The singularities of some invariant hypersurfaces*

by

George Kempf

The Johns Hopkins University

In his work [1] Igusa has required detailed knowledge of the resolution of the invariant conic in $\mathbb{A}^{27}$ for the group E_6 and the invariant quartic in $\mathbb{A}^{56}$ for the group E_7. In this paper we will compute embedded resolutions of these hypersurfaces.

§1. Numerical properties of resolutions

Let H be a hypersurface of a smooth variety X. Let $\pi: X_1 \to X$ be a proper birational morphism where X_1 is a smooth variety. Then we have the strict transform H_1 of H and also we have the divisor-theoretic inverse image $\pi^{-1}(H)$ of H. Then $\pi^{-1}(H) = H_1 + \sum n_i E_i$ where the E_i are effective (exceptional) divisors on X_1 and the multiplicities n_i are non-negative numbers.

The morphism π induces an $\mathcal{O}_{X_1}$-homomorphism $\pi^*\Lambda^n\Omega_X \longrightarrow \Lambda^n\Omega_{X_1}$ where $n = \dim X$. Thus we have an effective divisor $\mathrm{Ram}(\pi)$ on X_1 such that $\Lambda^n\Omega_{X_1} = (\pi^*\Lambda^n\Omega_X)(\mathrm{Ram}\ \pi)$. We may write $\mathrm{Ram}\ (\pi) = \sum m_i E_i$ where the m_i are non-negative integers. The pairs (n_i, m_i) of integers are called the numerical data for the morphism π and the divisor H.

If H_1 and the E_i's are smooth divisors meeting transversally, we will say that π is an embedded resolution. In this case the numerical data gives information about the singularities of H. For instance [2] H has rational singularities iff $n_i \le m_i$ for all i. We want to have a stronger condition

* $2n_i \le m_i$ for all i.

In practice we will construct π to be a succession of blowing up with smooth centers. Then we will have a sequence $X_n, \ldots, X_0 = X$ of smooth varieties, smooth centers $C_{n-1}, \ldots, C_0$ in X_i such that X_{i+1} is X_i with C_i blown up.

*Partly supported by NSF grant #MPS75-05578

Here we denote the projection $X_{i+1} \longrightarrow X_i$ by π_i^{i+1}. Similarly the composition $X_j \longrightarrow X_i$ given by $\pi_{j-1}^j \circ \dots \circ \pi_i^{i+1}$ is denoted by π_i^j. We have exceptional divisor ${}_iE = (\pi_{i-1}^i)^{-1}(C_{i-1})$ in X_i for $1 \le i \le n$. For any variety Y of X_i we will denote by Y_j the strict transform of Y by π_i^j. Thus X_n contains the exceptional divisors ${}_1E_n, \dots, {}_nE$.

In this situation we may compute the numerical invariants inductively. Assume that $(\pi_0^j)^{-1}H = H_j \; \varepsilon \sum_{1 \le i \le j} n_i({}_iE_j)$

and $\mathrm{Ram}\,(\pi_0^j) = \sum_{1 \le i \le j} m_i({}_iE_j)$.

Then $(\pi_0^{j+1})^{-1}H = H_j + \sum_{1 \le i \le j} n_i({}_iE_{j+1}) + x({}_{j+1}E)$

and $\mathrm{Ram}\,(\pi_0^{j+1}) = \sum_{1 \le i \le j} m_i({}_iE_{j+1}) + y({}_{j+1}E)$

where x is the multiplicity of $(\pi_0^j)^{-1}H$ at a generic point of C_j and $y = (\mathrm{cod}\, C_j - 1)$ plus $\sum_{C_j \subset {}_iE_j} m_i$.

§2. The group E_6.

We will be working with a fundamental representation of E_6 of dimension 27. Explicitly consider the vector spaces of triples (A,x,y) where A is a 6×6 skew-symmetric matrix and x and y are 6-vectors. Let $C(A,x,y)$ be the cubic form $\mathrm{Pf}(A) + y\,A\,{}^tx$ where Pf denotes the Pfaffian. Then the group E^6 is the connected subgroup of $GL(27)$ which leaves this form C invariant.

