective} \\ \text{curve in } P^3 \text{ of degree} \\ d \text{ and genus } g \end{array} \right\}$$

In 1882 G. Halphen [Hal] claimed a description of the set G(1,3). But his proof contained a serious gap, and it was only recently that a satisfactory proof was given by L. Gruson and Chr. Peskine, in [GP] . The result is the following:

Theorem 2.1 (Halphen, Gruson-Peskine) G(1,3) consists of all points $(d,g) \in Z^2$ such that $d \geq 1$, $g \geq 0$ and either

(i) $g = \frac{1}{2}(d - 1)(d - 2)$

or

(ii) $g = (a - 1)(b - 1)$, $d = a + b$ where a and b are integers

or

(iii) $g \leq \frac{1}{6} d(d - 3) + 1$

As we shall see below, there are several natural curve-generating algorithms available. While we have not undertaken to collect and classify all known such algorithms, we have studied and experimented with those which are implicitly given in the textbook [Har 4]. It came as a surprise that a selection of some os these algotithms was abel to fill the area described by (i) - (iii) above, at least for $d \leq 100$, cf [Ho 5]. Unfortunately we are not able to prove that this is the case for all d, this would yield a proof of the theorem of an elementary number theoretical mature. We now explain the algorithms.

A key tool in generating smooth curves in P^3 is the following proposition, see [Har 2]:

<u>Proposition 2.2</u> <u>The pairs</u> (d,g) <u>which corresponds to the smooth curves on the non singular cubic surface in</u> P^3 <u>have</u> (d,g) <u>given by</u>

$$d = 3a - \Sigma_{i=1}^{6} b_i$$
$$g = \binom{a-1}{2} - \Sigma_{i=1}^{6} \binom{b_i}{2}$$

where

$$a > 0,\ b_1 \geq \ldots \geq b_6 \geq 0$$
$$a \geq b_1 + b_2,\ 2a \geq b_1 + \ldots + b_5$$
$$a^2 > b_1^2 + \ldots + b_6^2$$

By this proposition a large number of (d,g) pairs are generated. However, there definitely are some gaps, showing some striking patterns. In part this is explained in [GP].

The next proposition makes it possible to obtain many curves of higher degree form lower degree ones.

<u>Proposition 2.3</u>. Let d_i, g_i $i = 1,2$ be such that $(d_i,g_i) \in G(1,3)$. Moreover, let a_i be integers such that

$$a_i \geq \begin{cases} g_i+3 & \text{if } g_i \geq 2 \\ 3 & \text{if } g_i = 1 \\ 1 & \text{if } g_i = 0 \end{cases}$$

Put

$$d = d_2a_1 + d_1a_2,\quad g = g_1 + g_2 - g_1g_2 + (a_1 + g_1 - 1)(a_2 + g_2 - 1)$$

Then $(d,g) \in G(1,3)$.

<u>Proof</u>. We need the following result, which is due to G. Halphen. For a proof, one may consult [Har 4] Chapther IV §6.

<u>Lemma</u>. <u>A curve</u> X <u>of genus</u> $g \geq 2$ <u>has a non-special very ample divisor of degree</u> d <u>if and only if</u> $d \geq g + 3$.

In particular it follows that there exists a very ample divisor D_i on X_i of degree a_i. Letting

$$Q = X_1 \times X_2$$

we get that

$$O_Q(D_1,D_2) = pr_1^* O_{X_1}(D_1) \otimes pr_2^* O_{X_2}(D_2)$$

is a very ample sheaf on Q. Hence the corresponding linear equivalence class of divisors contains non singular, irreducible curves by Bertini's Theorem. Let Y denote one of these. Then $J_Y = O_Q(-Y) = O_Q(-D_1,-D_2)$ is the Ideal sheaf of Y on Q. We thus have the exact sequence

$$0 \to O_Q(-D_1,-D_2) \to O_Q \to O_Y \to 0 .$$

The long exact cohomology sequence becomes

$$\begin{aligned} 0 &\to H^0(Q,O_Q(-D_1,-D_2)) \to H^0(Q,O_Q) \to H^0(Y,O_Y) \\ &\to H^1(Q,O_Q(-D_1,-D_2)) \to H^1(Q,O_Q) \to H^1(Y,O_Y) \\ &\to H^2(Q,O_Q(-D_1,-D_2)) \to H^2(Q,O_Q) \to 0 \end{aligned}$$

