Tangent Cone of a Gorenstein Singularity

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Summary: The self-duality of a Gorenstein Artin algebra A with maximal ideal m over a field $k \cong A/m$, carries over in a weaker form to the associated graded algebra $A^* = Gr_m A = \oplus A_i$, where $A_i = m^i/m^{i+1}$. If $j = \max \{i \mid A_i \neq 0\}$ is the socle degree of A, then A^* has a canonically defined decreasing sequence of ideals $A^* = C(0) \supset C(1) \supset \dots \supset C(j+1) = 0$, whose sucessive quotients $Q(a) = C(a)/C(a+1)$ are reflexive A^*-modules. Thus, for $a = 0,\dots,j$, $Q(a)_i = \mathrm{Hom}(Q(a)_{j-a-i},k)$ and up to a shift in grading $Q(a) = \mathrm{Hom}_{A^*}(Q(a),k)$. We define the i-th graded piece $C(a)_i \subset A_i$ of $C(a)$ by

$$C(a) = ((0:m^{j+1-a-i}) \cap m^i)/((0:m^{j+1-a-i}) \cap m^{i+1}).$$

When A is a quotient $A = R/I$ of the power series ring $R = k\{X\}$ with maximal ideal M, then $Q(a)$ as R^*-module depends only on $I \cap M^{j-a}$.

We apply this to construct families of relatively compressed Gorenstein algebras $B = R/J$ agreeing with A in top degrees: these are maximal length algebras B such that $J \cap M^{j-a} = I \cap M^{j-a}$ (work joint with J. Emsalem). Attention to the modules $Q(a)$ allows one to construct further large classes of examples of Gorenstein Artin algebra quotients of R with specified properties, step by step: one begins by choosing $I \cap M^j$, then $I \cap M^{j-1}, \dots, I \cap M^0 = I$.

We begin with the special case $A = R/I$, where $R = k\{x,y\}$; here the factors $Q(a)$ are isomorphic to graded complete intersection quotients of R^*. We place this fact in the context of work by Macaulay and others. In Section 3 we generalize to more than two variables - where $Q(a)$ need not be a Gorenstein R^*-module even when A is a complete intersection. In Section 4 we introduce the relatively compressed Gorenstein algebras. We end by showing how to use the $Q(a)$ and Macaulay's inverse systems to construct Gorenstein Artin quotients of R, having certain specified Hilbert functions.

1. Introduction.

Recall that the socle of a local Artin algebra A with maximal ideal m over a field $k = A/m$ (equicharacteristic case) is the k-vector space $\mathrm{Soc}\, A = (0:m) = \{a \mid ma = 0\}$ in A. The length $\ell(\mathrm{Soc}\, A)$ as a k-vector space is called the type of A. The algebra A is Gorenstein Artin when its type is one; then there is an integer j, called the socle degree of A, such that $\mathrm{Soc}\, A = m^j \neq 0$, but $m^{j+1} = 0$.

We suppose henceforth that A is a Gorenstein Artin local algebra as above, and that φ is a k-linear projection $\varphi: A \to k$ nonzero on Soc A. Define a pairing $\langle\cdot,\cdot\rangle_\varphi : A \times A \to k$ by $\langle a,b\rangle_\varphi = \varphi(ab)$.

It follows that the pairing is exact, and $(m^i)^{\perp} = (0:m^i)$. (See E-L.) Let A_i denote m^i/m^{i+1}, and $A^i = (0:m^{i+1})/(0:m^i)$; then, the associated graded algebra to A is $A^* = Gr_m A = \oplus A_i$, and its dual A^*-module is $\hat{A^*} \underset{\text{def}}{=} Hom_{A^*}(A^*,k) = Gr_{(0:m\bullet)}A = \oplus A^i$. The inner product $\langle\bullet,\bullet\rangle_q$ defines an isomorphism

(1.1) $\quad (A_i)^{\vee} = A^i$, via $(m^i/m^{i+1})^{\vee} = (m^{i+1})^{\perp}/(m^i)^{\perp}$.

When A is a graded Gorenstein Artin algebra $(A = \oplus A_i)$, then it is easy to see that $A^i = A_{j-i}$, hence that (see MAC2)

(1.2) $\quad (A_i)^{\vee} = A_{j-i}$ for A graded Gorenstein.

The Hilbert function of A is the series $H(A) = \sum \ell(A_i)z^i$; we sometimes omit z and write $H = (t_0,\dots,t_j)$ where $t_i = \ell(A_i)$, the length of A_i as a k-vector space. Naturally, when A is a graded Gorenstein Artin algebra, $H(A)$ is symmetric: $t_i = t_{j-i}$.

In the special case $A = k\{x,y\}/I$ is an Artin quotient of the power series ring in two variables, A is Gorenstein if and only if $I = (f,g)$ is a complete intersection. Thus, the graded Gorenstein quotients of $R = k\{x,y\}$ satisfy (here $\underline{d}$ is any sequence of d's)

(1.3) $\quad A = R/(f,g)$ where f,g are forms of degree $d \le d'$,
and $H(A) = (1,2,\dots,d,\underline{d},d-1,d-2,\dots,1)$, where $t_{d'} = d-1$.

We look for such graded Gorenstein algebras disguised as subquotients of the associated graded algebra A^* to a Gorenstein quotient A of R. On occasion A^* is itself graded Gorenstein - for example, when A is compressed, or when $A = R/(x^2+xy^2,y^3)$ and $A^* = R^*/(x^2,y^3)$. In general A^* is not a Gorenstein algebra, although A is Gorenstein. Thus, the Gorenstein Artin algebra $A = R/(xy,x^2+y^3)$ has associated graded algebra $A^* = R/(xy,x^2,y^4)$ - note that $y^4 = y(x^2+y^3)-x(xy)$ is in I and I^*. The Hilbert function $H(A) = H(A^*) = (1,2,1,1)$, from $A_0 = k$, $A_1 = \langle\bar{x},\bar{y}\rangle$, $A_2 = \langle\bar{y}^2\rangle$, $A_3 = \langle\bar{y}^3\rangle$, and is not symmetric. However, as we shall see, A^* does have two naturally defined graded Gorenstein subquotients, (recall that $R^* = k[x,y]$ a polynomial ring)

$Q(0) = R^*/(x,y^4)$, of Hilbert function $H(Q(0)) = (1,1,1,1)$.
$Q(2) = (x,y^4)/(xy,x^2,y^4)$, with $H(Q(2)) = (0,1,0)$.
(Note that $Q(2) \cong k = R^*/(x,y)$ after removal of the factor x.)

