A THEOREM ON THE MONODROMY OF ISOLATED SINGULARITIES

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INTRODUCTION . – Let $\pi : (X, x) \to (T, t)$ be a flat morphism between germs of smooth complex spaces, (Δ, t) its discriminant. Suppose that the fibre $(X_t, t) = \pi^{-1}(t)$ is a hypersurface with an isolated singularity at x. Then π induces a fibre bundle on $T - \Delta$ whose fibre M has the homotopy type of a bouquet of spheres of dimension $r = \dim X - \dim T$ (see [4]) ; the associated representation $\pi_1(T - \Delta, t) \to \mathrm{Aut}(H_r(M, \mathbb{Z}))$ is called the monodromy of π. By looking at the properties of a representation of $\pi_1(T - \Delta, t)$ in the case that π is semiuniversal, we show an irreducibility property of such a representation. As a consequence we get a no-splitting principle for a hypersurface isolated singularity, that extends a known result for curves [3] .

1 . THE MONODROMY OF π. – Let $\pi : (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$ be a flat morphism whose fibre $(X_0, 0) = \pi^{-1}(0)$ is a hypersurface with an isolated singularity at 0. Denote by $(\Delta, 0)$ the discriminant of π. Choose a small ball B around 0 in $\mathbb{C}^N$. Then there exists a contractible open neighborhood U of 0 in $\mathbb{C}^n$ such that $\pi : B \cap \pi^{-1}(U) \to U$ is a smooth proper map and π is of maximal rank on $\pi^{-1}(U - \Delta) \cap \mathring{B}$ and on $\pi^{-1}(U) \cap \partial B$. It follows that $\pi : B \cap \pi^{-1}(U - \Delta) \to U - \Delta$ is a differentiable fibre bundle, whose fibre M is a compact differentiable $2r$-dimensional manifold with boundary, where $r = N - n$; moreover since π is of maximal rank on $\pi^{-1}(U) \cap \partial B$ we may suppose that the group of the bundle is made with diffeomorphisms of M which are the identity on ∂M.

One knows (see [4]) that M is a parallelizable manifold which is homotopically equivalent to a bouquet of μ spheres of dimension r. In particular $H_r(M, \mathbb{Z})$ is a free module of rank μ over $\mathbb{Z}$. Let $p \in U - \Delta$ and identify M with $\pi^{-1}(p) \cap B$. Then there is a homomorphism $\pi_1(U - \Delta, p) \to \mathrm{Aut}\, H_r(M, \mathbb{Z})$. Letting U vary, one deduces a homomorphism $\sigma : \pi_1(\mathbb{C}^n - \Delta, 0) \to \mathrm{Aut}\, H_r(M, \mathbb{Z})$, where $\pi_1(\mathbb{C}^n - \Delta, 0)$ <u>denotes the local fundamental group of $\mathbb{C}^n - \Delta$ at 0</u>. (*)

(*) Remark that $\pi_1(\mathbb{C}^n - \Delta, 0)$ is defined up to an inner automorphism ant that the indeterminacy of σ is just that of an inner automorphism of $\pi_1(\mathbb{C}^n - \Delta, 0)$.

2 . THE PRESENTATION OF $\pi_1(\mathbb{C}^n - \Delta, 0)$. – Let $(\Delta, 0) \subset (\mathbb{C}^n, 0)$ be defined by an equation

$$w^m + a_1(z)w^{m-1} + \ldots + a_m(z) = 0$$

where $(w, z_1, \ldots, z_{n-1})$ are local coordinates on $(\mathbb{C}^n, 0)$ and $a_i(0) = 0$ for $i = 1, \ldots, m$.

Consider a nice stratification of Δ , say a stratification verifying Whitney's conditions. By the curve selection lemma one can see that there exists $\varepsilon_0 > 0$ s.th. for $0 < \varepsilon < \varepsilon_0$ the hypersurface $\|z\| = \varepsilon$ is transversal to each stratum. It follows that, if $U_\varepsilon = \{(w, z) \mid \|z\| < \varepsilon\}$ then for $0 < \varepsilon' < \varepsilon < \varepsilon_0$ the inclusion $U_{\varepsilon'} - \Delta \to U_\varepsilon - \Delta$ is a homotopy equivalence. Moreover let $\eta(\varepsilon) = \sup_{U_\varepsilon \cap \Delta} |w|$, $U_{\varepsilon,\eta(\varepsilon)} = \{(w, z) \in U_\varepsilon \mid |w| < \eta(\varepsilon)\}$. Then the inclusion $U_{\varepsilon,\eta(\varepsilon)} - \Delta \to U_\varepsilon - \Delta$ is a homotopy equivalence. Since $a_i(0) = 0$ for $i = 1, \ldots, m$ one has that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, so that $U_{\varepsilon,\eta(\varepsilon)}$ is an arbitrary small neighborhood of 0 in $\mathbb{C}^n$. In particular if $0 < \varepsilon < \varepsilon_0$ and $p \in U_\varepsilon - \Delta$, then $\pi_1(\mathbb{C}^n - \Delta, 0) \to \pi_1(U_\varepsilon - \Delta, p)$ is an isomorphism.

