EQUIMULTIPLICITY OF DEFORMATIONS OF CONSTANT MILNOR NUMBER

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Introduction.

In 1971 Zariski asked the famous question: Does topological equisingularity of hypersurface singularities imply equimultiplicity? His hope was to get an answer in a relatively short order, [42]. However 15 years later Zariski's question is still an open question. The aim of this note is to give a report on known partial results and some new progress we made in regard to the equimultiplicity problem.

After Zariski's work on plane curve singularities, the first step forward is due to A'Campo, [1]. His work on the zeta-function of the monodromy utilizing the concept of embedded resolutions implies that there is an affirmative answer in case of hypersurface singularities for which the topological Euler-characteristic of the complement of the projectivized tangent cone does not vanish, see Theorem 1.7. In fact, as far as we know this is the only essential contribution in this generality.

The main part of our paper actually concerns a slightly weaker question. Consider the semi-universal deformation of an isolated hypersurface singularity of Milnor number μ . Then it is well known that the parameter space S is smooth and that the Hilbert-Samuel function gives rise to an analytic stratification of the discriminant Δ of S. The stratum of highest multiplicity is of particular interest. It characterizes precisely the locus over which singularities occur with Milnor number equal to μ. Thus it is called the μ-constant stratum Δ_μ. According to the work of Lê-Ramanujam and Perron, one knows that Δ_μ is also the locus over which the topological equisingular type does not change. So the question arises whether the multiplicity at least does not change along Δ_μ.

The semi-universal deformation admits an unique singular section along the μ-constant stratum. Zariski proposed in [42] to check the

* Supported in part by the National Science Foundation

conditions of a Whitney stratification along the section since, due to Hironaka, an affirmative answer would imply equimultiplicity. However Briancon-Speder discovered first counterexamples showing that the Whitney conditions are toot strong. There was another approach based on the work of Wahl, [38], [39], utilizing Zariski's original definition of equisingularity, that of simultaneous resolution. Wahl introduced in [39] the functor ES of equitopological deformations of a good resolution which blow down to deformations of given singularity. Laufer's deep result in [17] shows that ES exactly describes the μ-constant deformations at least in dimension two. As a corollary, see 3.4, one obtains a nice partial answer but examples show that the topology of resolutions does not in general give sufficient informations in order to control the multiplicity under ES-deformations. The crucial point is that the topology does not reflect the fact that the singularity is a hypersurface singularity. Last summer the author obtained a paper of K.Altmann, [2], utilizing the concept of deformations of embedded resolutions which combines in some sense A'Campo's and Wahl's approach. This was our point of departure for this paper. Our main result can be roughly stated as follows. Let f=0 be a defining equation for a hypersurface singularity (X,p). The associated Newton polyhedron defines a partition Λ_0 of $\mathbb{R}^n_+$ into convex rational cones. Via toroidal embeddings, Λ_0 gives rise to a proper birational morphism π onto $\mathbb{C}^n$. If f is non-degenerated and commode, there always exists a subdivision, say Λ, of Λ_0 such that the corresponding morphism π defines an embedded resolution of (X,p) whose irreducible exceptional divisors are smooth and have only normal crossings.

Theorem (4.14). Suppose (X,p) is two-dimensional. If Λ_0 admits a "good" subdivision, then μ-constant deformations of (X,p) are equimultiple.

The precise meaning of "good" subdivision will be explained in §4.

Acknowledgement. Part of this research was done during the "Special Year of Algebraic Geometry" at the University of North Carolina at Chapel Hill. I would like to thank all participating mathematicians, especially the organizers J.Damon and J.Wahl, for many helpful discussions. I'm grateful to G.Pfister and M.Roczen for inviting me to the conference "Algebraic Geometry" at Berlin and making my stay so pleasant and stimulating. Finally I wish to thank K.Altmann to let have his manuscript.

§ 1 Topological equisingularity

By a n-dimensional singularity we always mean the germ of a pure k-dimensional reduced complex space X at an isolated singular point p.

1.1 Given an analytic embedding of X in some affine space $\mathbb{C}^N$. It is well known that X transversally intersects the boundary of a sufficiently small disc around p in an oriented, closed 2n-1-dimensional differentiable manifold which is up to orientation preserving diffeomorphism uniquely determined by the germ (X,p). This manifold is called the neighborhood-boundary of (X,p) and will be denoted by K_p. Further, the embedding $X \subset \mathbb{C}^N$ defines a differentiable embedding of K_p into the 2N-1-sphere S^{2N-1} which is uniquely determined up to isotopy. The pair (S^{2N-1},K_p) is called the link of the singularity (X,p) in $\mathbb{C}^N$. If the singularity is irreducible, i. e. K_p is connected, then (S^{2N-1},K_p) is also called the knot of the singularity. Two n-dimensional singularities (X,p) and (X',p') in $\mathbb{C}^N$ are called topologically (differentially) equisingular if there exists a homeomorphism (C^1-diffeomorphism) $f : U \to U'$ of open neighborhoods of the singular points in $\mathbb{C}^N$ which sends p into p' and $X \cap U$ into $X' \cap U'$.

1.2 If X is a sufficiently small representative of the germ (X,p), then X is topologically the cone over the neighborhood-boundary K_p. Thus it can be readily seen that topological equisingularity is equivalent to require the corresponding links are homeomorphic. As is well known, it may happen for irreducible singularities that K_p is homeomorphic to S^{2n-1}. In this case, according to results of Zeeman and Haefliger, (S^{2N-1},K_p) is homeomorphic to the trivial knot (S^{2N-1},S^{2n-1}) if the complex codimension $N-n > 1$. With other words, for non-hypersurface singularities the notion of topological equisingularity is not good enough to measure the complexity (e. g. the multiplicity) of the singularity.

1.3 On the other hand, differential equisingularity turns out to be a too strong equivalence relation. Clearly, differentially equisingular singularities have diffeomorphic links. However, the reversal is wrong. First of all there is an interesting non-trivial differential-topological obstruction. We observe that, if a diffeomorphism of the links induces a differential equisingularity, then the corresponding self-diffeomorphism of the sphere S^{2N-1} must extend to a

diffeomorphism of the disc D^{2N}. But as is well known, in 1956 Milnor exhibited orientation preserving self-diffeomorphisms of certain k-spheres which could not be extended. More precisely, if $k \geq 4$, the obstruction is given by an element in the group Θ^{k+1} of h-cobordism classes of k+1-dimensional oriented homotopy spheres, see [40]. We claim that the vanishing of this obstruction is even too weak for differential equisingularity. For example, choose
$X_1 = \{(t,0,0) \mid t \in \mathbb{C}\} \subset \mathbb{C}^3$ and $X_2 = \{(0,t^2,t^3) \mid t \in \mathbb{C}\} \subset \mathbb{C}^3$.
Clearly, X_1 is smooth at 0 and X_2 has an ordinary cusp at 0. The link of $(X_1,0)$ is the trivial knot (S^5,S^1). Further, the neighborhood boundary K_2 of $(X_2,0)$ is diffeomorphic to S^1 and hence, taking into account the results of Zeeman and Haefliger we mentioned above, the link of $(X_2,0) \subset (\mathbb{C}^3,0)$ is diffeomorphic to the trivial knot. Since $\Theta^6 = 0$, the corresponding self-diffeomorphism of S^5 extends to a diffeomorphism of $D^6 \subset \mathbb{C}^3$. However, such diffeomorphism does not send X_1 into X_2 because of following result due to Gau and Lipman.