By the Künneth-formula we get

$$H^0(Q,O_Q(-D_1,-D_2)) = H^0(O_{X_1}(-D_1)) \otimes H^0(O_{X_2}(-D_2)) = 0$$

and similarly

$$\begin{aligned} H^1(Q,O_Q(-D_1,-D_2)) &= H^0(X_1,O_{X_1}(-D_1)) \otimes H^1(X,O_{X_2}(-D_2)) \\ &\oplus H^1(X_1,O_{X_1}(-D_1)) \otimes H^0(X_2,O_{X_2}(-D_2)) = 0 \end{aligned}$$

while

$$H^2(Q,O_Q(-D_1,-D_2)) = H^1(X_1,O_{X_1}(-D_1)) \otimes H^1(X_2,O_{X_2}(-D_2))$$

We also get, in the same way

$$\begin{aligned} H^1(O_Q) &= H^1(O_{X_1}) \oplus H^1(O_{X_2}) \\ H^2(O_Q) &= H^1(O_{X_1}) \otimes H^1(O_{X_2}) \end{aligned}$$

By Riemann-Roch's theorem for the curves X_1 and X_2 we get

$$\dim H^0(X_i,O_{X_i}(-D_i)) - \dim H^1(X_i,O_{X_i}(-D_i)) = -a_i - g_i + 1$$

so that

$$\dim H^1(X_i, O_{X_i}(-D_i)) = a_i + g_i - 1$$

Hence

$$\dim H^2(Q, O_Q(-D_1, -D^2)) = (a_1 + g_1 - 1)(a_2 + g_2 - 1)$$

Moreover,

$$\dim H^1(Q, O_Q) = g_1 + g_2$$
$$\dim H^2(Q, O_Q) = g_1 g_2$$

Thus the long cohomology sequence yields

$$g_1 + g_2 - g(Y) + (a_1 + g_1 - 1)(a_2 + g_2 - 1) - g_1 g_2 = 0$$

so that the claimed formula for $g(Y)$ follows.

In order to show the formula for d, note that the projective embedding of Q corresponding to the linear system $[Y]$ is the composition

$$X_1 \times X_2 \hookrightarrow P^3 \times P^3 \hookrightarrow P^{15}$$

where the first embedding is the product of the two given embeddings, and the last is the Segre-embedding. Let s, t, τ denote, respectively, the pullbacks of the hyperplane classes of P^3 via the first and the second projection, and the hyperplane class of P^{15}. Then we have

$$[Y] = [D_1 \times X_2] + [X_1 \times D_2] \in A(P^3 \times P^3)$$

so that

$$[Y] = d_2 a_1 s^3 t^2 + d_1 a_2 s^2 t^3$$

Since $s^3 t^2$ and $s^2 t^3$ are both mapped to τ^{14}, we are done.

Using this proposition, we may cover all but very few of the gaps left by Proposition 2.2, the lemma and the usual description of (d, g) for plane curves and the curves on the quadratic surface. In fact, for $d \leq 100$ a total of 39 gaps remain. It is interesting to note that for $75 \leq d \leq 100$ there are only 5 gaps. This is not so surprising, for the algorithm becomes better and better the more points

(d_i, g_i) we may use. Thus one might exept that this proposition is capable of filling all gaps for all $d \gg 0$.

In order to fill the remaining gaps for $d < 100$, one may use a proposition which is capable of generating a large family of curves. The idea is to reembedd the cubic surface by some very ample linear system, and look at the new degrees aquired by the curves in Proposition 2.2.

In fact, a curve given by parameters $a, b_1, \ldots, b_6$ as Proposition 2.2 is a very ample divisor if and only if the inequalities there are all strict. Using this, one easily proves the following proposition in the same way as Proposition 2.2.

Proposition 2.4. Let $a, b_1, \ldots, b_6$, $\alpha, \beta_1, \ldots, \beta_6$ satisfy the conditions in Proposition 2.2, the latter with strict inequalities, and let $\sigma \in S_6$. Then $(d,g) \in G(1,3)$, where

$$d = a\alpha - \Sigma b_i \beta_{\sigma(i)}$$
$$g = \binom{a-1}{2} - \Sigma\binom{b_i}{2}$$

Using this proposition as well it is now possible to fill all the gaps for $d < 100$. In fact, I expect that Proposition 2.3 and 2.4 should be capable of filling all of $G(1,3)$.