NOTATIONS. – $U = U_\varepsilon$; $V = U \cap \{w = 0\}$; $\varphi : U \to V$ the projection. Fix $|w_0| \gg \varepsilon$.

For $z \in V$, L_z is the straight line $\varphi^{-1}(z)$ and $p_z = (w_0, z) \in L_{z_0}$.

Finally let Γ denote the discriminant of $\varphi : U \cap \Delta \to V$, $\tilde\Gamma = \varphi^{-1}(\Gamma)$.

Suppose $z_0 \in V - \Gamma$. then one has a diagram :

$$0 \to \pi_1(L_{z_0} - \Delta) \xrightarrow{j} \pi_1(U - \Delta \cup \tilde\Gamma) \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \pi_1(V - \Gamma) \to 0$$
$$\downarrow \alpha$$
$$\pi_1(U - \Delta)$$

where the base point is always p_{z_0} , j and α are induced by inclusions, β by φ and γ by the map $z \mapsto (w_0, z)$.

Remark that $\varphi : U - \Delta \cup \tilde T \to V - T$ is a fibre bundle with fibre $L_{z_0} - \Delta$ and that $z \to (w_0, z)$ induces a cross section of such a bundle. So from the homotopy sequence of a fibre bundle we get

1) the horizontal line is exact and $\beta \circ \gamma$ is the identity on $\pi_1(V - \Gamma)$.

Moreover one has obviously

2) α is surjective

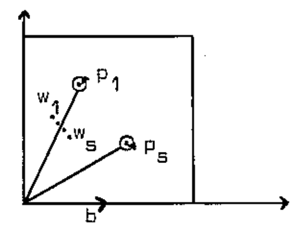
3) $\alpha \circ \gamma$ is the null homomorphism

Consider the sequence $0 \to \pi_1(V - \Gamma) \xrightarrow{\gamma} \pi_1(U - \Delta \cup \tilde\Gamma) \xrightarrow{\alpha} \pi_1(U - \Delta) \to 0$.

This must not be exact at $\pi_1(U - \Delta \cup \tilde\Gamma)$. Nevertheless one has

4) $\ker \alpha$ is generated by the conjugated of $\operatorname{Im} \gamma$.

Proof. – Obviously for $v \in \operatorname{Im} \gamma$ and $b \in \pi_1(U - \Delta \cup \tilde\Gamma)$ one has $b^{-1}vb \in \ker \alpha$. Let $b \in \ker \alpha$. Then $b = \partial c$ with $c \in \pi_2(U - \Delta , U - \Delta \cup \tilde\Gamma)$. Let us represent c by a map $\mathcal{C} : [0, 1] \times [0, 1] \to U - \Delta$ which is transversal to $\tilde\Gamma$. Then $\mathcal{C}^{-1}(\tilde\Gamma)$ is a finite set of points, let say $p_1, \ldots, p_s$. The following picture shows that b is equivalent in $\pi_1(U - \Delta \cup \tilde\Gamma)$ to a product $w_1 \ldots w_s$ of simple loops around $\tilde\Gamma$



w_i simple means that it is composed of an arc τ from p_{z_0} to a point near a regular point $\tilde p$ of $\tilde\Gamma$, a small circle around $\tilde\Gamma$ and then back with τ^{-1} . One can construct a cylinder in $U - \Delta \cup \tilde\Gamma$ whose boundaries are two circles, one being that of w_i , the other being in $V - \Gamma$. Choose an arc $\tilde\tau$ in $V - \Gamma$ from p_{z_0} to that circle, and call v_i the resulting simple loop ; if α_i is defined so that it follows τ and then a path along the cylinder from one circle to the other and then τ^{-1}, one realizes that $w_i = \alpha_i v_i \alpha_i^{-1}$. So b is a product of elements, each of which conjugated to an element of $\operatorname{Im} \gamma$.

COROLLARY. – $\ker \alpha \cap \ker \beta$ is the minimal normal subgroup N of $\ker \beta$ that contains the elements of the form $bvb^{-1}v^{-1}$ with $b \in \ker \beta$, $v \in \operatorname{Im} \gamma$.

Proof. – Let $b \in \pi_1(U - \Delta \cup \tilde\Gamma)$, $v \in \operatorname{Im} \gamma$.