1.4 Proposition. ([12]). If two isolated singularities (X,p) and (X',p') are differentially equisingular, then mult(X,p) = mult(X',p').

Remarks. (i) Above example shows that the assertion in 1.4 is wrong if one only requires topological equisingularity.
(ii) It would be interesting to know of examples of singularities with diffeomorphic links but for which the obstruction in Θ^{k+1} does not vanish.

1.5 According to Milnor, [25], the link of a hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$ is fibred, i. e. the complement $S^{2n-1} - K_p$ admits a C^∞-fibration, the so called Milnor fibration, over S^1 such that the closure $\overline{F}$ of the fibre F is a real 2n-dimensional manifold with boundary K_p. A characteristic diffeomorphism $g : F \to F$ of the Milnor fibre is called the geometric monodromy of (X,p). The induced automorphism $h := g^* : H^*(F;\mathbb{C}) \to H^*(F;\mathbb{C})$ is the monodromy of (X,p) which plays a fundamental role in the study of hypersurface singularities. It can be readily sun that the monodromy only depends on the topologically equisingular type.

The following observation is due to Deligne and can be readily obtained from the work of A' Campo in [1].

1.6 Lemma. Let h be the monodromy of a hypersurface singularity (X,p). Then

$$\mathrm{mult}(X,p) \leq \inf\{s \in \mathbb{N} \mid \Lambda(h^s) \neq 0\}$$

where $\Lambda(h^s)$ denotes the Lefschetz number of the s-th power of h.

Proof. Assume $f(z_0^1,\ldots,z_n)$ is a defining equation for given hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$ with $z_1(p) = \ldots = z_n(p) = 0$. Let U be a small open neighborhood of 0 in $\mathbb{C}^n$ such that p is the only singular point of X in U. According to Hironaka, there exists an embedded good resolution of the singularity, i. e. there exists a complex manifold M together with a proper surjective map $\pi : M \to U$ which fulfills following properties:

(i) $\pi|M-\pi^{-1}(0) : M-\pi^{-1}(0) \to U-\{0\}$ is biholomorphic.

(ii) The irreducible components of the exceptional divisor $E := (f\circ\pi)$ are smooth and have only normal crossings

The decomposition of E into irreducible components can be written as

$$E = \tilde{X} + \sum_{i=1}^{r} m_i E_i \tag{1.6.1}$$

where $\tilde{X}$ is the strict transform of X with respect the modification map π. Now we assert that

$$\mathrm{mult}(X,p) = \inf\{m_i \mid 1 \leq i \leq r\}. \tag{1.6.2}$$

We may assume $m_1 = \inf\{m_i \mid 1 \leq i \leq r\}$. Consider a generic line L in $\mathbb{C}^n$ passing through 0 such that $(L\cdot X)_0 = m$, $m := \mathrm{mult}(X,p)$. The strict transform of L, say $\tilde{L}$, intersects the support of E, say Supp(E), at an unique point q. Let $q \in E_{i_0}$, then $\tilde{L}\cdot E \geq m_{i_0}$ and $\tilde{L}\cdot E = (L\cdot X)_0 = m$ since $\pi|_{\tilde{L}} : \tilde{L} \to L$ is an isomorphism. On the other hand we may choose a line $\tilde{L}_1$ on M which transversally intersects supp(E) at an unique smooth point p of supp(E) with $p \in E_1$. Hence we have $\tilde{L}_1\cdot E = m_1$. The image $L_1 := \pi(\tilde{L}_1)$ is a curve on $\mathbb{C}^n$ meeting X at 0. Thus we have $m_1 = (L_1\cdot X)_0 \geq m$ proving the formula in (1.6.2).

In order to compute the Lefschetz numbers $\Lambda(h^s)$, A' Campo introduced following subsets of supp(E):

$$E^{(k)} := \{x \in \pi^{-1}(0) \mid E \text{ is locally given by } z^k = 0 \text{ at } x\}.$$

Let $\chi_T(E^{(k)})$ denote the topological Euler characteristic, then A' Campo, [1], has shown that for $s \geq 1$:

$$\Lambda(h^s) = \sum_{k|s} k\, \chi_T(E^{(k)}). \tag{1.6.3}$$

Putting this formula together with (1.6.2) we are done.

As an application we obtain following partial answer to Zariski's question, compare also the comment of Lê and Teissier in [21].

1.7 Theorem. Given two n-dimensional hypersurface singularities (X,p) and (X',p') embedded in $\mathbb{C}^n$. Let V, resp. V', denote the projectivized tangent cone which is a subvariety of $\mathbb{P}^{n-1}$. If $\chi_T(\mathbb{P}^{n-1}-V), \chi_T(\mathbb{P}^{n-1}-V') \neq 0$, then topological equisingularity of (X,p) and (X',p') implies $\mathrm{mult}(X,p) = \mathrm{mult}(X',p')$.

Proof. Let m be the maximal ideal in $\mathcal{O}_{X,p}$. Then the projectivized tangent cone to X at p is defined as $V := \mathrm{Proj}(\bigoplus_{i=0}^{\infty} m^i/m^{i+1})$. In other words, the projectivized tangent cone is the exceptional divisor in the blow-up of X at p, and it is a subvariety of the projectivized Zariski tangent space which may be identified with $\mathbb{P}^{n-1}$. Notice that the multiplicity of X at p is equal to the degree of the projective variety V.

Now take an embedded good resolution of (X,p) which factorizes via the blow-up of $\mathbb{C}^n$ at 0. By hypothesis, it can be readily seen from (1.6.3) that $\mathrm{mult}(X,p) = \inf\{s \in \mathbb{N} \mid \Lambda(h^s) \neq 0\}$ where h denotes the monodromy. Since analogous arguments work for (V',p'), the assertion is a consequence of the topologically equisingular invariance of the monodromy.

As is well known, Zariski's question can be affirmatively answered in the case of plane curve singularities. Above result provides a nice proof, see [1].

1.8 Corollary. Topologically equisingular plane curve singularities are equimultiple.

Proof. Recall the multiplicity of a given isolated singularity can be computed in terms of the multiplicities of the irreducible branches and their intersection numbers. Thus it suffices to check irreducible singularities since a topological equisingularity preserves the irreducible branches. We observe that the projectivized tangent cone of an irreducible plane curve singularity is a one-point space and hence

$\chi_T(\mathbb{P}^2-V) = 1$. So we may apply above result.

1.9 Remark. In general one even does not know whether $\chi_T(\mathbb{P}^n-V)$ is an invariant of the topologically equisingular type.

§ 2 The stratum of constant Milnor number.

2.1 There is a more algebro-geometric notion of equisingularity via deformation theory. Let $\varphi : (X,p) \to (T,0)$ be a deformation of (X,p) over a reduced space germ $(T,0)$ together with a section $\sigma : (T,0) \to (X,p)$. Then φ is said to be a topologically equisingular (TE)-deformation along σ if the singularities $(X_t,\sigma(t))$ are topologically equisingular for all $t \in T$ near 0.