The subquotient $Q(a)$ has effective socle degree $j' = j-a$, in the sense that $(Q(a)_i)^{\vee} = Q(a)_{j'-a}$, in analogy to (1.2) above. In the example $j = 4$, and $(Q(2)_1)^{\vee} = k^{\vee} = k = Q(2)_{4-2-1}$.

In Section 1 we assume $A = k\{x,y\}/I$ and find in A^* a decreasing

sequence of ideals whose successive quotients $Q(a) = C(a)/C(a+1)$ are graded Gorenstein algebras after a shift in grading. Section 3 gives the general definition of $Q(a)$ for any codimension, and gives examples. In Section 4 we consider quotients A of a fixed power series ring R and study the family of maximal length Gorenstein algebras $B = R/J$ such that J agrees with the ideal I defining A in degrees above j'. We announce (work joint with J. Emsalem) the existence result specifying the Hilbert function of such <u>relatively compressed Gorenstein algebras</u> and describe the family of B. Finally, we give examples illustrating how to build Gorenstein algebras with the Hilbert functions $H(Q(a))$ specified. Much of this article announces and gives context for results that will be proven and generalized in a forthcoming article (I5). We give here a practical guide to using the results.

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<u>2. The intersection of two plane curves</u>, after F.H.S. Macaulay, C.A. Scott, et al. From 1902 to 1905, first Macaulay, then Scott, then again Macaulay studied quotients $A = R/I$ of the power series ring $R = k\{x,y\}$, especially the complete intersection quotients. In his definitive article MAC1 Macaulay relates the number of generators $v(I)$ and type of A, showing $v(I) = \text{type } A + 1$; he uses this to delimit the Hilbert function of A. These algebras were studied more recently by J. Briançon, and by this author, from a parametrization viewpoint. In particular we studied the family Gor H parametrizing Gorenstein quotients of R having Hilbert function H, and the morphism from Gor H to the variety G_H - parametrizing graded algebras of Hilbert function H - taking the point parametrizing A to that parametrizing A^*. (See BR, I-1, I-4). Here we reinterpret and refine these results.

<u>2.1. The Hilbert function.</u> An Artin quotient $A = R/I$ of R such that order $I = d$ (so $m^d \supset I$ but $m^{d+1} \not\supset I$) has Hilbert function

(2.1) $\quad H(A) = (1,2,\dots,d,t_d,\dots)$, with $d \ge t_d \ge t_{d+1} \dots$.

<u>Lemma 2.1.(Macaulay).</u> If I has $v = v(I)$ generators, then $|t_i - t_{i+1}| \le v-1$.

Macaulay's proof uses type $A = v-1$, and is based on studying the inverse system for I, namely the dual module $\hat{A} = Hom(A,k)$ regarded as quotient of $\hat{R}$. For a proof using the determinantal structure theorem for height two ideals, see BR or I-1, where it is further shown that (2.1) and the condition of the Lemma characterize those sequences possible as $H(A)$ for an algebra $A = k\{x,y\}/I$.

Thus, in the special case that A is Gorenstein (so c.i.), $H(A)$ satisfies $|t_i - t_{i\pm1}| \le 1$. The Hilbert function has an initial rising segment, then breaks up into platforms - level sequences - and regular flights of descending stairs, each step of height one.

<u>2.2. The level sequences of $H(A)$, and A^*.</u> When $t_i = t_{i+1}$, the following Lemma shows that A^* in degrees $i, i+1$ is determined by a single form $h \in R$. That form is the only connection between the portion of A^* in degrees greater than $i+1$, and the portion before degree i.

<u>Lemma 2.2.1.(Old)</u> If $H(A)$ satisfies $t_i = t_{i+1} = s$, then there is a form h of degree s in x,y (so in R_s) such that

$$I_i = (h) \cap R_i, \quad I_{i+1} = (h) \cap R_{i+1}, \quad \text{and} \quad (h) \supset I_u \quad \text{for} \quad u < i.$$

<u>Proof</u>. One way to show this elementary result is to use the properties of the simplicity $\tau(V)$ of a vector subspace of R_i, when $R = k\{x,y\}$. By definition, $\tau(V) = \ell(R_1 V) - \ell(V)$, where $R_1 V$ denotes the vector space span of $\{xv, yv \mid v \in V\}$. (See I-2, and I-1 p. 56-57.) One shows that $\tau(V) = v(\overline{V})$, the number of generators of the "ancestor ideal" $\overline{V} = (V) + \langle V:R_1\rangle + \ldots + \langle V:R_i\rangle$, with $V:R_u = \{f \mid R_u f \subset V\}$. Here for $V = I_i$, the equality $t_i = t_{i+1}$ implies that $\tau(V) = 1$, hence that $\overline{V}$ is principal: $\overline{V} = (h)$. Since $\overline{V}$ is also the largest ideal of R such that $\overline{V} \cap M^i = (V)$ (here M is the maximal ideal of R), it follows that $(h) \supset I_u$ for $u < i$. □

<u>Example 2.2.2</u>. The Gorenstein algebra $A = R/(xy, x^2+y^3)$ has $H(A) = (1,2,1,1)$, with $t_2 = t_3 = 1$. There is a form $h = x$ of degree one such that $I_2 = (x) \cap R_2 = \langle x^2, xy\rangle$, and $I_3 = (x) \cap R_3$.

