MORSE THEORY ON SINGULAR SPACES

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Introduction.

Let X be a space, $f : X \to \mathbb{R}$ a map. Denote by X_a the set $\{x \in X \mid f(x) \leq a\}$. Morse theory is concerned with the homotopy type of $X_a \subsetneq X_b$ for real numbers $a < b$ when X is a differentiable manifold and f is a proper differentiable generic map.
Here we treat the case that X has isolated singularities. The main applications are to complex spaces. For example applying this theory we can prove that a Stein space X with isolated singularities is homotopically equivalent to a CW complex of dimension $n = \dim_{\mathbb{C}} X$. Also Lefschetz type theorems are available. They depend in general on the kind of singularities of X.

An example : let X be a complex projective algebraic variety with isolated singularities, X_0 an hyperplane section and let

$$\sigma_i \ : \ H_i(X_0, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

be the homomorphism induced by the inclusion $X_0 \subsetneq X$; in general nothing can be said on the σ_i. However if the singularities of X are "good" (for example if they are of complete intersection type) then the usual Lefschetz theorem holds, i.e. σ_i is an isomorphism for $i < \dim_{\mathbb{C}} X_0$ and surjective for $i = \dim_{\mathbb{C}} X_0$.

In the sequel we give the definitions and we state the theorems. Proofs are only sketched ; details will appear in a forthcoming paper in Annali Scuola Normale Superiore, Pisa.

1. Let X be a locally closed set in $\mathbb{R}^N$. Suppose that there exists a discrete subset $\Sigma \subset X$ such that $X - \Sigma$ is a differentiable submanifold of $\mathbb{R}^N$ of dimension $n > 0$. For $x \in X - \Sigma$ denote by $T_x(X)$ the tangent space to X at x. We shall say that X is a space with isolated singularities iff

$$\lim_{x \to y} \sin(T_x(X), x-y) = 0$$

for all $y \in \Sigma$, where $\sin(H,v)$ denotes the sinus between the vector v and the linear subspace H in $\mathbb{R}^N$.

Remarks.

a) We suppose X embedded in some $\mathbb{R}^N$ for simplicity only ; actually every construction or result in the sequel will depend only on the struture given by the sheaf $\mathcal{E}_X$ of germs of differentiable functions on X.

b) Every analytic space with isolated singularities is a space with isolated singularities (see H. Whitney,"Tangents to analytic variety", Ann. of Math., 81 (1965), 547).

From now on $X \subset \mathbb{R}^N$ is a space with isolated singularities of dimension n.

Let $Z(X,x)$ denote the Zariski tangent space of X at x ; namely $Z(X,x)$ is the vector subspace of $\mathbb{R}^N$ of all vectors c such that $df_x(v) = 0$ for every differentiable function f vanishing on X.
An n-plane Π in $\mathbb{R}^N$ is said to be a tangent plane to X at x iff there exists a sequence of regular points (x_ν) in X such that $x_\nu \to x$ and $T_{x_\nu}(X) \to H$, the latter limit being made in the Grassmanian of n-planes in $\mathbb{R}^N$. The set of all tangent planes to X at x will be denoted by $T_x(X)$; it is easily recognized as a closed subset of the Grassmanian of n-planes in $Z(X,x)$.
Remark that if x is a regular point of X, then $Z(X,x)$ is the usual tangent space $T_x(X)$ and that it is also the single tangent n-plane to X at x.

Denote by $Z'(X,x)$ the vector space of linear forms on $Z(X,x)$, by $D(X,x)$ the set of all $l \in Z'(X,x)$ identically vanishing on some element of $T_x(X)$ and $\Omega(X,x) = Z'(X,x) - D(X,x)$.

Proposition 1.

If X is a real analytic space, then $D(X,x)$ is closed without interior. Moreover if f can be endowed with a complex analytic structure near x, then $\Omega(X,x)$ is connected.

Both assertions are proved by looking at $T_x(X)$. One proves that $T_x(X)$ is a closed set of Hausdorff dimension less or equal to n-1 ; from that the first assertion follows easily. The second is easier, since in that case $T_x(X)$ is an analytic space of dimension less or equal to n-2 in the appropriate

Grassmanian.

Let $f : X \to \mathbb{R}$ be a differentiable function. The differential df_x of f at a point x is a well defined element of $Z'(X,x)$. We shall say that f is regular at x iff $df_x \in \Omega(X,x)$; otherwise x is said to be a critical point of f. A critical point x of f will be said non degenerate iff x is a simple point of X and (as usual) the Hessian $H(f)_x$ is non singular.

Definition.