The main part of our paper is devoted to following weaker version of Zariski's question.

(Q) Are topologically equisingular deformations of hypersurface singularities equimultiple?

2.2 Definition. Let $\varphi : (X,p) \to (D,0)$ be a deformation of a hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$. Then ρ is called a 1-parameter deformation with trivial section $\bar{D}$ if following conditions are satisfied:

(i) D is an open disc neighborhood of 0 in $\mathbb{C}$.

(ii) X is a reduced hypersurface in some open neighborhood $W \times D$ of $0 \in \mathbb{C}^n \times \mathbb{C}$ given by an equation $F(z,t) = 0$ such that the fibers $X_t = (\mathbb{C}^n \times \{t\}) \cap X$ are hypersurfaces having isolated singular points along the trivial section $\bar{D} = \{(0,t) \mid t \in D\}$ and that $X_0 \cong X$.

(iii) $\varphi : X \to D$ is given by the projection $(z,t) \to t$.

Remark. With regard to above question, by base change via a resolution of the parameter space, it suffices to study 1-parameter deformations which are topologically equisingular along given trivial section.

2.3 Regarding this, Zariski proposed to check whether a 1-parameter TE-deformation $\varphi : X \to D$ of (X,p) is geometrically equisingular along its trivial section $\bar{D}$; e. e. whether the pair $(X^0,\bar{D})$, $X^0 := X-\mathrm{Sing}(X)$, satisfies the Whitney conditions a) and b) in a neighborhood of p. Due to Thom and Mather, [34], [24], one knows that geometric equisingularity implies topological equisingularity. Moreover, a result due to Hironaka says that geometrically equisingular deformations are equimultiple.

2.4 Unfortunately this approach does not work. For this let us briefly recall the crucial numerical characterization of geometric and topological equisingularity. By definition, the Milnor number $\mu(X,p)$ of a hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$ is given by the multiplicity of the jacobian ideal of a defining function f, i. e. $\mu(X,p) := \mathrm{mult}_{\mathcal{O}_n}(J(f)) - \dim \mathcal{O}_n/J(f)$ where $\mathcal{O}_n := \mathbb{C}\{z_1,\dots,z_n\}$. The Milnor number is actually a topological invariant since $\mu(X,p)$ equals the n-th Betti number of the Milnor fiber, [25].

The intersection of (X,p) with a generic i-dimensional hyperplane of $(\mathbb{C}^n,0)$, $0 < i < n$, yields again a hypersurface singularity. The corresponding Milnor number is denoted by $\mu^{(i)}(X,p)$. So one obtains a sequence $\mu^*(X,p) = (\mu^{(n)}(X,p),\dots,\mu^{(0)}(X,p))$ of integers which turn out to be invariants of the singularity (X,p), [30]. We observe that $\mu^{(n)}(X,p) = \mu(X,p)$, $\mu^{(1)}(X,p) = 1 + \mathrm{mult}(X,p)$ and $\mu^{(0)}(X,p) = 1$. Following result summarizes deep works of several people.

2.5 Theorem. Let $\varphi : (X,p) \to (T,0)$ be a deformation of a hypersurface singularity (X,p) over a reduced space germ $(T,0)$ together with a section $\sigma : (T,0) \to (X,0)$. Then we have:

(i) is topologically equisingular along φ if and only if φ is μ-constant along σ, i. e. $\mu(X_t, \sigma(t)) = \mu(X,p)$ for all t near 0.

(ii) Suppose T is smooth and 1-dimensional. Then φ is geometrically equisingular along σ if and only if φ is μ^*-constant along σ.

Comments. The μ-constancy of topological equisingularity is clear. If $\dim X \neq 2$, the other direction was done by Lê and Ramanujam, [20]. The proof follows rather easily from the h-cobordism theorem which is not valid in real dimension 4. In fact, the surface case turned out

to be a very deep result which was finally proved by Perron,[27]. Concerning the second assertion Briancon-Speder,[7], at first could show that the numerical condition is necessary for geometric equisingularity. Again the other direction is more difficult and was settled some years later by Teissier,[33]

2.6 As a corollary, Briancon-Speder, [6], could point out a first example showing that geometric equisingularity is stronger than topological equisingularity and is in particular not suitable to answer our equimultiplicity problem. More precisely, the hypersurface

$$\overline{X} := z_0^3 + z_1^9 + z_1 z_2^4 + t\, z_0 z_2^3 \subset \mathbb{C}^3 \times \mathbb{C}$$

has along the trivial section $t \to (0,t)$ two-dimensional quasi-homogeneous hypersurface singularities of multiplicity 3 such that $\mu^*(\overline{X}_0,0) = (56,7,2,1)$ and $\mu^*(\overline{X}_t,0) = (56,6,2,1)$ for $t \neq 0$. One observes that the 2-dimensional generic hyperplane section yields a plane curve singularity which is irreducible for $t = 0$ and has two branches for $t \neq 0$.

2.7 Above example of a μ-constant deformation is given by a "linear" perturbation. In fact our first step is to show that any "linear" TE-deformation of a hypersurface singularity is equimultiple. More precisely, consider a 1-parameter deformation $\varphi : X \to D$ which is μ-constant along its trivial section $\overline{D}$. Let $F(z,t) = 0$ be a defining equation for the hypersurface $X \subset W \times D \subset \mathbb{C}^n \times \mathbb{C}$. We may write

$$F(z,t) = f(z) + t^{\alpha}\cdot h(z,t) \text{ with } \alpha \geq 1 \text{ and } h(z,t) = \sum_{i\geq 1} t^{i-1} h_i(z)$$

where $h_1 \neq 0$ and $f(z) = 0$ is a defining equation for X. Let us call h_1 to be a <u>first order term</u> of φ. It is clear that φ is equimultiple along $\overline{D}$ if and only if

$$\nu(h_i) \geq \mathrm{mult}(X,p) \text{ for all } i \geq 1$$

where $\nu(h_i)$ denotes the order of h_i in z. Here we set $\nu(h_i) = \infty$ if $h_i = 0$.

2.8 <u>Proposition.</u> Let $(X,p) \subset (\mathbb{C}^n,0)$ be a hypersurface singularity, and let $\varphi : X \to D$ be a 1-parameter deformation of (X,p). If φ is μ-constant along trivial section $\overline{D}$, then

$$\nu(h_1) \geq \mathrm{mult}(X,p)$$

for every first order term h_1 of φ.

<u>Proof.</u> Let $F(z,t) = 0$ be a defining equation for $X \subset \mathbb{C}^n \times \mathbb{C}$. We shall use the following valuation test for μ-constant deformations: due to Lê, Saito and Teissier, [33]:

(2.8.1) φ is μ-constant along $\overline{D}$ if and only if for every holomorphic curve $\gamma : (\mathbb{C},0) \to (\mathbb{C}^n \times \mathbb{C},0)$

$$\nu(\frac{\partial F}{\partial t} \circ \gamma) \geq \inf \{\nu(\frac{\partial F}{\partial z_i} \circ \gamma) \mid 1 \leq i \leq n\}.$$

In [33], one actually only finds the estimate $\nu(\frac{\partial F}{\partial t} \circ \gamma) \geq \inf\{\ldots\}$. But it is not difficult to see that the weaker inequality is equivalent to the strong inequality we assert above, [13].