§ 3. Surfaces. Having seen the efficiency by which rather elementary curve generating algerithms cover $R(1,3)$, it lies near to implement similar algorithms for surfaces and tabulate the numerical invariants thus generated.

To this end we consider the numerical invariants consisting of the degrees of all monomials in the Chern-classes of the smooth surface X with respect to the given embedding:

$$e_{ij}(X) = \deg\,(c_1(X)^i\, c_2(X)^j),\quad i + 2j < 2$$

So

$$e_{00}(X) = d = \deg\,(X)$$
$$e_{10}(X) = \deg\,(c_1(X)) = -\deg\,(K_X) = -K_X \cdot H$$
$$e_{20}(X) = \deg\,(c_1(X)^2) = c_1(X)^2 = K_X{}^2$$
$$e_{01}(X) = \deg\,(c_2(X)) = c_2(X)$$

Here K_X is the canonical divisor and as usual we do not distinguish between the zero cycle and its degree in the notation. The divisor H is the pullback of a general hyperplane. Put

$$(e_0, e_1, e_2, e_3) = (e_{00}, e_{10}, e_{20}, e_{01})$$

With this choice of invariants the first problem now becomes to determine

$$R(2,N) \subset Z^4 .$$

It is sometimes convenient to replace e_{01} by the Euler characteristic since by Noethers Formula we have

$$\chi = \frac{1}{12}(c_1^2 + c_2) = \frac{1}{12}(e_{20} + e_{01}) .$$

In particular

(3.1) $\quad e_{20} + e_{01} \equiv 0 \pmod{12}$

Further, let X_H denote the generic sectional curve, i.e. $X \cap H$ where H is a hyperplane in general position. The genus of X_H is then given by

$$g(X_H) = 1 + \frac{1}{2}(e_{00} - e_{10})$$

as we find by a simple computation. Thus

(3.2) $\quad e_{00} + e_{10} \equiv 0 \pmod 2$

The congruences (3,1) and (3,2) have a common source: A formal computation with Riemann Roch yields that if X has the invariants $e_{00}, \ldots, e_{01}$, then the Hilbert Polylomial has the form

$$HP_X(t) = \frac{1}{2} e_{00} t^2 + \frac{1}{2} e_0 t + \frac{1}{12}(e_{20} + e_{01})$$

and the conditions express precicely the requirement that

$$HP_X(m) \in Z \quad \text{for all} \quad m = 0, 1, 2, \ldots\ldots$$

In this way we obtain n such conditions if $\dim(X) = n$.

Another class of conditions are inequalities which come from several sources. First of all, in the terminology of [Ho 3] the embedding obstructions must be non negative. (Note the change of sign from [Ho 2].) This yields the inequalities

$$(3.3)\quad e_{00}^2 - 10\, e_{00} + 5\, e_{10} - e_{20} + e_{01} \geq 0$$

and

$$(3.4)\quad 6e_{00} - 4e_{10} + e_{01} \geq 0$$

Similarly the theory from [Ho 1] yields inequalities of the form

$$\delta_s = \sum_{i=s}^{n} \binom{i+1}{s+1} \deg\, (c_{n-i}(\Omega_X)) \geq 0$$

when X is of dimension n and $s = o, \ldots, n$. We get for surfaces the inequalities

$$(3.5)\quad 3e_{00} - 2\, e_{10} + e_{01} \geq 0$$

$$(3.6)\quad 2e_{00} - e_{10} \quad \geq 0$$

However, the condition $g(X_H) \geq 0$ yields the sharper

$$(3.6)'\quad e_{10} \leq e_{00} + 2\ .$$

Next, we get an inequality which is best expressed for the set $S(2,N)$, invariants of those surfaces which are strictly contained in P^N. Then X_H is strictly contained in P^{N-1}, and we may apply the Castelnuovo Bound to X_H with $r = N - 1$. See [ACGH] p 116 for the form of this result we employ, here actually weakening the bound by taking $m = \dfrac{e_{00} - 1}{r - 1}$ rather than [ ] of this, and replacing ε by $r - 2$. A computation then yields the condition