We will need to know some transformations in E_6. First of all $SL(6)$ is contained in E^6. Let M be an element of $SL(6)$. Then $M\cdot(A,x,y) = (MA{}^tM, xM^{-1}, yM^{-1})$ is a linear transformation of $\mathbb{A}^{27}$ which leaves C invariant. Also $SL(2)$ is contained in E^6. For N in $SL(2)$ set $N\,(A,x,y) = (A,z,w)$ where $\binom{z}{w} = N\binom{x}{y}$. These operations also leave C invariant. The diagonals of $SL(2)$ and $SL(6)$ together form a maximal torus of E^6 for which the action on $\mathbb{A}^{27}$ is diagonal. We need to know some more erotic one-parameter subgroups of E_6.

Let $\binom{s_1,\dots,s_r}{t_1,\dots,t_r}$ be zero if the s_i's are not a permutation of the t_i's and sign (permutation) is they are. Let $f_1<f_2<f_3$ be a triple of elements of $[1,6]$

and let $f_4<f_5<f_6$ be the complementary triple. Then we have a one-parameter subgroup $T_{f_1f_2f_3}$ of E_6 given by

$$T_{f_1f_2f_3}(\lambda)(a_{ij}) = a_{ij} + \lambda\varepsilon\binom{i\ j\ k}{f_1f_2f_3}x_k$$

$$T_{f_1f_2f_3}(\lambda)(y_k) = y_k + \lambda\binom{i\ j\ k}{f_4f_5f_6}a_{ij}$$

$$T_{f_1f_2f_3}(\lambda)(x_k) = x_k$$

where $\varepsilon = \begin{pmatrix} f_1f_2f_3f_4f_5f_6 \\ 1\ 2\ 3\ 4\ 5\ 6 \end{pmatrix}$.

We also have one-parameter subgroups $S_{f_1f_2f_3}$ where

$$S_{f_1f_2f_3}(\lambda)(a_{ij}) = a_{ij} - \lambda\binom{i\ j\ k}{f_1f_2f_3}y_k$$

$$S_{f_1f_2f_3}(\lambda)(y_k) = y_k$$

$$S_{f_1f_2f_3}(\lambda)(x_k) = x_k + \lambda\binom{i\ j\ k}{f_4f_5f_6}a_{ij} \quad \text{with } i<j\ .$$

We want to resolve the singularities of the cone $H = (C=0)$. Let $X_0=X$ be $\mathbb{A}^{27}$. The blowing up as described is

<u>Theorem</u> a) The center C_0 is the origin which has codimension 27 and a triple point of H.

b) The center C_1 is the singular locus of H_1 which has codimension 10 in X, and H_1 has rank 10 double points along H_1. Furthermore H_1 and ${}_1E$ meet transversally.

c) H_2 is smooth and $H_{2ij}E_2$ and ${}_2E$ meet transversally.

<u>Corollary</u>. $(\pi_0^2)^{-1}H = H_2 + 3({}_1E_2) + 2({}_2E)$

and $\mathrm{Ram}(\pi_0^2) = 26({}_1E_2) + 9({}_2E)$.

<u>Proof</u>. As $\mathbb{A}^{27}$ is spanned by extreme weight vectors for E_6, we need blow-up only one coordinate as all of them are isomorphic under the action of the Weyl group. Let B = {skew-symmetric 6×6 matrices where $a_{56} = 1$}. Consider the quadratic transformation $\sigma\colon \mathbb{A}^1 \times B \times \mathbb{A}^6 \times \mathbb{A}^6 \longrightarrow \mathbb{A}^{27}$ given by $\sigma(\lambda,b,x,y) = (\lambda b, \lambda x, \lambda y)$. Then π is locally isomorphic to σ and $\sigma{\times}C = \lambda^3C(b,x,y)$. Thus $H_1 = (C(b,x,y) = 0)$ is transversal to ${}_1E = (\lambda=0)$. We need to analyze $C(b,x,y)$.

Let D be the subset of B where $b_{ij} = 0$ if i or j is 5 or 6 but $\{i,j\} \neq \{5,6\}$. Let K be the upper diagonal subgroup of SC(6) with 1 on the diagonal and only non-zero entrees, $m_5^1,\ldots,m_5^4,m_6^1,\ldots,m_6^4$.