We keep the notation of 2.7. Then it is clear that $\tilde{F}(z,t) = f(z) + t \cdot h(z,t) = 0$ also defines a μ-constant deformation along the trivial section $\overline{D}$. Choose $a \in \mathbb{C}^n$ near 0 such that $h_1(a) \neq 0$. Define $\gamma : \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$ by $\gamma(s) = (sa,0)$. Then $\frac{\partial \tilde{F}}{\partial t} \circ \gamma(s) = h_1(sa)$ and $\frac{\partial \tilde{F}}{\partial z_i} \circ \gamma(s) = \frac{\partial f}{\partial z_i}(sa)$. Then it follows that $\nu(h_1) = \nu(\frac{\partial \tilde{F}}{\partial t} \circ \gamma)$ and $\nu(\frac{\partial \tilde{F}}{\partial z_i} \circ \gamma) = \mathrm{mult}(X,p) - 1$. Now apply (2.8.1) and we are done.

2.9 We can give a geometric interpretation of above result. Let $\delta : X \to U$ denote a sufficiently small representative of the semi-universal deformation of a given hypersurface singularity (X,p). Then U is a small neighborhood of 0 in $\mathbb{C}^{\tau}$ where τ is the Tjurina number. Recall, if $f(z_1,\ldots,z_n) = 0$ is a defining equation for X, then $\tau = \dim \mathcal{O}_n/(f,J(f))$. The image of the critical set of δ is an analytic subset of U which is called the discriminant of δ and is denoted by Δ. It is an irreducible reduced hypersurface of U and the Hilbert-Samuel function induces an analytic stratification of Δ where a stratum Δ_m, $m \in \mathbb{N}$, is given by the set of points $x \in \Delta$ with $\mathrm{mult}(\Delta,x) = m$. Suppose $t \in \Delta_m$, then m can be computed in terms of Milnor numbers as follows,[29]: $m = \sum_x \mu(X_t,x)$, $x \in X_t$. In particular, $\mathrm{mult}(\Delta,0) = \mu := \mu(X,p)$ and because of the semi-continuity property of the Milnor number, it is clear that $\Delta_m = \emptyset$ for $m > \mu$. The stratum Δ_μ of highest multiplicity provided with the reduced structure is called the <u>μ-constant stratum.</u> Due to results of Lê and Lazzeri, [19], the singular locus over Δ_μ does not split, i. e.

$$\Delta_\mu = \{t \in \Delta \mid \mathrm{Sing}(X_t) = \{x_t\} \text{ and } \mu(X_t,x_t) = \mu\}.$$

Hence it can be readily seen that there exists an unique singular section of $\delta_\mu : X^{(\mu)} \to \Delta_\mu$ whose image is denoted by $\bar{\Delta}_\mu$. By 2.5 we observe that δ_μ is a TE-deformation along $\bar{\Delta}_\mu$. Moreover it can be easily deduced from [29] that δ_μ together with the section $\bar{\Delta}_\mu$ is a semi-universal TE-deformation with respect to reduced base spaces.

Let $m = \mathrm{mult}(X,p)$, then consider the set

$$\Delta_{EM} := \{t \in \Delta \mid \mathrm{mult}(X_t,x) = m \text{ for all } x \in \mathrm{Sing}(X_t)\}.$$

It is well known that Δ_{EM} is an analytic subset of Δ. Provided with the reduced structure, it is called the <u>equimultiplicity stratum</u> of the semi-universal deformation of (X,p). So our question (Q) is equivalent to the question whether or not

$$\Delta_\mu \subset \Delta_{EM}?$$

From Proposition 2.8 we immediately obtain following partial answer, compare also [13].

<u>2.10 Theorem.</u> If the μ-constant stratum Δ_μ of a hypersurface singularity (X,p) is smooth, then the tangent space $T_0\,\Delta_\mu$ of Δ_μ at the distinguished point 0 is a subspace of $T_0\,\Delta_{EM}$.

As an application we can affirmatively answer our question (Q) in the quasihomogeneous case. This is due to Greuel, [13], and O'Shea, [26].

<u>2.11 Theorem.</u> Let (X,p) be a quasihomogeneous hypersurface singularity. Then $(\Delta_\mu,0) \subset (\Delta_{EM},0)$, i. e. every μ-constant deformation of (X,p) is equimultiple.

<u>Proof.</u> By hypothesis, X can be defined by a quasihomogeneous polynomial $f \in \mathbb{C}[z_1,\dots,z_n]$ with weights $w_1,\dots,w_n$ and weighted degree $\mathrm{wt}(f) = d$. Let $\delta : X \to U$ be a representative of the semi-universal deformation. We may assume that X is a hypersurface in $\mathbb{C}^n \times \mathbb{C}^\tau$ which is given an equation $F(z,t) = f(z) + \sum_i t_i g_i(z)$, $1 < i < \tau$, where $g_1,\dots,g_\tau$ are monomials which represent a $\mathbb{C}$-linear basis of $\mathcal{O}_n/(f,J(f))$. A deep result of Varchenko, [36], says that

$$\Delta_\mu = \{t \in U \mid t_i = 0 \text{ if } \mathrm{wt}(g_i) < d\}.$$

Thus our assertion immediately follows from 2.8.

<u>2.12 Remark.</u> It was conjectured that the μ-constant stratum should be always smooth. Very recently Luengo, [23], gave first counterexamples, see also [15] for more details concerning this subject.

<u>§ 3 The functor ES.</u>

Zariski's algebraic approach to equisingularity arose from his conception of simultaneous resolution of plane curve singularities. Since in higher dimensions there is no canonical resolution procedure available, it is not at all clear what the right meaning of simultaneous resolution should be. Regarding this, one should compare Teissier's discussion of various notions of simultaneous resolution in [32]. On the other hand the philosophy of Wahl's work on equisingularity, [38], [39], is to study deformations of resolutions which blow down to deformations of corresponding singularities.

<u>3.1</u> Let (X,p) be a normal n-dimensional singularity, and let $\sigma : \tilde{X} \to X$ be a resolution with exceptional set $A = \sigma^{-1}(p)$. We may assume that X is s Stein space and that $\tilde{X}$ is a 1-convex manifold. Then every deformation of the germ (X,A) over a complex space germ $(T,0)$ can be given by a 1-convex flat map $\Phi : \tilde{X} \to T$ with $\tilde{X} = \tilde{X}_0$ after possibly shrinking of M. Then Φ factorizes via the relative Remmert quotient $\bar{\sigma} : \tilde{X} \to X$, i. e. there exists following commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\bar{\sigma}"] \arrow[dr, "\Phi"'] & & X \arrow[dl, "\bar{\Phi}"] \\ & T & \end{tikzcd}$$

We say Φ blows down if $\bar{\Phi}$ defines a deformation of (X,p) over $(T,0)$.

<u>3.2</u> Now assume that $\sigma : \tilde{X} \to X$ is a good resulution, i. e. the irreducible components of A are smooth and have only normal crossings. Recall Wahl's definition, [39], of the functor ES:

$$ES_{\tilde{X}}(T,0) = \{\text{isomorphism classes of deformations of } \tilde{X} \text{ over } (T,0) \text{ which blow down and induce locally trivial deformations of } A \text{ over } (T,t_0)\}.$$

The elements of $ES_{\tilde{X}}(T,0)$ are called <u>equisingular deformations</u> of the resolution germ $(\tilde{X},A)$ over $(T,0)$.