$$(3.7)\quad e_{00}^2 - 4e_{00} + 9 - 3N + (N - 2)e_{10} \geq 0$$

Taking $N = 4$ we get the inequality

$$(3.7)'\quad e_{00}^2 - 4e_{00} - 3 + 2e_{10} \geq 0\ .$$

We finally have a condition for $R(2,4)$ which we do not make explicit: Unless

$$(3.8)\quad 1 + \frac{1}{2}(e_{00} - e_{10}) \leq \frac{1}{6}e_{00}(e_{00} - 3) + 1$$

we have either

$$(3.9)\quad 1 + \frac{1}{2}(e_{00} - e_{10}) = \frac{1}{2}(e_{00} - 1)(e_{00} - 2)$$

or

$$1 + \frac{1}{2}(e_{00} - e_{10}) = (a - 1)(b - 1)\ ,\ e_{00} = a + b$$

where a and b are integers. (See Theorem 2.1).

The condition (3.9) is of course equivalent to the assertion that X is contained in a linear 3-subspace of P^N: The generic section is then a planar curve. (3.9) can be written more neatly as

$$(3.11)\quad e_{00}^{\ 2} - 4e_{00} + e_{10} = 0$$

which completely characterizes the smooth 2-dimensional hypersurfaces. Similarly for higher dimensions.

For a 2-dimensional hypersurface we get

$$c_1(X) = (4 - d)i^*(h)\ ,\ c_2(X) = (d^2 - 4d + 6)i^*(h^2)$$

where $d = e_{00}$ is the degree, i the embedding and h class of a hyperplane. Thus

$$\begin{aligned} &e_{00} = d\ ,\ e_{10} = d(4 - d)\ ,\ e_{20} = d(4 - d)^2 \\ &e_{01} = d(d^2 - 4d + 6) \end{aligned}$$

Comparing with (3.11), we find that for $(e_{00}, \ldots, e_{01}) \in R(2,N)$, the relation (3.11) implies the relations

$$\begin{aligned} e_{10} &= e_{00}(4 - e_{00})\ , \\ e_{20} &= e_{00}(4 - e_{00})^2\ , \\ e_{01} &= e_{00}(e_{00}^{\ 2} - 4e_{00} + 6) \end{aligned}$$

This observation illustrates the incompleteness of the conditions above. Thus for instance if $n = 3$, $d = e_{00} = 3$, we must have

$$e_{10} = 3\ ,\ e_{20} = 3\ ,\ e_{01} = 9\ .$$

However, in the range

$$e_{01} \leqslant 10 \ , \ e_{20} \geqslant -15$$

we find a total of 18 vectors $(e_{00}, \ldots, e_{01})$ which satisfy the conditions as well as the full strength of Castelnuovo's Bound from [ACGH] We know in this case that X must be of general type, we then get additorial inequalities

$$e_{01} > 0 \ ,$$

cf. [BPV] Proposition 2.4 ch. VII and the Miyaoka-ineqality

$$c_{20} \leqslant 3e_{01} \ ,$$

The first of these excludes two cases, the last none.

Thus clearly new conditions are required to get a better hold on the domains $S(2,N)$ for $N \geqslant 3$, even in the case of surfaces of general type.

At the other end of the problem lies of course the project of testing out surface generating algorithms. But since the domain for the invariants given above is clearly much too big, one must at first test the algorithms against a domain where there is more information. It lies near to do the following:

i) Select those surfaces in the output from the algorithms which are minimal of general type,

and

ii) Study the projection of $S(2,N)$ in the c_{20} , c_{01} plane, i.e. find where they lie in the so called "surface-geography".

For some details of a beginning of this we refer to [Ra].