Claim. The multiplication $K\times D \to B$ sending (k,d) to $k\cdot d\cdot {}^tk$ is an isomorphism This claim is clear by matrix multiplication. Using the claim we have an isomorphism $K \times \{D \times \mathbb{A}^6 \times \mathbb{A}^6\} \longrightarrow B \times \mathbb{A}^6 \times \mathbb{A}^6$ given by inclusion of K in E^6. Now $C(kd^tk, xk^{-1}, yk^{-1}) = C(d,x,y)$ by the invariance of C. This last expression involves few variables than $C(b,x,y)$. Next we will use the exotic transformations to remove more variables. Let $f_1<f_2<f_3$ be a subset of $[1,4]$. The transform $T_{f_1,f_2,f_3}(\lambda)$ leaves $B \times \mathbb{A}^6 \times \mathbb{A}^6$ invariant. If $f_4,5,6$ is the complementary subset, we have $T_{f_1,f_2,f_3}(\lambda)y_{f_4} = y_{f_4} + \lambda(\begin{smallmatrix}5 & 6 & f_4\\ f_4 & 5 & 6\end{smallmatrix})$ and $T_{f_1,f_2,f_3}(\lambda)y_* = y_*$ if $* = 5$ or 6. Thus if V is the subset of $\mathbb{A}^6$ where $y_1 = y_2 = y_3 = y_4 = 0$. we have an isomorphism $\mathbb{A}^4 \times B \times \mathbb{A}^6 \times V \longrightarrow B \times \mathbb{A}^6 \times \mathbb{A}^6$ given by the action of the above 4 one-parameter subgroups. Similarly with the S transformation if $W = \{x_1 = x_2 = x_3 = x_4 = 0$ in $\mathbb{A}^6\}$, $\mathbb{A}^4 \times B \times W \times V \longrightarrow B \times \mathbb{A}^6 \times V$ is an isomorphism. The function $C(b,x,y)$ on $B \times W \times V$ is just $Pf(B_{ij_{1\le ij\le 4}}) + x_5y_6 - y_5x_6$ which is a rank 10 quadric. Thus the local equation of H_1 is a rank 10 quadric and it has singular locus $0 = B_{ij} = x_k = y_\ell$ for $1\le i,j\le 4$ and $5\le k,\ell\le 6$. It is elementary to check that H_2 is smooth and transversal to ${}_1E_2$ and ${}_2E$, The only remarkable fact is the transformation in E_6 induce the group D_5 of this quadratic. Q.E.D.

3, The group E_7.

We will be working with a fundamental representation of E_7 of dimension 56, Explicitly consider the space of pairs (Z,Y) of skew-symmetric 8×8 matrices. Let

$$Q(Z,Y) = Pf(Z) + Pf(Y) - \frac{1}{4}Tr(ZYZY) + \frac{1}{16}(Tr(ZY))^2.$$

Then the group E_7 is the connected subgroup of GL(56) which leave invariant Q and the skew-symmetric form $\langle(Z,Y),(Z',Y')\rangle = Tr(ZY'-Z'Y)$. There is an obvious inclusion of SL(8) in E_7 given by the action $A \cdot (Z,Y) = (AZ^tA, {}^tA^{-1}YA)$

for A in SL(8). The diagonal of SL(8) is a maximal torus in E_7 for which the action on A^{56} is diagonalized. We also have exotic transformations in E_7. Let $f_1<f_2<f_3<f_4$ be a subset of $[1,\ldots,8]$ with complement $f_5<f_6<f_7<f_8$. Let U_{f_1,f_2,f_3,f_4} be the one-parameter subgroup defined by

$$\begin{cases} U_{f_1,f_2,f_3,f_4}(\lambda)(Z_{ij}) = Z_{ij} + \lambda \sum_{m,n} \binom{i\ j\ m\ n}{f_1 f_2 f_3 f_4} Y_{m,n} \\ \text{and} \\ U_{f_1,f_2,f_3,f_4}(\lambda)(Y_{ij}) = Y_{ij} - \lambda\,\varepsilon \sum_{m,n} \binom{i\ j\ m\ n}{f_5 f_6 f_7 f_8} Z_{m,n} \end{cases}$$

where $\varepsilon = \binom{f_1,\ldots,f_8}{1,\ldots,8}$.