As above, let $\bar{\Phi} : X \to T$ denote the blown down deformation of an ES-deformation $\Phi : \tilde{X} \to T$. By definition, there exists a subspace

$A \subset \tilde{X}$ such that $\Phi|A : A \to T$ is a locally trivial deformation of A over T. If T is reduced we observe that there exists an unique singular section $\gamma : (T,0) \to (X,p)$ which is given by the image $\bar{\sigma}(A)$ of the relative Remmert quotient $\bar{\sigma} : \tilde{X} \to X$ of the 1-convex map Φ.

Question: Suppose (X,p) is a hypersurface singularity of dimension $n > 2$. Is the blowing down of an ES-deformation of a good resolution $\tilde{X}$ of X over a reduced space germ topologically equisingular along the canonical section?

By Theorem 2.5, it suffices to check that the blown down deformation $\bar{\Phi} : X \to T$ is μ-constant along the section $\bar{T} := \gamma(T)$. If n is even, the Milnor number μ can be computed in terms of Chern numbers of $\tilde{X}$ and the genus $p_g(X,p)$ of the singularity which is due to Looijenga, [22]. Thus $\bar{\Phi}$ is μ-constant if the genus does not change along $\bar{T}$. It is well known that the genus-function is upper semicontinuous along $\bar{T}$ and that it is constant if $n = 2$, [28]. This gives an affirmative answer to above question if $n = 2$. But in dimension 2, a much stronger statement is true.

3.3 Theorem. (Laufer [17]). Let $\varphi : X \to T$ be a deformation of a 2-dimensional hypersurface singularity (X,p) over a reduced parameter space T together with a section $\gamma : (T,0) \to (X,p)$. Then φ is topologically equisingular along γ if and only if (φ,γ) can be obtained by blowing down an ES-deformation of any good resolution of X.

3.4 Corollary. Let $\sigma : \tilde{X} \to X$ be a resolution of a 2-dimensional hypersurface singularity (X,p). Let Z be the fundamental cycle on $\tilde{X}$. If $\mathrm{mult}(X,p) = -Z\cdot Z$, then every μ-constant deformation of (X,p) is equimultiple.

Proof. Recall that Z is the minimal effective exceptional divisor on $\tilde{X}$ having non-positive intersection number with each irreducible component of the exceptional set $A = \sigma^{-1}(p)$. It is well known that $\mathrm{mult}(X,p) \geq -Z\cdot Z$, [37]. Further it is easy to check that the self-intersection number of the fundamental cycle on each other resolution of (X,p) is equal to $Z\cdot Z$. Thus we may assume that $\sigma : \tilde{X} \to X$ is a good resolution. Let $\varphi : \tilde{X} \to T$ be an ES-deformation of $\tilde{X}$. By definition, Z lifts to $\tilde{X}$, i. e. there exists a subspace $Z \subset \tilde{X}$ such that the restriction $\varphi|Z : Z \to T$ is a locally trivial deformation of Z. Clearly, Z_t is the fundamental cycle on $\tilde{X}_t$. Thus blowing down of φ yields an equimultiple deformation of (X,p).

3.5 Examples. (3.5.1) Every μ-constant deformation of a rational or minimally elliptic hypersurface singularity is equimultiple. Since this assertion is true for all double points, it remains to check minimally elliptic singularities of multiplicity > 3. But for those we can apply above corollary, [18]. Rational and minimally elliptic hypersurface singularities include the uni- and biomodal singularities in the sense of Arnold.

(3.5.2) Let $(X,p) \subset (\mathbb{C}^3,0)$ be a hypersurface singularity which resolves in one blow up of the maximal ideal. Then μ-constant deformations of (X,p) are equimultiple.

For this, let $f = 0$ be an equation of X in a small neighborhood U of 0 in $\mathbb{C}^3$. Let $\pi : M \to U$ be the blow up of U at 0. Then the decomposition of the exceptional divisor $E = (f\circ\pi)$ into irreducible components is given by $\tilde{X} + m\cdot\mathbb{P}^2$ where $m = \mathrm{mult}(X,p)$. By assumption $\sigma := \pi|_{\tilde{X}} : \tilde{X} \to X$ is a resolution with exceptional set $A = \tilde{X}\cap\mathbb{P}^2 = V_{red}$ where V is the projectivized tangent cone. It can be readily seen that $V = V_{red}$ and that $m_p\cdot\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-V)$. Thus it follows from [37] that $V = Z$ and $m = -Z\cdot Z$.

(3.5.3) Finally we wish to mention some interesting examples of 2-dimensional hypersurface singularities which resolve in one blow up.

(i) $f(x,y,z) = x^k+y^{k+1}+z^k+y^{k-2}z^2$ where the projectivized tangent cone V is an irreducible plane curve of geometric genus $g = \frac{1}{2}(k-1)(k-2) - [\frac{k}{2}]$ having exactly one singular point which is of type A_{k-1}.

(ii) $f(x,y,z) = y(xy^3+z^4)^2+x^9+y^{10}$ where V is irreducible, $g(V) = 10$ and V has one singular point which is of type A_{35}.

(iii) $f(x,y,z) = y^3(xy^2+z^3)^2+x^5z^4+(x+y+z)^{10}$ where V is irreducible, $g(V) = 6$ and V has exactly 3 singular points which are of type A_{18}, E_8 and isomorphic to the quasihomogeneous plane curve singularity given by a local equation $u^4+v^7 = 0$.

We should mention that the μ-constant stratum of the first example is smooth, [15], and singular for the other two cases, [23].

3.6 Remarks. Recall that there exists a good resolution $M \to X$ such

that $m_p \cdot \mathcal{O}_M$ is locally principal. Then there exists an effective exceptional divisor D on M, the so called <u>maximal ideal cycle</u>, with $D > Z$ and $\mathcal{O}_M(-D) = m_p \cdot \mathcal{O}_M$, [41]. Consider an ES-deformation $\varphi : \mathcal{M} \to T$. As above there exists a locally trivial lifting $\mathcal{D} \to T$ of D over T but if $D > Z$, then $\mathcal{D}_t$, $t \neq 0$, does not need to be the maximal ideal cycle on $\mathcal{M}_t$.

For instance, consider the example of Briancon-Speder we mentioned in 2.6:

$$F(z,t) = z_0^3 + z_1^9 + z_1 z_2^4 + t \cdot z_0 z_2^3.$$

Let X_0 be the hypersurface given by the equation $F(z,0) = 0$. Then the weighted dual graph of a good resolution $M \to X_0$ with $m_0 \cdot \mathcal{O}_M$ locally principal is given by $\overset{-1}{\bullet}\!\!-\!\!\overset{-4}{\bullet}\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\overset{-3}{\bullet}$ [3]. Number the vertices by A_1, A_2, A_3, A_4 starting with the -1-vertex. Then A_2 corresponds to a smooth curve of genus 3, the genera of the other curves are equal to 0. The maximal ideal divisor D is given by $D = A_1 + A_2 + 3A_3 + A_4$. Since $F(z,t)$ defines a μ-constant deformation, we know by Laufer's result that this deformation can be obtained from an ES-deformation $\mathcal{M} \to D$ of M. It turns out that $m_0 \cdot \mathcal{O}_{\mathcal{M}_t}$ is locally principal, too, but for $t \neq 0$ the maximal ideal cycle D_t is given by $D_t = 2A_1 + A_2 + 2A_3 + A_4$ and hence different from the lifting of D to $\mathcal{M}_t$.