A Morse function on X is a proper differentiable map $f : X \to \mathbb{R}$ with only non degenerate critical points.

It is easy to see that if f is a Morse function, the set of its critical points is discrete on X. Also the usual density theorems for Morse functions are available in view of proposition 1, if X is a real analytic space.

2. Let x be an isolated singularity of $X \subset \mathbb{R}^N$, $f : X \to \mathbb{R}$ a differentiable function regular at x, $f(x) = 0$.

Notations : $B(\varepsilon)$ the open ball of radius ε around x, $D(\varepsilon) = \overline{B(\varepsilon)}$,

$$S(\varepsilon) = D(\varepsilon) - B(\varepsilon) \quad , \quad X_\eta = \{y \in X \mid f(y) = \eta\} .$$

Proposition 2.

There exists $\varepsilon_0 > 0$ and a continuous function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ so that :

i) $X_0 \cap B(\varepsilon_0)$ is a space with isolated singularities, whose only singularity is x.

ii) for $0 < \varepsilon < \varepsilon_0$, S_ε is transversal to X and X_0.

iii) for $0 < \varepsilon < \varepsilon_0$, $0 < \eta < \eta(\varepsilon)$ the set $M(\varepsilon,\eta) = X_{-\eta} \cap D(\varepsilon)$ is a smooth compact manifold with boundary whose diffeomorphism class depends only on X, x and f ; it will be called the vanishing manifold of f at x in X and denoted by $M(X,x,f)$.

ii) and iii) have usual proofs. The proof of i) is just the following lemma on angles between linear spaces.

Lemma 3.

Let H be an h-plane, K an hyperplane and v a vector in $\mathbb{R}^N$.
Suppose

$$\sin(v,H) < \varepsilon_1,\ \sin(v,K) < \varepsilon_2,\ \sin(v,H\cap K) > \eta$$

where $0 < \varepsilon_1 < \eta$. Then

$$\sin(H,K) < (2\varepsilon_1+\varepsilon_2)\cdot(\eta^2-\varepsilon_1^2)^{-1/2}$$

Proposition 4.

Let f, g be differentiable functions on X, both regular at x.
If df_x and dg_x belong to the same connected component of $\Omega(X,x)$, then the vanishing manifolds $M(X,x,f)$ and $M(X,x,g)$ are diffeomorphic.

The proof is based on the study of the map $\pi : X \times \mathbb{R} \to \mathbb{R}^3$ defined by $\pi : (x,\lambda) \to (\|x\|^2,\ f(x)+\lambda g(x),\lambda)$; lemma 3 gives a uniformity which implies that π is of maximal rank on .....etc....

In view of proposition 1, if X carries a complex analytic structure, the preceding theorem assures that $M(X,x,f)$ does not depend on f at all. In that case we can say some more. Let $X \subset \mathbb{C}^N$. Then $T_x(X)$ is a subset of a complex Grassmanian. A holomorphic function $g : X \to \mathbb{C}$ with $g(x) = 0$ will be said regular at $x \in X$ iff dg_x (the complex differential of g at x) is not identically zero on any element of $T_x(X)$.
Let $0 < \varepsilon \ll 1$ and $0 < \eta \ll 1$ and define $M_{\mathbb{C}}(X,x,g) = \{y \in X \mid g(y)=\eta\} \cap D(\varepsilon)$.
Then $M_{\mathbb{C}}(X,x,g)$ is a compact manifold with boundary that does not depend on ε, η or g. Moreover the function $\tilde{g}$ = real part of g is a function $X \to \mathbb{R}$ which is regular at x.

Proposition 5.

$M_{\mathbb{C}}(X.x,g)$ is a deformation retract of $M(X,x\ \tilde{g})$.

The proof is just the construction of a vector field on $M(X,x,f)$ whose flow gives the required deformation ; it is constructed with the gradient of the function $(g^1)^2$ where g^1 is the imaginary part of g.

3. Let $f : X \to \mathbb{R}$ be a Morse function. For $a \in \mathbb{R}$ denote by M_a the set $\{y \in X \mid f(y) \leq a\}$. Let x be a singular point of X, $f(x) = c$; suppose $a < c < b$ and that $f^{-1}([a,b]) - \{x\}$ does not countain singular points of X or critical points of f.
We shall describe the map $M_a \hookrightarrow M_b$ in the homotopical category. First let us recall some definition. Let Z be a topological space, Y a subset in Z. In the dijoint union of Z with $\{Y\}$ (a one-point space) identify $\{Y\}$ with all $y \in Y \subset Z$. The resulting space is denoted by Z/Y. In particular $Z\times[0,1]/Z\times\{1\}$ is called the cone over Z and denoted by C(Z). Next identify in the dijoint union of Z with C(Y) each $y \in Y \subset Z$ with $y \times \{0\} \in C(Y)$. The resulting space is denoted by C(Z,Y).