If $t_i = t_{i+1}$ is greater than t_{i+2}, the associated graded ideal I^* defining A^* requires a generator in degree $i+2$: for the ideal (I_{i+1}) generated by I_{i+1} is $(h) \cap M^{i+1}$, and its degree- $i+2$ piece has colength t_{i+1} greater than t_{i+2}. The ideal I^* requires two initial generators, the first in degree d and the second in the first degree i at least d where $t_i \ne d$. Subsequently, I^* requires a generator following each constant sequence of $H(A)$ of height t_i less than d - a generator for each such platform. For a sequence H satisfying (2.1) and $|t_i - t_{i+1}| \le 1$ for each i, we define a number

$$v(H) = 2 + \#(\text{platforms of height less than } d \text{ in the graph of } H)$$

which is also the minimum possible number of generators $v(J)$ of a graded ideal J defining an algebra B of Hilbert function H. We

now characterize those graded ideals J defining associated graded algebras $B = A^*$ of complete intersections. Recall that Gor H in the Grassmanian $\text{Grass}(n, R/M^n)$ with $n = |H| = \Sigma t_i$ parametrizes Gorenstein quotients of R having Hilbert function H; and G_H in the same Grassmanian is the projective variety parametrizing graded quotients $B = R/J$ having Hilbert function H.

<u>Theorem 2.2A.</u> The graded algebra $B = R/J$ of Hilbert function H is the associated graded algebra $B = A^*$ of a complete intersection A iff H satisfies (2.1) and $|t_i - t_{i+1}| \le 1$ for all i, and $v(J) = v(H)$. The family of such graded algebras is open dense in G_H, hence is locally an affine space, of dimension $(2d - v(H) + (d - t_d))$.

<u>Theorem 2.2B.</u> (Fiber of Gor H over G_H) The family of complete intersections A having given associated graded algebra B is an open-dense in an affine space $\mathbb{A}^N$ of dimension $N = (n - 3d + v(H) - (d - t_d))$.

<u>Theorem 2.2C.</u> The family Gor H is open dense in the irreducible family Z_H parametrizing quotients of R having Hilbert function H. Thus, Gor H is covered by opens in an affine space $\mathbb{A}^{n-d}$.

<u>Proof.</u> Most of the statements above are special cases - for the restricted class of Hilbert functions with steps $e_{i+1} = t_i - t_{i+1}$ no greater than one - of Theorems 2.11, 2.12, 3.13, 3.14, and Proposition 4.4 of I-1. The first statement of Theorem 2.2A here is novel. The methods of Sections 2.3 and 3 below can be used to show $B = A^*$ implies $v(J) = v(H)$, as well as the Macaulay conditions on $H(A)$ for A Gorenstein (see I-5). We give a proof of Theorem 2.2A similar to that of the Proposition 4.4 cited; it suffices to refine that proof.

Suppose $B = A^*$, with A c.i., and assume for the moment that char $k = 0$. Then there is a choice of elements, say x, y, of R_1 such that I_i has cobasis $\langle x^{t_i - 1} y^{i+1-t_i}, \ldots, y^i\rangle = P_i$ in each R_i. Furthermore, I has standard generators $f_i = x^i y^{k_i} + c_i$, with c_i in $P_{i+k_i} + P_{i+k_i+1} + \ldots$, where $k_d = 0$, and $k_{i-1} \ge k_i + 2$; here k_i is the length of the height-i horizontal bar in the graph of H. The relations among the f_i have standard form

$$r_i = y^{w_i} f_i - x f_{i-1} + a_i f_{i-2} + \ldots, \quad \text{with } w_i = k_{i-1} - k_i,\ a_i \in k[y].$$

By assumption, $v(I) = 2 = \dim_k \langle f_0, \ldots, f_d\rangle / \langle r_1', \ldots, r_d'\rangle$ where $r_i' =$ class of r_i in $\oplus(R/MR)f_i$. Because of the triangular nature of the matrix $r' = r_1', \ldots, r_d'$, its rank is $d-1$ implies $a_i' \ne 0$, $i = d, \ldots, 2$. The associated graded ideal I^* has standard generators the initial forms $F_d, \ldots, F_0$ of $f_d, \ldots, f_0$. When degree $f_{i-1} =$ degree $f_{i-2} - 1$, then the new standard relation $(y^{w_i} F_i - x F_{i-1} + \underline{a}_i F_{i-2} + \ldots)$ has the class $\underline{a}_i^! = a_i^! \neq 0$. It follows that $v(I^*) \leq v(H)$; since $v(H)$ is the minimal value of $v(J)$ for graded ideals J defining an algebra B of Hilbert function H, also $v(I^*) = v(H)$, as claimed. If char $k \neq 0$, the standard basis $f_d, \ldots, f_0$ has the same initial degrees or orders as when char $k = 0$, but different x-degrees; the relation r_i still has the features needed to prove $v(I^*) = v(H)$.

If $B = R/J$ is graded of Hilbert function H and $v(J) = v(H)$, it is not hard to construct a complete intersection $A = R/I$ with $A^* = B$. One chooses first f_0, then $f_1, \ldots, f_d$ such that $F_i =$ in f_i is given by J, and such that each f_{i-2} immediately following a platform of H, has nonzero coefficient $a_i^!$ in $r_i^!$. The nonzero choice of $a_i^!$ is possible since $a_i^!$ is simply related to the affine space parameters of the fiber of Z_H (family of all algebras R/I of Hilbert function H) over G_H - see the proof of Prop. 4.4 cited. □

2.3. The Gorenstein factors $Q(a)$ of A^*. We now interpret $v(I^*) = v(H)$ in a way showing the fine structure of A^* when A is a complete intersection Artin quotient of $R = k\{x,y\}$.