Then $bvb^{-1} = (b.\gamma\beta(b^{-1})).(\gamma\beta(b).v.\gamma\beta(b^{-1})).(\gamma\beta(b).b^{-1}) = \bar b.\bar v.\bar b^{-1}$

where $\bar v \in \operatorname{Im} \gamma$, $\bar b \in \ker \beta$. So if $b \in \ker \alpha$, one has from 4) and this remark that $\bar{b} = b_1 v_1 b_1^{-1} \dots b_s v_s b_s^{-1}$ with $b_i \in \ker \beta$, $v_i \in \operatorname{Im} \gamma$. Moreover $b \in \ker \beta$ implies $\beta(v_1 \dots v_s) = 1$ and hence $v_1 \dots v_s = 1$, since β is injective on $\operatorname{Im} \gamma$. Let $n_i = b_i v_i b_i^{-1} v_i^{-1} \in N$; then $b = n_1 v_1 \dots n_s v_s = v_s^{-1} \dots v_1^{-1} n_1 v_1 \dots n_s v_s$. From this and the remark that $v \in \operatorname{Im} \gamma$, $n \in N$ implies that $v^{-1} n v \in N$ one deduces $b \in N$.

Let R be a straight line in V s. th. $\pi_1(R - \Gamma) \to \pi_1(V - \Gamma)$ is surjective and let H denote $\varphi^{-1}(R)$. From the preceding result we know that $\pi_1(H - \Delta)$ and $\pi_1(U - \Delta)$ have the same generators with the same relations, hence they are isomorphic. This is the local version of a theorem of Zariski – Van Kampen on the presentation of the fondamental group of $\mathbb{P}_n$ minus a hypersurface ; compare the article of Cheniot [2] in this volume. Now $H = \mathbb{C} \times E$ where $E = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ and $\Delta_1 = \Delta \cup H$ is defined by some equation $w^m + b_1(z) w^{m-1} + \dots + b_m(z) = 0$, with the b_i holomorphic on E. Suppose that π is versal. Then (see [1], [6]) one may suppose that Δ_1 has the following properties :

i) it is irreducible

ii) it has only cusps or ordinary double points as singularities, with distinct images on E.

iii) it is flat on the z-direction, i.e. z is a transversal parameter at any point of Δ_1.

Let $\sigma : \tilde{\Delta}_1 \to \Delta_1$ be the normalization of Δ_1 and consider $\tau = \varphi_o \sigma : \tilde{\Delta}_1 \to E$. Then τ is only ramified at points that correspond to cusps of Δ_1, and the ramification index at those points is two.

LEMMA. – The permutation group of the Riemann surface Δ_1 is the full group of permutations of r elements.

Proof. – It is sufficient to remark that a) it is transitive (because of the irreducibility of Δ_1) and b) it is generated by transpositions (because Δ_1 is simply ramified over distinct points).

Let γ be a simple loop in $L_{z_o} - \Delta$ from z_o turning positively around some element of $L_{z_o} \cap \Delta$. Its image in $\pi_1(U - \Delta)$ will be called a geometric generator.

THEOREM 1. – i) Let γ_1, γ_2 be geometric generators ; there exists $\delta \in \pi_1(U - \Delta)$ s. th. $\gamma_1 \delta = \delta . \gamma_2$.

ii) Suppose that Δ is not smooth, and let γ be a geometric generator ; there exists a geometric generator γ' s. th. $\gamma . \gamma' . \gamma = \gamma' . \gamma . \gamma'$.

Proof. – i) Remark first that any two loops around the same z_i are conjugated in $\pi_1(L_{z_o} - \Delta)$ and hence also in $\pi_1(U - \Delta)$. Then the preceding lemma assures that if $z_i, z_j \in L_{z_o} \cap \Delta$ and γ is a loop around z_i, then γ is equivalent in $\pi_1(U - \Delta)$ to some loop around z_j.

ii) Since Δ is not smooth, Δ_1 must be ramified somewhere so that Δ_1 has at least a cusp. Suppose that z_o is near that ramification point and that γ is a loop around a point near the corresponding cusp. Then the loop γ' that goes like γ until that cusp and then links the other point near the cusp, satisfies the required relation. In general one knows from i) that γ is conjugated to such an element. Then it remains only to remark that each conjugated to a geometric generator is a geometric generator.