Proposition 6.

i) There exist a compact set K in $f^{-1}(a)$ and a homeomorphism $M_a/K \to M_c$ that is the identity on $M_{a-\varepsilon}$ for some $0 < \varepsilon \ll 1$.

ii) M_c is a deformation retract of M_b.

iii) K can be chosen homeomorphic with the vanishing manifold $M(X,x,f)$.

The proofs of i) ii) are similar. Start with the gradient of f on X, normalized so that if $\sigma(t)$ is an integral of it, then $\frac{d}{dt} f(\sigma(t)) = 1$. Then multiply by a function equal to 1 on $f^{-1}([a,b])$ and 0 out of $f^{-1}(]a-\varepsilon,b+\varepsilon[)$ for small ε. The vector field one gets is not defined at x. However studying the associated flow, one sees that each integral starting on M_a can be extended to the interval $[0,c-a]$ to obtain a continuous map $\rho : M_a \times [0,c-a] \to M_c$. From this it is easy to show ii). Then consider $\rho_{c-a} : M_a \to M_c$. Define $K = \rho_{c-a}^{-1}(x)$; ρ_{c-a} is a diffeomorphism between $M_a - K$ and $M_c - \{x\}$ and this proves i).
To prove iii) remark first that for small ε, $M_c \cap D(\varepsilon)$ is the cone over $M_c \cap S(\varepsilon)$ and $f^{-1}(c) \cap D(\varepsilon)$ the cone over $f^{-1}(\varepsilon) \cap S(\varepsilon)$; this follows from the proposition 2 applying standard techniques.Moreover from the description above one has that $M_c/f^{-1}(c) \cap D(\varepsilon)$ is homeomorphic with $M_a/M(X,x,f)$. The following lemma shows that M_c is homeomorphic with $M_c/f^{-1}(c) \cap D(\varepsilon)$ and so concludes the proof.

Lemma 7.

Let Y be a compact manifold with boundary, H the set

$$\{(y,t) \in C(Y) \mid y \in \partial Y,\ 1/2 \leq t \leq 1\}.$$

There exists a homeomorphism between $C(Y)$ and $C(Y)/H$ which is the identity on $Y\times\{0\}$.

Remark now that $M_a \to M_a/M(X,x,f)$ is homotopically equivalent to $M_a \hookrightarrow C(M_a, M(X,x,f))$. Also, if $K \subset M(X,x,f)$ is a deformation retract of $M(X,x,f)$, then $M_a \hookrightarrow C(M_a, M(X,x,f))$ is homotopically equivalent to $M_a \hookrightarrow C(M_a, K)$, so that the former proposition gives the following result

Theorem 8.

Let $K \subset M(X,x,f)$ be a deformation retract of $M(X,x,f)$. Then $M_a \hookrightarrow M_b$ is homotopically equivalent to $M_a \hookrightarrow C(M_a, K)$.

Remark now that if $H \subset M_a$ is obtained from a non empty subset $K \subset M_a$ by adjoing a cell e_λ, then $C(M_a, H)$ is obtained from $C(M_a, K)$ by adjoing a cell $e_{\lambda+1}$. This proves the following

Theorem 9.

Let $M(X,x,f)$ retract with a deformation on a finite spherical complex obtained from a point p by adjoining successively cells $e_{i_1}^{(1)}, \dots, e_{i_r}^{(r)}$.

Then $M_a \hookrightarrow M_b$ is homotopically equivalent to adjoining successively cells $e_{i_1+1}^{(1)}, \dots, e_{i_r+1}^{(r)}$ to M_a.

The same remark applied to a Morse function on $M(X,x,f)$ gives the following

Theorem 10.

Let $\varphi : M(X,x,f) \to \mathbb{R}$ be a differentiable function with the following properties :

i) φ has critical points $p_0, p_1, \dots, p_r$, each non degenerate and not in $\partial M(X,x,f)$.

ii) $\partial M(X,x\ f)$ is a fibre of φ.

iii) $\varphi(p_0) \leq \dots \leq \varphi(p_r)$

Denote by λ_i the index of φ at p_i, $i = 1, \dots, r$. Then $M_a \hookrightarrow M_b$ is homotopically equivalent to attaching successively cells $l_{\lambda_1+1}^{(1)}, \dots, l_{\lambda_r+1}^{(r)}$ to M_a.

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