Example 2.3.1. Let $I = (x^3+xy^2+y^4,\ x^2y)$. Then $I^* = (x^3+xy^2,\ x^2y,\ xy^3, y^7) = (x^3+xy^2,\ x^2y,\ y^7)$, since $xy^3+y^5 = y(x^3+xy^2+y^4)-x(x^2y)$, and $y^7 = y^2(xy^3+y^5)-x(xy^3+y^5)+y^2(x^2y)$. Since $H = (1,2,3,2,1,1,1)$ and $v(H) = 2+1 = 3$, the number of generators $v(I^*) = v(H)$ is minimal; in particular, the standard generator xy^3 of I^* is in the ideal $(x^3+xy^2,\ x^2y)$ of the first two generators.

We now define two Gorenstein subquotients $Q(0)$ and $Q(2)$ of A^*, having Hilbert functions $H(Q(0)) = (1,1,1,1,1,1,1)$ and $H(Q(2)) = (0,1,2,1,0)$. Since $t_4=t_5=t_6=1$, there is a degree-one form $h = x$ determining I_4, I_5, I_6 and dividing I_2, I_3. Thus, we choose

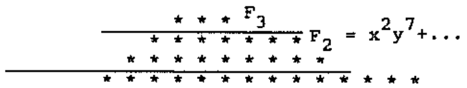
$$Q(0) = A^*/(\bar{x}) = R^*/(x,y^7)\ . \quad \text{(Here } \bar{x} \text{ is the class of } x \text{ in } A^*.)$$
$$Q(2) = (\bar{x}) = xR^*/(x^3+xy^2,\ x^2y) \cong R^*/(x^2+y^2,\ xy).$$

The isomorphism $Q(2)$ to a Gorenstein algebra requires a shift in grading by one - from division by x. That $(x^2+y^2,\ xy)$ defines a Gorenstein algebra or complete intersection is equivalent to the standard generator xy^3 of I^* being in the ideal of $(x^3+xy^2,\ x^2y)$. Thus, $v(I^*) = v(H)$, the minimum possible value, translates into A^* having here two, in general $v(H)-1$, natural graded complete intersection subquotients $Q(a)$.

We now define $Q(a)$, using the standard generators $F_d =$ in $f_d, \ldots, F_0 =$ in f_0 of I^*. We assume for convenience that char $k = 0$, so the standard generators f_i for I and F_i for I^* satisfy

$$f_i = x^i y^{k_i} + c_i, \text{ where } c_i \in P \cap M^{i+k_i}$$
$$F_i = x^i y^{k_i} + C_i, \text{ where } C_i \text{ is the degree } (i+k_i) \text{ part of } c_i.$$

We let $G_i = F_i/y^{k_i}$ in R_i. Consider a platform of H - a constant subsequence $t_u = t_{u+1} = \ldots = t_w$ where $d \geq t_{u-1} > t_u = t_w > t_{w+1}$. We assign to the platform the label $a = j-(u+t_u-1)$; the portion of the graph of H between this platform - extended left - and the next higher platform has center $(j-a)/2$. We assign the label $a = 0$ to the zero tail $t_{j+1} = t_{j+2} = \ldots = 0$ of H. The labels are integers between 0 and j, but not all such integers are used. Denote by $h(a)$ the degree-t_u form $h(a) = G_{t_u}$ determining $I_u, \ldots, I_w$ as in the Lemma of Section 2.2 : so $I_u = (h(a)) \cap R_u$. Let $t_{u'}, \ldots, t_{w'}$ be the next platform to the left, labelled a'. Denote by $F(a)$ the standard generator F_i, $i = t_{u'}-1$ of I^* having degree $w'+1$; denote by $G(a) = h(a')$ the form $G_{i'}$, $i' = t_{u'}$. If there is no further platform to the left, take $F(a) = F_{d-1}$ and $G(a) = G_d = F_d$ as we did for $Q(2)$ in Example 2.3.1. See Figure 1.



$a = 3$, $F(3) = F_2$, $G(3) = G_3$

$H(Q(3)) = (0,1,2,2,2,2,2,2,2,1,0,0,0,0)$

Figure 1. $H(Q(3))$ as area between platforms $a = 3$ and a' (=5).

Theorem 2.3. If A is a Gorenstein quotient of $R = k\{x,y\}$, then A^* has a descending sequence of principal ideals $A^* = C(0) \supset C(1) \supset \ldots \supset C(j+1) = 0$, whose successive quotients $Q(a) = C(a)/C(a+1)$ are A^*-modules isomorphic as R^*-modules to graded complete intersection quotients of R^*. If H has a platform labeled a, then

$$C(a) = (h(a))\, A^*, \quad \text{and} \quad C(a+1) = (G(a))\, A^*,$$
$$Q(a) = (h(a))\, A^*/(G(a)\, A^*\ .$$
$$H(Q(a)) = (\underbrace{0,\ldots,0}_{t_u},1,\ldots,\underbrace{t_u,-t_u,\ldots,t_u,-t_u}_{\deg F(a)\,-\,\deg G(a)},\ldots,1,\underbrace{0,\ldots,0}_{t_u}).$$

The Hilbert function $H(Q(a))$ gives the height of the portion of the graph of H between the a-th platform - extended left - and the next higher platform. As R^*-module, $A^*/C(a+1) = R^*/\overline{C}(a+1)$, where

$$\overline{C}(a+1) = (G(a),\ F(a),\ F(a-1), \ldots,\ F(0))\ . \quad \text{Likewise,}$$
$$Q(a) = R^*\, h(a)/(G(a), F(a))$$
$$\cong R^*/((G(a)/h(a)),(F(a)/h(a))) \ - \text{with a shift in grading.}$$

If H does not have a platform labeled a, then $Q(a) = 0$, and $C(a) = C(a+1) = C(a''+1)$, where a'' is the next smaller integer occuring as a platform label.