The moral is that the topology of the resolution does not in general give enough informations in order to control the multiplicity under ES-deformations. The crucial point is that the topology "forgets" the fact that the singularity is a hypersurface singularity. But, as we noticed at the beginning, this is the key point which guarantees equimultiplicity. So our new viewpoint is to look at deformations of good embedded resolutions.

§ 4 Deformations of embedded resolutions.

<u>4.1</u> Let X,Y be complex spaces and $A \subset X$, $B \subset Y$ closed analytic subspaces. By $F : (X,A) \to (Y,B)$ we denote the germ of a holomorphic map $X \to Y$ taking A into B. Then Def_F is the functor of deformations of the holomorphic map germ F. A representative of an element of the set $\mathrm{Def}_F(T,t_0)$ is given by a commutative diagram of holomorphic maps

$$\begin{tikzcd} \overline{X} \arrow[rr, "\overline{F}"] \arrow[dr, "\varphi"'] & & \overline{Y} \arrow[dl, "\psi"] \\ & T & \end{tikzcd}$$

where φ, resp. ψ, is a deformation of a small neighborhood

of A in X, resp. of B in Y, and $\overline{F}|\overline{X}_{t_0}$ is a representative for the germ F. It is clear when two diagrams define the same element in $\mathrm{Def}_F(T,t_0)$.

<u>4.2</u> Now consider a hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$ and an embedded good resolution $\pi : M \to U$ with exceptional divisor $E = \overline{X} + \sum_i m_i E_i$, $1 \leq i \leq r$, where U is an open Stein neighborhood of 0 in $\mathbb{C}^n$. As before we write $E^0 = E_1 \cup \ldots \cup E_r$, $A := \overline{X} \cap E^0$, $\sigma := \pi|\overline{X}$. So π defines a closed embedding $J_\pi : (\overline{X},A) \to (M,E^0)$ of space germs.

<u>4.3</u> <u>Definition.</u> Def_{J_π} is called the functor of <u>embedded deformations</u> of the embedded good resolution $\pi : M \to U$ of (X,p).

<u>Remark.</u> Let
$$\begin{tikzcd} \overline{\mathcal{X}} \arrow[rr, "J_\pi"] \arrow[dr, "\varphi"'] & & \mathcal{M} \arrow[dl, "\psi"] \\ & T & \end{tikzcd}$$
be a representative of an element of $\mathrm{Def}_{J_\pi}(T,0)$. Then we may assume that $\overline{\mathcal{X}}_0 = \overline{X}$, $\mathcal{M}_0 = M$, φ, ψ are flat 1-convex maps and that $\overline{J}_\pi : \overline{\mathcal{X}} \to \mathcal{M}$ is a closed embedding, [8], [15].

<u>4.4</u> <u>Lemma.</u> Let $\pi : M \to U$ be an embedded good resolution of a hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$. Then an embedded deformation of π over $(T,0)$ simultaneously blows down to a deformation of (X,p) in $(\mathbb{C}^n,0)$, i. e. the exists a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{X}} \arrow[rr, "\overline{J}_\pi"] \arrow[d, "\overline{\sigma}"] \arrow[ddr, "\varphi"'] & & \mathcal{M} \arrow[d, "\overline{\pi}"] \arrow[ddl, "\psi"] \\ X \arrow[r] \arrow[dr, "\overline{\varphi}"'] & U \times T \arrow[d, "pr"] & \\ & T = T & \end{tikzcd}$$

such that $\overline{\sigma}$ and $\overline{\pi}$ are the relative Remmert quotients and that $\overline{\varphi}$ defines a deformation of (X,p).

<u>Proof.</u> Since $R^1\pi_* \mathcal{O}_M = 0$ it follows from [28] that the deformation $\psi : \mathcal{M} \to T$ simultaneously blows down to a deformation of U over T. On the other hand we know, [3], that, after possibly shrinking of U, any deformation of U is trivial. This yields the right-hand triangle diagram. In order to show that $\varphi : \overline{\mathcal{X}} \to T$ blows down, we have to check that the map $\Gamma(A, \mathcal{O}_{\overline{\mathcal{X}}}) \to \Gamma(A, \mathcal{O}_{\overline{X}})$ is onto, compare [28] and [14]. But this immediately follows from the surjectivity of $\Gamma(E^0, \mathcal{O}_M) \to \Gamma(E^0, \mathcal{O}_M)$.

By uniqueness of the relative Remmert quotient we get the commutativity of above diagram.

Remarks. (i) Suppose $X \subset U$ is given by an equation $f(z) = 0$, $z \in U$. Then we may assume that $\mathcal{X} \subset U \times T$ is given by an equation $\bar{f}(z,t)=0$ with $\bar{f}(z,0)=f(z)$. Further it is clear that the zero-divisor $\mathcal{E} := (\bar{f}\circ\bar{\pi})$ on M defines a flat lifting of E over T.
(ii) Denote by $B_{(M,E^0)}$, resp. $B_{(\tilde{X},A)}$, the functor of deformations of the germ (M,E^0), resp. $(\tilde{X},A)$, which blow down to deformations of $(\mathbb{C}^n,0)$, resp. (X,p). Then above lemma says that there exist natural transformations $\mathrm{Def}_{J_\pi} \to B_{(M,E^0)}$ and $\mathrm{Def}_{J_\pi} \to B_{(\tilde{X},A)}$.

4.5 Definition. We keep our notations. Let $\pi : (M,\tilde{X}) \to (U,X)$ be an embedded good resolution of an irreducible hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$. Then $\mathrm{ESE}_{(M,\tilde{X})}$ is the functor of complex space germs defined by

$$\mathrm{ESE}_{(M,\tilde{X})}(T,t_0) = \{\text{embedded deformations of } \pi \text{ over } T \text{ such that the induced deformations of } (M,E^0) \text{ and } (\tilde{X},A) \text{ are equisingular in the sense of 3.2}\}/\text{equivalence}.$$

In other words, $\mathrm{ESE}_{(M,\tilde{X})}$ is the full subfunctor of Def_{J_π} such that the natural transformations $\mathrm{ESE}_{(M,\tilde{X})} \to B_{(M,E^0)}$ and $\mathrm{ESE}_{(M,\tilde{X})} \to B_{(\tilde{X},A)}$ factorize via $\mathrm{ES}_{(M,E^0)}$ and $\mathrm{ES}_{(\tilde{X},A)}$.
The representatives of elements of $\mathrm{ESE}_{(M,\tilde{X})}(T,t_0)$ are called embedded equisingular (short: ESE) deformations of π.