All conditions above as well as the full strength of Castelnuovo's Bound is contained in the following computer program, written in SIMULA 76. It takes an input the value of N, $d = e_{00} = \deg(X)$, an <u>upper</u> bound M_{01} for e_{01} and a <u>lower</u> bound M_{20} for e_{20} . It returns a list of those $(e_{00}, e_{10}, e_{20}, e_{01})$ where $e_{00} = d$, $e_{20} \geqslant m_{20}$, $e_{01} \leqslant M_{01}$. We include a sample output.

```
BEGIN INTEGER E00,E10,E20,E01,N,D,MAXE01,MINE01,MAXE10,MINE10,
   MAXE20,MINE20,KSI,SECTGEN,A,PLANEGEN,QUBGEN,M,EPS,PI,C,TOP;
   OUTTEXT("GIVE N,D,MAX E01,MIN E20"); OUTIMAGE;
   N := ININT;
   D := E00 := ININT;
   MAXE01 := ININT;
   MINE20 := ININT;
   MINE10 := ENTIER(-(D**2 - 4*D +9 - 3*N)/(N-2));
   MAXE10 := D + 2;
   PLANEGEN := (D-1)*(D-2)/2;
   QUBGEN := ENTIER(D*(D-3)/6 + 1);
   M := ENTIER(D-1/N-2);
   EPS := D-1 - M*(N-2);
   PI := M*(M-1)*(N-2)/2 + M*EPS;
   FOR E10 := MINE10 STEP 1 UNTIL MAXE10 DO
   BEGIN
      IF MOD(E00 + E10,2) NE 0 THEN GOTO OUT2;
      SECTGEN := 1 + (E00 - E10)/2;
      IF N EQ 3 THEN
      BEGIN IF SECTGEN NE PLANEGEN THEN GOTO OUT2;GOTO T; END;
      C := 0; TOP := 0;
      IF SECTGEN GT PI THEN GOTO OUT2;
      IF SECTGEN GT QUBGEN THEN
      BEGIN FOR A := 1 STEP 1 UNTIL ENTIER(D/2) + 1 DO
        BEGIN IF (A-1)*(D-A-1) EQ SECTGEN THEN C := 1; END;
        TOP := 1;
      END;
      IF C EQ 0 AND TOP EQ 1 THEN GOTO OUT2;
T:    MINE01 := MAX(2*E10 - 3*D, 4*E10 - 6*D);
      FOR E01 := MINE01 STEP 1 UNTIL MAXE01 DO
      BEGIN
        MAXE20 := D**2 - 10*D + 5*E10 + E01;
        FOR E20 := MINE20 STEP 1 UNTIL MAXE20 DO
        BEGIN
          IF MOD(E20 + E01,12) NE 0 THEN GOTO OUT1;
          KSI := (E20 + E01)/12;
          OUTINT(D,10);
          OUTINT(E10,10);
          OUTINT(E20,10);
          OUTINT(E01,10);
          OUTINT(KSI,10);
          OUTINT(SECTGEN,10);
          OUTINT(PLANEGEN,10);
          OUTINT(QUBGEN,10);
            OUTIMAGE;
OUT1: END;
        END;
OUT2:   END;
END;
```

GIVE N,S,MAX E01,MIN E20
3 3 10 -15

3	3	-9	-3	-1	1	1	1
3	3	-10	-2	-1	1	1	1
3	3	-11	-1	-1	1	1	1
3	3	-12	0	-1	1	1	1
3	3	-13	1	-1	1	1	1
3	3	-14	2	-1	1	1	1
3	3	-15	3	-1	1	1	1
3	3	-3	3	0	1	1	1
3	3	-4	4	0	1	1	1
3	3	-5	5	0	1	1	1
3	3	-6	6	0	1	1	1
3	3	-7	7	0	1	1	1
3	3	-8	8	0	1	1	1
3	3	-9	9	0	1	1	1
3	3	3	9	1	1	1	1
3	3	-10	10	0	1	1	1
3	3	2	10	1	1	1	1

```
GIVE N,D,MAX E01,MIN E20
4 3 10 -15
```

3	5	-14	2	-1	0	1	1
3	5	-2	2	0	0	1	1
3	5	-15	3	-1	0	1	1
3	5	-3	3	0	0	1	1
3	5	-4	4	0	0	1	1
3	5	8	4	1	0	1	1
3	5	-5	5	0	0	1	1
3	5	7	5	1	0	1	1
3	5	-6	6	0	0	1	1
3	5	6	6	1	0	1	1
3	5	-7	7	0	0	1	1
3	5	5	7	1	0	1	1
3	5	-8	8	0	0	1	1
3	5	4	8	1	0	1	1
3	5	-9	9	0	0	1	1
3	5	3	9	1	0	1	1
3	5	-10	10	0	0	1	1
3	5	2	10	1	0	1	1
3	5	14	10	2	0	1	1

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