Proof. The proof merely interprets the condition $v(I^*) = v(H)$, following Example 2.3.1. A second proof, directly relying on the construction of $Q(a)$ in Section 3 below, will appear in I-5.

3. The reflexive factors Q(a) of A*.

We assume that A is a Gorenstein Artin quotient of $R = k\{x_1,\dots,x_r\}$. We define a sequence of ideals in A^*, $A^* = C(0) \supset C(1) \supset \dots \supset C(j+1) = 0$ whose successive quotients $Q(a) = C(a)/C(a+1)$ are the reflexive A^*-modules we call factors of A^*. In the special case $R = k\{x,y\}$, these are the graded complete intersections $Q(a)$ of Section 2; but when $R = k\{x,y,z\}$ the module $Q(a)$ may have type (length of socle) two, even when A is a complete intersection (Example 3.2).

Consider a table $T = T_{uv}$ of ideals in A whose u,v entry is

$$T(u,v) = T_{uv} = (0:m^{j+1-u}) \cap m^v, \quad u,v = 0,\dots,j+1.$$

Here j is the socle degree of A. The main diagonal of T has entries $T_{uu} = m^u$, while the a-th subdiagonal below the main diagonal has entries $T(v+a,v) = (0:m^{b-v}) \cap m^v$, where $b = j+1-a$. We define

$$C(a)_i = T(i+a,i)/T(i+a,i+1), \text{ a subspace of } A_i,$$
$$= (0:m^{b-i}) \cap m^i/(0:m^{b-i}) \cap m^{i+1}.$$
$$Q(a)_i = C(a)_i/C(a+1)_i = T(i+a,i)/(T(i+a,i+1)+T(i+a+1,i))$$
$$= (0:m^{b-i}) \cap m^i/((0:m^{b-i}) \cap m^{i+1} + (0:m^{b-i-1}) \cap m^i).$$

Thus, $C(a) \subset A^*$ is defined piece-by-piece, dividing each entry of the a-th subdiagonal of T by the entry horizontally to the right; while $Q(a)$ is defined by dividing the entry in the ath subdiagonal by that horizontally to the right, and the one below. An A^* action on $C(a)$ and $Q(a)$ is defined by lifting first to A, then requotienting - so the action defined is nontrivial. Note that $Q(a)_i = 0$ if $i > j' = j-a$.

Theorem 3.1A. The A^*-module $Q(a)$ is reflexive: $\mathrm{Hom}_{A^*}(Q(a),k) = Q(a)$, up to a shift in grading; and the pairing $\langle \cdot,\cdot \rangle$ defines the isomorphism $\mathrm{Hom}((Q(a)_i),k) = Q(a)_{j'-i}$, where $j' = j-a$. $Q(0)$ is the unique graded Gorenstein algebra of socle degree j that is a quotient of A^*.

Theorem 3.1B. If $A = R/I$, define $\overline{C}(a)_i = \overline{T}(i+a,i)/\overline{T}(i+a,i+1)$, where $\overline{T}_{uv} = (I:M^{j+1-u}) \cap M^v$, with M the maximal ideal of R. Then $\overline{C}(a)$ is the union of associated graded ideals J^* to ideals J defining a Gorenstein quotient $B = R/J$ and satisfying $J \cap M^b = I \cap M^b$, $b = j+1-a$. Also $Q(a) = \overline{C}(a)/\overline{C}(a+1)$, and depends only on $I \cap M^{j-a}$.

The proof of Theorem 3.1A uses the properties of the pairing $\langle \cdot,\cdot \rangle$ defined in Section 0. The proof that $C(a)$ depends only on $I \cap m^b$ uses in addition the Macaulay inverse system. The proof of Theorem 3.1B is an extension of the proof of Theorem 4.1 below.

Example 3.2. Complete intersection having Q(2) non-Gorenstein. Suppose $R = k\{x,y,z\}$, and consider the complete intersection $A = R/I$, $I = (z^2+x^3, zx+y^4, zy)$. Then $I^* = (z^2, zx, zy, x^4, x^3y, y^5)$, and $H(A) = (1,3,3,4,3,2,1)$; the socle degree is $j = 6$. The graded Gorenstein algebra $Q(0) = R^*/(z,x^3,y^5)$, whose Hilbert function is $H(Q(0)) = (1,2,3,3,3,2,1)$. The reflexive A^*-module $Q(2) = \langle \overline{z}, \overline{x}^3 \rangle$, with trivial action of the maximal ideal m of A^*; thus $H(Q(2)) = (0,1,0,1,0)$, with $j' = 4$. Notice that $Q(2)$ has length two socle, so is not isomorphic to a Gorenstein algebra, though A is a C.I.

It is possible for Gorenstein algebras with the same Hilbert function to have different Hilbert function decompositions $DH(A) = + H(Q(a))$ (See Examples 4.2.1, 4.2.2). I believe further study of the fibering $\mathrm{Gor}\ H \longrightarrow G_H$ will involve the finer structure on G_H from DH.

4. Relatively compressed Gorenstein algebras.

Suppose that $R = k\{x_1,\dots,x_r\}$ is the power series ring over an infinite field k, and that $A = R/I$ is a Gorenstein quotient of socle degree j. Consider the family of Gorenstein quotients $B = R/J$ such that $J \cap M^{j+1-a} = I \cap M^{j+1-a}$, for a fixed integer a; we call such an algebra B a continuation of A (or of $M^{j+1-a} \twoheadrightarrow m^{j+1-a}$) into degrees less or equal $j-a$. We term B a relatively compressed continuation of A if length B is maximum among this family. We shall see in Theorem 4.1 that length B is maximum among these continuations of A iff the factor $Q(a)$ for B satisfies, $H(Q(a))$ has the maximum possible value, $H_{max}Q(a))$, specified in Lemma 4.1.1.