Remark. Suppose the diagram
$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr] & & M \arrow[dl] \\ & T & \end{tikzcd}$$
defines an embedded deformation of π. Then it can be readily seen that it defines an embedded equisingular deformation if and only if there is subspace, say $\bar{\mathcal{E}}$, of M which is locally trivial over T and lifts the reduced exceptional divisor $\bar{E} = E_{red}$ of π.

4.6 Theorem. Let $\pi : (M,\tilde{X}) \to (U,X)$ be an embedded good resolution of an irreducible hypersurface singularity $(X,p) \subset (\mathbb{C}^n,0)$, and let $\bar{\varphi} : (X,p) \to (D,0)$ be a 1-parameter deformation of (X,p). Suppose $\bar{\varphi}$ arises by blowing down an ESE-deformation of π. Then there exists an unique singular section, say $\bar{D} \subset X$, which φ is equimultiple and topologically equisingular along.

Proof. Suppose $f(z) \rangle 0$ is a defining equation for $X \subset U$. Then we may assume that $\mathcal{X} \subset U \times D$ is given by an equation $\bar{f}(z,t) = 0$ with $\bar{f}(z,0) = f(z)$. Further let
$$\begin{tikzcd} \tilde{X} \arrow[rr, "J_\pi"] \arrow[dr, "\varphi"'] & & M \arrow[dl, "\psi"] \\ & D & \end{tikzcd}$$
be an ESE-deformation of π which blows down to $\bar{\varphi}$. Then $\mathcal{E} := (\bar{f}\circ\bar{\pi})$ defines a flat lifting of the exceptional divisor E of π over D, compare the first remark to 4.4. Here $\bar{\pi} : M \to U \times D$ is the relative Remmert quotient with respect to ψ. On the other hand, there exists a locally trivial lifting, say $\bar{E}$, of E_{red} over D, see above remark. Hence we have a locally trivial lifting $\mathcal{A} \subset \mathcal{X}$ of A over D. Then $\bar{D} := \bar{\sigma}(\mathcal{A})$ is the singular section of the deformation $\bar{\varphi}$ where $\bar{\sigma} : \mathcal{X} \to X$ is the relative Remmert quotient with respect to φ. It is obvious that $\mathrm{supp}(\mathcal{E}_t) = \bar{E}_t$, $t \in D$. Thus it is easy to check $\mathcal{E}$ is even a locally trivial lifting of E. So, if $E = \tilde{X} + \sum_i m_i E_i$, $1 \leq i \leq r$, then we have $\mathcal{E}_t = \tilde{X}_t + \sum_i m_i \bar{E}_{t,i}$, $1 \leq i \leq r$. Now our assertion follows from (1.6.2) and A'Campo's formula for the Milnor number, [1], $\mu = (-1)^n(-1+ \sum_{k\geq 1} k\cdot\chi(E^{(k)})$.

Putting this together with Laufer's result, see 3.3, we get following answer to our question (Q).

4.7 Proposition. Let (X,p) be a two-dimensional irreducible hypersurface singularity. Then TE-deformation of (X,p) are equimultiple if there exists a good embedded resolution $\pi : (M,\tilde{X}) \to (U,X)$ such that the transformation $\mathrm{ESE}_{(M,\tilde{X})} \to \mathrm{ES}_{(\tilde{X},A)}$ is onto.

4.8 In case of plane curve singularities, it follows immediately from the work of Wahl in [38] that the transformation ESE $\to$ ES is surjective.

To study it in higher dimensions, we need informations about the functor ESE. We omit the subcript $(M,\tilde{X})$ if it is clear which spaces we consider. Let $\Theta_M(\log \bar{E})$ denote the dual of the sheaf of one-forms with logarithmic poles along the reduced exceptional divisor $\bar{E}$. Recall $\Theta_M(\log \bar{E})$ is locally free of rank n and there exists an exact sequence
$$0 \to \Theta_M(\log \bar{E}) \to \Theta_M \to \bigoplus_i N_{E_i} \oplus N_{\tilde{X}} \to 0$$
where N_{E_i}, resp. $N_{\tilde{X}}$, denotes the normal sheaf of E_i, resp. $\tilde{X}$, in M.

Since each element of $ESE(T,t_0)$ can be represented by a deformation of M to which each irreducible component of the exceptional divisor lifts and the blowing down condition is automatically satisfied, we deduce from [39]:

(4.8.1) First order deformation space $ESE(\mathbb{C}[\varepsilon]) = H^1(\Theta_M(\log \bar{E}))$.

(4.8.2) Formal smoothness of ESE is obstructed by elements in $H^2(\Theta_M(\log \bar{E}))$.

In particular we can follow the arguments in [39] to show that ESE has a hull in the sense of Schlessinger. Actually we prove in [15] following much deeper result.

4.9 Theorem. ([15]). $ESE_{(M,\tilde{X})}$ has versal deformations.

Remark to the proof. Using the recent deep work of Bingener and Kosarew, [5], we at first show that the functors $ES_{(\tilde{X},A)}$ and $ES_{(M,E^0)}$ have semi-universal deformations. Then our approach is to follow Flenner's proof of the existence of a versal deformation of a holomorphic map between compact spaces, [10]. However $(\tilde{X},A)$ and (M,E^0) are only space germs along compact spaces. This causes serious technical difficulties since we cannot apply Douady's result on the representability of the morphism functor, compare [10]. For further details we have to refer to [15].

4.10 The straightforward approach to handle the surjectivity of the transformation ESE → ES is to solve at first the corresponding infinitesimal problem. Given an element $\xi \in ESE(T)$ where T is a 0-dimensional space germ. Then ξ defines an element, say ζ of $ES(T)$ where $ES = ES_{(\tilde{X},A)}$ with $A = \tilde{X} \cap E^0$. Let $T \to T'$ be a small extension, i. e. $\mathcal{O}_{T'} \to \mathcal{O}_T$ is a surjection of Artinian algebras with one-dimensional kernel (ε), and suppose there exists an extension $\zeta' \in ES(T')$ of ζ. Does there exist an extension $\xi' \in ESE(T')$ of ξ which induces ζ'?

For this consider following commutative diagram of sheaves with exact rows:

$$
(*)\qquad
\begin{array}{ccccccccc}
0 & \to & \Theta_M(\log E^0)(-\tilde{X}) & \to & \Theta_M(\log \bar{E}) & \to & \Theta_{\tilde{X}}(\log A) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \Theta_M(\log E^0)(-\tilde{X}) & \to & \Theta_M(\log E^0) & \to & \Theta_M(\log E^0)\otimes \mathcal{O}_{\tilde{X}} & \to & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\bar{\delta} & & \\
0 & \to & \mathcal{O}_M^n(-\tilde{X}) & \to & \mathcal{O}_M^n & \to & \mathcal{O}_{\tilde{X}}^n & \to & 0
\end{array}
$$