We want to make an embedded resolution of the quartic hypersurface $H = (Q=0)$ in $X = X_0 = \mathbb{A}^{56}$. We will do this resolution by a sequence of blowing-up

<u>Theorem</u> a) The first center C_0 is the origin in X_0. Then C_0 has codimension 56 and H has multiplicity four at C_0.

b) The next center C_1 is the singular locus of the singular locus of H_1. Now C_1 has codimension 28 and H_1 has rank one double points along C_1. Also C_1 is transversal to ${}_1E$.

c) H_2 has two components in its singular locus. Let C_2 be the component of codimension 11. Then C_2 is transversal to ${}_1E_2$ and ${}_2E$ and H_2 has rank 11 double points generically along C_2.

d) The hypersurface H_3 is singular in codimension 3 in X_3. Let C_3 be its singular locus. Then H_3 has rank three double points along C_3. Furthermore $C_3 \subset {}_2E_3$ and C_3 is transversal to E_3 and ${}_3E$.

e) Let C_4 be ${}_2E_4 \cap H_4$ with its reduced structure. Then C_4 has codimension 2 and H_4 is smooth and C_4 is transversal to ${}_1E_4$, ${}_3E_4$ and ${}_4E$.

f) H_5 is smooth and meets ${}_2E_5$ transversally. Let C_5 equal $H_5 \cap {}_2E_5$. Then $C_5 \subset {}_5E$.

g) H_6 and the exceptional divisors are smooth and meet transversally.

Corollary. $(\pi_0^6)^{-1}H = \overline{H_6 + 4\,{}_1E_6 + 2\,{}_2E_6} + 2\,{}_3E_6 + 2\,{}_4E_6 + {}_5E_6 + 4\,{}_6E$

and

$$\mathrm{Ram}(\pi_0^6) = 55\,{}_1E_6 + 27\,{}_2E_6 + 10\,{}_3E_6 + 29\,{}_4E_6 + 28\,{}_5E_6 + 56\,{}_6E \,.$$

Proof. In the first blowing up we are magnifying the behavior at zero. Again as $\mathbb{A}^{56}$ is spanned by extreme vectors we need only look at the part of the blow-up where $Z_{7,8}$ has been inverted. Let B be the subset $Z_{7,8} = 1$ of $\mathbb{A}^{28}$. Then $(\pi_0^1)^{-1}(Z_{7,8} \neq 0)$ is isomorphic to $\mathbb{A}^1 \times B \times \mathbb{A}^{28}$, where the projection π_0^1 in these coordinates is $\pi_0^1(\lambda,b,Y) = (\lambda b,\lambda Y)$. As Q is homogeneous of degree 4, $(\pi_0^1)^*Q(\lambda,b,Y) = \lambda^4 Q(b,Y)$. Then H_1 is $Q(b,Y) = 0$ in local coordinates which is transversal to ${}_1E$ which is $\lambda = 0$. We may use the unipotent transformations in SL(8) to get $B_{ij} = 0$ if i or j equals 7 or 8 but not both. Let D be the subset of B defined by these inequalities. Let H be the subgroup of SL(8) of elements with ones on the diagonal and zero except for h_{ij} where $i = 7$ or 8 and $j \in [1,\ldots,6]$. Then the action of E_7 on $\mathbb{A}^{56}$ gives an isomorphism ψ: $H \times (D \times \mathbb{A}^{28}) \longrightarrow B \times \mathbb{A}^{28}$. As Q is invariant under E_7, $\psi^{-1}(Q)(h,d,Y) = Q(d,Y)$, thus ψ gives local coordinates such that the equation of H_1 is simpler. We will further simplify by using the exotic transformations. Let $f_1<f_2<f_3<f_4$ be a subset of $[1,\ldots,6]$. We can use this transformation to reduce to the case where $Y_{f_5,f_6} = 0$. Thus by a sequence of such transformations we can reduce to the case where $Y_{ij} = 0$ if i or j is not contained in {7,8}. Let $A_{ij} = b_{ij}$ for $1\le i,j\le 6$, $x_i = Y_{i,7}$ and $y_i = Y_{i,8}$ where $1\le i\le 6$ and $a = Y_{7,8}$. Then by direct calculation we have $Q(b,Y) = \mathrm{Pf}(A) + yA^t x - \frac{1}{4}a^2 = C(A,x,y) - \frac{1}{4}a^2$ where C is the cubic for E_6.