Lemma 4.1.1. Consider the family of Gorenstein quotients $B = R/J$ such that $H(Q(0)),\dots, H(Q(a-1))$ are specified. Let H' denote the sum $H(Q(0))+\dots+H(Q(a-1))$, and t_i' its i-th coefficient. Define

$$H_{max}(Q(a)) = (t_0'', t_1'', \dots) \quad \text{where} \quad t_i'' = \ell(R_i)-t_i' \text{ if } i \leq j'/2,$$
$$= t_{j'-i}'' \text{ otherwise.}$$

$$H_{max}(B,a) = H' + H_{max}(Q(a)).$$

Then $H(B) \le H_{max}(B,a)$ termwise. Should there be equality, then $H(Q(a)) = H_{max}(Q(a))$, and $Q(a+1) = Q(a+2) = \dots = 0$.

Proof indication. That $H(Q(a)) \le H_{max}(Q(a))$ is evident from the symmetry of $H(Q(a))$ about $j'/2 = (j-a)/2$, and the inequality $\sum H(Q(u)) = H(B) \le H(R)$. The proof that $H(B) \le H_{max}(B,a)$ requires in addition inequalities arising from $\overline{C}(u)$ being an ideal in R^*, namely that $\ell(\overline{C}(u)_i) \le \ell(\overline{C}(u)_{i+1})$, and the fact, $Q(u) = \overline{C}(u)/\overline{C}(u+1)$.

Example 4.1.1. Let $R = k\{x,y,z\}$ and suppose B satisfies $H(Q(0)) = (1,1,1,1,1,1,1)$, $H(Q(1)) = (0,1,2,2,1,0)$. Let $a = 2$. Then $Q(2)$ satisfies, $t_0 = 0$, $t_1 \le (3-1-1) = 1$, $t_2 \le (6-1-2) = 3$ (see Figure 2). If these take on their maximum values, then the rest of $H(Q(2))$ is determined by the symmetry around degree $j'/2 = 2$, and $Q(3) = 0$.

$H(R) =$	1	3	6	10	...			
$H(Q(0)) =$	1	1	1	1	1	1	1	
$H(Q(1)) =$	0	1	2	2	1	0		$H' = (1,2,3,3,2,1,1)$
$H_{max}(Q(2)) =$	0	1	3	1	0			
$H_{max}(B,2) =$	1	3	6	4	2	1	1	

Figure 2. Hilbert function of a relatively compressed B, with $a = 2$.

Theorem 4.1. (with J. Emsalem) Relatively compressed algebras. Fix a Gorenstein quotient $A = R/I$, and consider the family of Gorenstein quotients $B = R/J$ such that $J \cap M^{j+1-a} = I \cap M^{j+1-a}$, and length B is maximum. Any such relatively compressed B has the Hilbert function $H(B) = H_{max}(B,a)$ specified in Lemma 4.1, with $H(Q(0)),\dots,H(Q(a-1))$ taken from A (or B). The family of all such relatively compressed continuations of A is parametrized by a variety that is locally an affine space of dimension $(\ell(R/M^b) - \ell(B/M^bB))$. For such B the associated graded algebra B^* depends only on $J \cap M^{j-a}$.

The proof uses the inverse system of B, comparing it with that for A: one needs to choose the degree j-a piece generically, showing that $C(a+1)$ is then zero, and $C(a)$ maximal. We use a theorem of A. Miri, stating that a specified space of higher derivates in R_u of a generic degree-j' form in $\mathfrak{R}_{j'}$, (see Section 4.2) intersects properly a given subspace of $\mathfrak{R}_{j'-u}$ (M Proposition II.3.1).

Example 4.1.2. If we choose $a = 0$, and A arbitrary of socle degree j (A doesn't affect B in this case), we find the family of compressed Gorenstein quotients of R having socle degree j (See I-3).

Example 4.1.3. Let $a = j-2$, and $A = R/(x_1,\dots,x_{r-1},x_r^{j+1})$, we find a family of "stretched" Gorenstein Artin algebras with Hilbert function $(1,r,1,\dots,1)$, as in [S]. Choosing other values for a, one can parametrize quotients of R having Hilbert function $H = (1,r,\ell(R_2),\dots,\ell(R_t),\ell(R_{t-1}),\dots,r,1,\dots,1)$.

The relatively compressed Gorenstein algebras form a much broader class of algebras than the compressed algebras - they even include "stretched" algebras! They are not so extremal as the compressed algebras - and we know little about their minimal resolutions as R-modules. A key feature is we know they are there!

Both Lemma 4.1.1 and Theorem 4.1 can be generalized beyond Gorenstein algebras. Choose an algebra $A = R/I$ of socle-type E (see I3 or E-I), and consider quotients $B = R/J$ of the same socle type such that $J \cap M^{j+1-a} = I \cap M^{j+1-a}$, where j is the largest socle degree. There is a maximum possible Hilbert function, realized by relatively compressed continuations of A, which form an algebraic family similar to that described for A Gorenstein in Theorem 4.1.

4.2. Building a Gorenstein algebra.

We now turn to a practical method of constructing Gorenstein algebras having certain prescribed Hilbert function decomposition DH(A), specifying each $H(Q(a))$. Although we used it to find each example here, it is not infallible (see Example 4.2.3), and finding examples is partially an intuitive realm.

We need the inverse system to A. Let $\mathfrak{R}$ denote the polynomial ring $\mathfrak{R} = k[X_1,\dots,X_r]$, upon which R acts as higher order partial differential operators without coefficients. In other words, if K and L are multi-indices, then $x^K \cdot X^L = X^{L-K}$, understood to be zero if any index of L-K is negative; one extends the action bilinearly to one of R on $\mathfrak{R} = k[X]$.