Note that $\bar{E} = E^0 + \tilde{X}$. Thus $\Theta_M(\log \bar{E})$ is a subsheaf of $\Theta_M(\log E^0)$. Then easy local computation show that $\Theta_M(\log E^0)(-\tilde{X})$ is a subsheaf of $\Theta_M(\log \bar{E})$ with cokernel $\Theta_{\tilde{X}}(\log A)$. This yields the upper part of the diagram. Let $x_1,\dots,x_n \in \Gamma(M, \mathcal{O}_M)$ be the global section defined by local coordinates on $U \subset \mathbb{C}^n$. Then the map δ is defined by taking local derivations and successively evaluate them on the global functions x_i, $1 \le i \le n$. From $(*)$ we obtain following commutative diagram of exact cohomology sequences:

$$
\begin{array}{ccccccc}
H^1(\Theta_M(\log \bar{E})) & \to & H^1(\Theta_{\tilde{X}}(\log A)) & \to & H^2(\Theta_M(\log E^0)(-\tilde{X})) & \to & H^2(\Theta_M(\log \bar{E})) \\
\downarrow & & \downarrow\bar{\delta}^* & & \downarrow\delta^* & & \downarrow \\
0 & \to & H^1(\mathcal{O}_{\tilde{X}}^n) & \to & H^2(\mathcal{O}_M^n(-\tilde{X})) & \to & 0
\end{array}
$$

Since $ES(\mathbb{C}[\varepsilon]) = \operatorname{Ker} \bar{\delta}^*$, see [39], we observe that formal smoothness of the transformation ESE → ES, i. e. the existence of ξ', is obstructed by elements in $\operatorname{Ker} \delta^*$. Actually a much stronger result is true.

4.11 Theorem. Let $(X,p) \subset (\mathbb{C}^n,0)$ be an irreducible hypersurface singularity. Suppose there exists an embedded good resolution $\pi : (M,\tilde{X}) \to (U,X)$ with exceptional divisor E such that $H^2(\Theta_M(\log \bar{E})) = 0$ and $\operatorname{Ker}(\Theta_M(\log E^0)(-\tilde{X})) \to \mathcal{O}_M^n(-\tilde{X})) = 0$, then we have:

(i) The transformation $ESE_{(M,\tilde{X})} \to ES_{(\tilde{X},A)}$, $A = \tilde{X} \cap E^0$, is surjective.

(ii) If $n = 3$, then TE-deformations of (X,p) are equimultiple.

Proof. The cohomological vanishing conditions imply that ESE is formally smooth, see (4.8.2), and that ESE → ES is formally smooth. Hence ES is formally smooth. By Theorem 4.9 we know that both functors ESE and ES have versal deformations. Thus one can readily deduce from Satz 2.4 in [4] and Satz 5.2 in [11] that ESE and ES are smooth. But this clearly implies the surjectivity of the transformation ESE → ES. Presumably one should be able to show the smoothness of this transformation. The second statement is an immediate consequence of 4.7.

4.12 There is an interesting class of hypersurface singularities for which embedded good resolutions can be obtained via the Newton polyhedron. Let $f(z) = 0$ be a defining equation of an irreducible hypersurface singularity (X,p). Suppose $f(z) = \Sigma a_k z^k$, $k \in \mathbb{N}^n$. Then $\Gamma_+(f) :=$ convex hull of $\bigcup_k (k+\mathbb{R}_+^n)$, $a_k \neq 0$, is called the Newton polyhedron of f. The union of the compact boundary faces of $\Gamma_+(f)$ is denoted by $\Gamma(f)$ and is called the Newton boundary. For a face σ of

$\Gamma(f)$, we write $f_\sigma(z) := \Sigma a_k z^k$, $k \in \sigma$. Then f is said to be <u>non-degenerated</u> if for each face σ of $\Gamma(f)$ the polynomials f_σ and df_σ have no common zeros on $\mathbb{C}^{*n}$. Zhe Newton polyhedron gives rise to a decomposition Λ_0 of $\mathbb{R}^n_+$ into conved rational cones. By [16], there is a corresponding toroidal embedding Y_{Λ_0} together with a proper birational map $\pi : Y_{\Lambda_0} \to \mathbb{C}^n$. By $\pi : M_{\Lambda_0} \to U$ we denote the restriction of π to 1-convex neighborhood M_{Λ_0} of $E^0 = \pi^{-1}(0)$. A subdivision Λ of Λ_0 is said to be <u>smooth</u> if the corresponding map $\pi : M_\Lambda \to U$ is an embedded good resolution of the hypersurface singularity defined by f at 0. Due to Varchenko, [35], there always exist smooth subdivisions of Λ_0 if f is non-degenerated and <u>commode</u>, i. e. $f(0,\ldots,0,z_i,0,\ldots,0) \not\equiv 0$ for $1 \leq i \leq n$.

In his recent thesis, [2], Altmann has studied our cohomological vanishing conditions in 4.11 for this class of hypersurface singularities. His main result depends on the existence of a "good" smooth subdivision.

<u>4.13 Definition.</u> ([2]. Suppose f is non-degenerated and commode. Then a smooth subdivision Λ of Λ_0 is called <u>good</u> if following condition is satisfied for $1 \leq k \leq n$:

Let a be an one-dimensional cone of Λ_0 such that a and e_k are contained in a cone of Λ_0. Then there exists a cone of Λ which contains $P_k(a)$ and e_k where $P_k(a)$ is given as follows: The cone a is uniquely determined by a vector $(a_1,\ldots,a_n) \in \mathbb{Z}^n_+$ with relatively prime coordinates. Set $d := \gcd(a_1,\ldots,\hat{a}_k,\ldots,a_n)$. Then $P_k(a)$ is the unique element of $\mathbb{Z}^n$ with $P_k(a) = \frac{1}{d}(a_1,\ldots,a_k+s,\ldots,a_n)$ where $0 \leq s \leq d$ and $s \in \mathbb{Z}$.

<u>4.14 Theorem.</u> Let (X,p) be a two-dimensional hypersurface singularity which can be defined by a non-degenerated and commode function. If the complex Λ_0 of convex rational cones associated to the Newton boundary $\Gamma(f)$ admits a good smooth subdivision, then μ-constant deformations of (X,p) are equimultiple.

<u>Proof.</u> Let $\pi : (M,\tilde{X}) \to (U,X)$ be an embedded good resolution of (X,p) arising from a smooth subdivision of Λ_0. Then Altmann shows that $H^2(\Theta_M(\log \tilde{E})) = 0$, [2]. If Λ is even good, then he can prove the vanishing of the obstruction space $\mathrm{Ker}(H^2(\Theta_M(\log E^0)) \to H^2(\mathcal{O}^n_M(-\tilde{X})))$, [2 ;Satz 4.4]. Now the result follows from 4.11.

<u>4.15 Remarks and Examples.</u> (i) We observe the existence of a good smooth subdivision implies that the μ-constant Stratum is smooth, see also [15].

(ii) If each of the coordinate axes of $\mathbb{R}^3$ is intersected by exactly one 2-dimensional face of the Newton boundary Γ, it is always possible to find a good smooth subdivision of the complex Λ_0 associated to Γ, [2; Satz 4.5].

(iii) The polynomial $f(x,y,z) = x^5+y^6+z^5+y^3z^2$ is non-degenerated and commode but $\Gamma(f)$ does not satisfy above condition. However it is easy to check that Λ_0 admits a good smooth subdivision. However, it turns out Λ_0 of the polynomial $x^6+y^7+z^6+y^4z^2$ does not admit a good smooth subdivision. On the other hand we know by (3.5.3) that μ-constant deformation of the singularity defined by g are equimultiple. This shows that if we even restrict us to singularities given by non-degenerated and commode equations above result is still incomplete with respect to our key question (Q).

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