3 . PICARD-LEFSCHETZ THEORY . – This theory describes the monodromy of a semiuniversal deformation in the following way (see [5]) : the fibre over each simple point of Δ has just an isolated singularity of the type $\sum_0^r x_i^2 = 0$; hence to each geometric generator γ, there is associated a vanishing cycle $e \in H_r(M, \mathbb{Z})$ uniquely determined up to the sign by γ. The action of γ on $H_r(M, \mathbb{Z})$ is given by the Picard-Lefschetz formula

$$(*) \qquad h \to h + (-1)^s \langle e, h \rangle e \quad , \quad h \in H_r(M, \mathbb{Z})$$

where $s = (r+1).(r+2)/2$ and $\langle \, , \, \rangle : H_r(M, \mathbb{Z}) \times H_r(M, \mathbb{Z}) \to \mathbb{Z}$ is the cap product. Moreover $\langle e, e \rangle$ is zero if r is odd and $2.(-1)^{r.(r+1)/2}$ if r is even.

Let $\{z_1, \dots, z_m\} = L_{z_0} \cap \Delta$ and choose simple loops $\gamma_1, \dots, \gamma_m$ in such a way that γ_i turns positively around z_i and γ_i, γ_j don't intersect outside z_0. Call e_i the vanishing cycle associated to γ_i. Then $\gamma_1, \dots, \gamma_m$ generate freely $\pi_1(L_{z_0} - \Delta)$ and $e_1, \dots, e_m$ are a base of $H_r(M, \mathbb{Z})$ over $\mathbb{Z}$.

THEOREM 2. – Let I, J be a partition of $\{1, \dots, m\}$. There exist $i \in I$ and $j \in J$ s. th. $\langle e_i, e_j \rangle \neq 0$.

Proof. – From formula $(*)$ one gets that the images $\bar{\gamma}_i$, $\bar{\gamma}_j$ of γ_i, γ_j in $\mathrm{Aut}\, H_r(M, \mathbb{Z})$ commute if $\langle e_i, e_j \rangle = 0$. Suppose that this happens for all $i \in I$ and $j \in J$. Fix $i_1 \in I$, $j_1 \in J$; because of theorem 1 one can write $\gamma_{i_1} \delta = \delta . \gamma_{j_1}$ where $\delta \in \pi_1(U - \Delta)$ and hence δ is a product of γ_i, γ_j. Since each $\bar{\gamma}_i$ commutes with each $\bar{\gamma}_j$ one can write $\bar{\delta} = \bar{\delta}_J . \bar{\delta}_I$ where $\bar{\delta}_I$ is a product of $\bar{\gamma}_i$ and $\bar{\delta}_J$ a product of $\bar{\gamma}_j$. So one has $\bar{\delta}_J^{-1} . \bar{\gamma}_{i_1} . \bar{\delta}_J = \bar{\delta}_I . \bar{\gamma}_{j_1} . \bar{\delta}_I^{-1}$ and hence $\gamma_{i_1} = \gamma_{j_1}$. This equality, with the help of formula $(*)$ and the fact that $e_{i_1} \neq e_{j_1}$, gives $\langle e_{i_1}, h \rangle = 0$ for all $h \in H_r(M, \mathbb{Z})$. This cannot happen. In fact theorem 1 says that there exists a geometric generator γ' such that $\gamma_{i_1} . \gamma' . \gamma_{i_1} = \gamma' . \gamma_{i_1} . \gamma'$; if e' denotes the vanishing cycle associated to γ' one sees from formula $(*)$ that this relation is equivalent to $\langle\ ', \ _{i_1} \rangle = \pm 1$ and this concludes the proof.

COROLLARY. – The set of points where Δ is locally reducible is contained in the set of points where Δ has smaller multiplicity than at the origin.

Proof. – Let $t' \in \Delta$ where Δ has irreducible components $\Delta_1, \dots, \Delta_s$, $s \geq 2$. Then $\pi^{-1}(t')$ has s singular points $x_1, \dots, x_s$ s.th. the multiplicity m_i of Δ_i at t' is the number of vanishing cycles at x_i, $i = 1, \dots, s$. Choose L_{z_0} near t' and define $\{z_1, \dots, z_m\} = \Delta \cap L_{z_0}$, $I = \{1, \dots, m\}$, $I_\alpha = \{i \in I \mid z_i \in \Delta_\alpha\}$, $\alpha = 1, \dots, s$. Suppose $\sum_1^s m_i = m$; it follows that $(I_\alpha)_\alpha$ is a partition of I. Let e_i denote the vanishing cycle at z_i. One has $\langle e_i, e_j \rangle = 0$ for $i \in I_\alpha$, $j \in I_\beta$ and $\alpha \neq \beta$; in fact

e_i, e_j have representative cycles lying in disjoint balls around x_α, x_β respectively. This cannot happen because of the theorem 2, so that the multiplicity $\sum_1^s m_\alpha$ of Δ at t' must be less than m.

Remark. – This result can be expressed in the following way : if a deformed fibre of a hypersurface isolated singularity has more than one singularity, then the direct sum module of vanishing cycles at those singularities is a proper submodule of that of vanishing cycles at the original singularity.

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