Now it is clear that the singular locus of H_1 is locally $0 \times \{\text{singular locus of } C^3\} \times$ removed variables. So the singular locus C_1 of the singular locus is given by $a = A = x = y = 0$ which is codimension 28. Clearly H_1 has rank 1 souble points along C_1. Now blowing up C_1 in the coordinate a produced a smooth H_2 which does not meet the new exceptional divisor. Blowing up the other coordinates is more interesting. Note that E_6 acts on $\mathbb{A}^1 \times \mathbb{A}^{27} = \{(a,A,x,y)\}$ by the previous action on the second factor and by the trivial one on the first. In fact one can check that this action is induced by part of the E_7-action on $\mathbb{A}^{56}$. As in the last proof we may only blow up the $A_{5,6}$-coordinate as all other are equivalent under the action of E_6. Using the elimination of variables as before and choicing appropriate local coordinates (z_i), a local equation of H_2 is $0 = -\frac{1}{4}a^2 + \lambda \sum_{1\leq i\leq} z_i z_{10-i}$ where $\lambda = 0$ is the local equation of the exceptional divisor ${}_2E$. Thus we are down to a hypersurface in 12 variables. The singular locus of H_2 locally is $\{a = \lambda = \sum z_i z_{10-i} = 0\} \cup \{a = z_1 = \ldots = z_{10} = 0\}$. This last component is C_2 and it has codimension 11 and is transversal to ${}_2E$. Also H_2 has rank 11 double points along C_2 generically.

Next we blow up C_2. Again blow-up in variable a we have H_3 is smooth and does not meet the divisor ${}_2E_3$ and ${}_3E$. Blowing up a variable z_j (say $j = 10$) the equation of H_3 becomes $0 = -\frac{1}{4}a^2 + \lambda\,(\sum_{1\leq i\leq 4} z_i z_{10-i} + z_9)$ where $\lambda = 0$ is ${}_2E_3$. Let $\sum_{1\leq i\leq 4} z_i z_{10-i} + z_9 = z$ be a new coordinate. The equation H_3 is $0 = -\frac{1}{4}a^2 + \lambda z$. The singular locus C_3 of H_3 is given by $0 = a = \lambda = z$ and has codimension 3 in X_3. Furthermore H_3 has rank 3 double points along C_3. Clearly H_3 is transversal to ${}_1E_3$ and ${}_3E$ and, hence, so is C_3. Note that C_3 is contained in ${}_2E_3$.

Now we can blowup C_3. Clearly H_4 is smooth and meets ${}_1E_4$, ${}_3E_4$ and ${}_4E$ transversally. We need to see how H_4 meets ${}_2E_4$. Blowup the coordinate a H_4 does not meet ${}_2E_4$ in this open. Blowup the coordinate λ ${}_2E_4$ does not meet this open. Finally blowing the coordinate z, H_4 has equation $-\frac{a^2}{4} + \lambda = 0$ where $\lambda = 0$ is the local equation of ${}_2E_4$. Thus $(H_4 \cap {}_2E_4)_{red} = C_4$ is given by $a = \lambda = 0$. After blowing up C_4 it is elementary to check that all the divisors H_5, ${}_1E_5$, ${}_2E_5$, ${}_3E_5$, ${}_4E_5$ and ${}_5E$ meet transversally except unfortunately $H_5 \cap {}_2E_5 = {}_2E_5 \cap {}_5E = C_5$. Then C_5 is of codimension two. Blowup C_5 and then all intersections are transversal. Q.E.D.

References

[1] J. Igusa, Forms of higher degree, Tata Inst. Lecture Notes 59, Springer-Verlag (1978) (esp. pp. 122-128).

[2] G. Kempf, Varieties with rational singularities, Lefschetz conference Mexico City Dec. 1984, to appear.

G. Kempf, The Johns Hopkins University
Dept. of Mathematics, Baltimore, MD 21218, U.S.A.