Lemma (Macaulay). There is a one-to-one correspondence of sets between

$$\left\{\begin{array}{l}\text{Socle-degree } j \text{ Gorenstein}\\ \text{quotients } A = R/I\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}R\text{-cyclic submodules } \hat{A} = R\cdot\mathcal{F}\\ \text{of } \mathfrak{R} \text{ with degree } \mathcal{F} = j.\end{array}\right\}$$

The correspondence is given by

$$A = R/I \longrightarrow \{f \in \mathfrak{R} \text{ such that } I\cdot f = 0\} = \hat{A}$$

$$R/\mathrm{Ann}\,\mathcal{F} \longleftarrow \text{Generator } \mathcal{F} \text{ of } \hat{A}$$

In the above correspondence, the Hilbert function $H(A)$ is the same as that of the R-module $\hat{A}$. We give first two examples, Gorenstein algebra quotients of $R = k\{x,y,z\}$ having the same Hilbert function, but different Hilbert function decompositions $DH = \{H(Q(a))\}$. We build these algebras by writing the generator $\mathcal{F}$ of $\hat{A}$ as a sum of divided powers of linear forms (or generalized sum, see I-3). In the two examples, the degree $j-a$ power sum determines the dual module to $Q(a)$, as submodule of A. Here $j = 7$, $H(A) = (1,3,6,8,6,4,2,1)$.

<u>Example 4.2.1.</u> Given $H(Q(0)) = (1,2,3,3,3,3,2,1)$, and $H(Q(1)) = (0,1,3,5,3,1)$, while $Q(2) = Q(3) = \ldots = 0$. The graded Gorenstein algebra $Q(0)$ is that determined by the top degree form $\mathcal{F}_j$ of $\mathcal{F}$, in the above correspondence: the dual module of $Q(0)$ is $R\cdot\mathcal{F}_j$. Its Hilbert function beginning $1,2,\ldots$ implies that $\mathcal{F}_j$ involves only two variables, say x,y. That the maximum value of $H(Q(0))$ is 3 implies (in two variables, see I-3) that $\mathcal{F}_j$ can be written as a generalized sum of three divided powers: we choose $\mathcal{F}_7 = X^7+Y^7+[X+Y]^7$. That $t_1(A) = 3$, and the maximum value of $H(Q(1))$ is 5, suggest that $\mathcal{F}_6$ be the sum of five (divided) sixth powers of linear forms involving the third variable, Z. We choose (Note $[X+Y]^7 = X^7+X^6Y+X^5Y^2+\ldots+Y^7$)

$$\mathcal{F} = X^7+Y^7+[X+Y]^7+Z^6+[Z+X]^6+[Z-X]^6+[Z+Y]^6+[Z-Y]^6.$$

We can verify that $\mathcal{F}$ determines an algebra A with the given DH by checking first that $R\cdot\mathcal{F}_7$ has the Hilbert function $H(Q(0))$, then checking that $R(\mathcal{F}_7+\mathcal{F}_6) = R\mathcal{F}$ has the sum Hilbert function $H(Q(0)) + H(Q(1)) = H(A)$. This works because $H(Q(1))$ is symmetric about $(j-1)/2$, so cannot be written as a sum involving nonzero $H(2), \ldots, H(j)$ with symmetries about $(j-2)/2$ or lower numbers. Thus the two Hilbert function checks determine DH. Finding I can be harder!

Verifying $H(A)$ involves taking the various "higher partials" - without coefficients, lining up the resulting polynomials by top degree, and finding the length of the vector space of top-degree-i forms. For example, to verify $t_5 = 4$, note that $x^2\cdot\mathcal{F}$, $xy\cdot\mathcal{F}$, $y^2\cdot\mathcal{F}$, $z\cdot\mathcal{F}$ have linearly independent top degree forms, namely those of

$$X^5+[X+Y]^5+[Z+X]^4+[Z-X]^4,\quad [X+Y]^5,\quad Y^5+[X+Y]^5+[Z+Y]^4+[Z-Y]^4,$$
$$Z^5+[Z+X]^5+[Z-X]^5+[Z+Y]^5+[Z-Y]^5.$$

All higher partials of $\mathcal{F}$ yield lower-degree polynomials, and lower partials $(1,x,y)$ are accounted for in $t_7 = 1$, $t_6 = 2$.

The ideal $I = \text{Ann}\,\mathcal{F}$ has seven generators, $I = (zxy,\ xy(x-y),\ (z^2y^2-z^2x^2)+2(x^5-y^5),\ zy^3-z^3y,\ zx^3-z^3x,\ 4z^4-5(z^2y^2+z^2x^2)+2(x^5+y^5-2x^4y) -4x^5y,\ x^6+y^6-3x^5y-2x^4z^2)$. The algebra $Q(0) = R^*/(z,\ xy(x-y),\ x^6+y^6-3x^5y)$, and $Q(1) = (z, xy(x-y), x^6+y^6-3x^5y)/I$.

<u>Example 4.2.2.</u> Given $H(Q(0)) = (1,2,3,4,4,3,2,1)$, $H(Q(1)) = (0,1,2,3,2,1,0)$ and $H(Q(2)) = (0,0,1,1,0,0)$ summing to $H(A) = (1,3,6,8,6,4,2,1)$ as in Example 4.2.1, we choose for similar reasons

$$\mathcal{F} = X^7+Y^7+[X+Y]^7+[X-Y]^7+Z^6+[Z+X]^6+[Z-X]^6+Y^2Z^3.$$

Here the term Y^2Z^3 is added so that yz and yz^2 derivates of $\mathcal{F}$ will yield YZ^2 and YZ in $\hat{A}$ independent of $R\cdot(\mathcal{F}_7+\mathcal{F}_6)$. That $DH(R/\text{Ann}\,\mathcal{F})$ is the specified decomposition is checked by finding successively $H(R\cdot\mathcal{F}_7) = H(Q(0))$ specified, $H(R\cdot(\mathcal{F}_7+\mathcal{F}_6)) = H(Q(0)) + H(Q(1))$, and $H(R\cdot\mathcal{F}) = H(A)$. This checking is not hard. We calculate $I = (zxy,\ zy^2-z^4+z^2x^2,\ 3z^3y-3y^6+2x^6-2y^7,\ xz(x^2-z^2),\ xy(x^2-y^2),\ 6z^4-9z^2x^2+3x^5-3x^3y^2-2x^6+4y^7,\ 6(x^5+y^5-x^3y^2)-3x^4y-4x^6+8y^7$. Here $Q(0) = R^*/(z, xy(x^2-y^2), 2(x^5+y^5-x^3y^2)-x^4y)$; $Q(1) = (z,xy(x^2-y^2),2(x^5+y^5-x^3y^2)-x^4y)/(I^*,zy)$; and $Q(2) = \langle\overline{zy}, \overline{zy}^2\rangle$. Can $Q(a)$ be related to the Pfaffian determining A? As in two variables, is there a "standard" resolution (not minimal) of A that relates to $Q(a)$?

<u>Example 4.2.3.</u> Let $R = k\{x,y\}$, and choose $\mathcal{F} = Y^6+Y^4X$. Then $H(Q(2)) = (0,1,1,1,0)$ is <u>nonzero</u> although $\mathcal{F}_{j-2} = 0$. Here, $Q(0) = R^*/(x,y^7)$, of Hilbert function $(1,1,1,1,1,1,1)$, and $Q(1) = Q(3) = \ldots = 0$, so $H(A) = (1,2,2,2,1,1,1)$. Also $A = R/(x^2,xy^3-y^5)$, $A^* = R^*/(x^2,xy^3,y^7)$ and $Q(2) = x\,A^* = \langle\overline{x},\overline{xy},\overline{xy^2}\rangle$.

The last example shows that the choice of $\mathcal{F}_{j-1}$ may affect not just $Q(1)$, but higher $Q(a)$. It also illustrates that the simplest choice of $\mathcal{F}_{j-2}$ is not 0, but rather $\mathcal{F}_{j-2}$ generic, (here X^4 is generic enough!), yielding $Q(0)$ unchanged, $Q(1) = 0$, but $H(Q(2)) = H_{max}(Q(2)) = (0,1,2,1,0)$, if $\mathcal{F} = Y^6+Y^4X+X^4$.

<u>Note 4.2.4. Magic Squares: duality and two way sums of DH.</u>

The talk and an earlier version of this article was subtitled "magic squares", referring to striking numerical properties of the table N whose entries N_{uv} are the lengths $\ell(\overline{C}(v)_{u-v})$. It was through trying to understand these properties - not exactly a usual magic square - that I noted the decomposition DH, a first difference of the columns of N; the symmetry there suggested the existence of the reflexive modules $Q(a)$.

In terms of DH, the magical property is this. Suppose a sequence H of integers, ending in degree j, is a sum $|DH| = H(0) + \ldots + H(j)$ of sequences $H(a)$, such that $H(a)$ is symmetric about $(j-a)/2$ (and, of course, zero after degree $j-a$). Then, if DH is arranged as rows $H(0), H(1), \ldots$ of a matrix, the diagonals opposite to the main diagonal sum to $H^{\vee}$, which is the reverse of H. (See Figure 3).

In our context, this summing of diagonals to $H^\vee$ arises from the reflexivity $Q(a) \cong Q(a)^\vee$, the direct sum $\widehat{A^*} = \oplus Q(a)^\vee$. and the identity $H(\widehat{A^*}) = H(A^*) = + H(Q(a))$.

$$
\begin{array}{ccccccccc}
 & & H & = & \underline{1} & \underline{4} & \underline{4} & \underline{2} & \underline{1} \\
 & & & & 1 & 2 & 3 & 2 & 1 \\
 & & & & 0 & 1 & 1 & 0 & \\
 & & & & 0 & 1 & 0 & & \\
\hline
1 & 2 & 4 & & 4 & 1 & = H^\vee & &
\end{array}
$$

Figure 3. Magic square.

References

B Briançon, J., Description de $Hilb^n C\{x,y\}$, Invent. Math. 41 (1977), 45-89.

E-L Eisenbud, D. and Levine, H., An algebraic formula for the degree of a map germ, Annals of Math. 106 (1977), 19-44.

E-I Elias, J. and Iarrobino, A., The Hilbert function of a Cohen-Macaulay local algebra, preprint (1985), to appear, J. Algebra

F-L Fröberg, R. and Laksov, D., Compressed algebras, p. 121-151 in Conf. on Complete Intersections, S.Greco and R.Strano,ed. SLN #1092. Springer Lecture Notes #1092 (1984).

I-1 Iarrobino, A., Punctual Hilbert Schemes, A.M.S. Memoir #188 (1977).

I-2 .... Ancestor ideals , preprint (1975)

I-3 .... Compressed algebras, Trans. A.M.S. 285 (1984), 337-378.

I-4 ....Appendix: Hilbert function of C[x,y]/I to Deforming complete intersection Artin algebras, in Singularities A.M.S. PSPM #40, Vol. I, 593-608 (1983).

I-5 ....Gorenstein Artin algebras, and their associated graded algebras, preprint (1986)

MAC1 Macaulay, F.S., On a method for dealing with the intersections of two plane curves, Trans. A.M.S. 5(1904), 385-410.

MAC2 .... The Algebraic theory of Modular Systems, Cambridge Univ. Press, 1916 (reprint Stechert-Hafner, 1964).

M Miri, A. Artin modules having extremal Hilbert series: compressed modules, Ph.D. thesis, Northeastern Univ. (1985)

S Sally, J. Stretched Gorenstein rings, London Math. Soc.(2), 20, (1979), 19-26

SC Scott, C.A., On a recent method for dealing with the intersections of plane curves, Trans. A.M.S. 3 (1902), 216-263.

SCH Schenzel, P., Uber die freien auflösungen extremaler Cohen-Macaulay Ringe, J. Algebra 64(1980), 93-101.

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