A NOTE ABOUT PLURIGENERA

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First, given a Cohen-Macaulay isolated quasi homogeneous singularity, I describe the process of the resolution of singularities based on Demazure's work and generalize the work of Orlik and Wagreich in dimension two.

After using this resolution of singularities I calculate explicitly the plurigenera for a Gorenstein quasi-homogeneous isolated singularity in terms of an invariant, called the index of regularity of the Hilbert function.

In the second part, similar calculations are given for plurigenera of singularities of complete intersections which are generic in the sense of Newton polyhedra. Proofs need the explicit process of the resolution of singularities developed by the author in [M2]. In this note I present only results, proofs will appear in forthcoming paper.

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1. GRADED RINGS, HILBERT'S FUNCTION (cf. [M1]).

Let $A = \bigoplus_{m\in\mathbb{Z}} A_m$ be a graded ring such that $A_m = 0$ for $m < 0$, $A_0 = k$ a field and $A_i \cdot A_j \subset A_{i+j}$ for all i, j. We work in the category of A-graded modules of finite type. Let M, N be two modules in this category, $\mathrm{Hom}_A(M,N)$ denotes the graded module of A-morphisms; $\mathrm{Ext}^i_A(M,N)$, the left-derived functors of $\mathrm{Hom}_A(M,N)$, are graded. We denote by $M[m]$ the shift of the module M, i. e. $M[m]_j = M_{m+j}$ for all $j \in \mathbb{Z}$.

Let $B = k[X_1,\dots,X_n]$ be the graded ring of polynomials with the graduation given by $\deg X_i = e_i \geq 1$, $i = 1,\dots,n$.

In the following we will work in the category of B-graded modules of finite type.

<u>Definition.</u>

<u>The Hilbert function:</u> $H(M,m) = \dim_k M_m$

<u>The Poincare Series:</u> $F(M,\lambda) = \sum_{m\in\mathbb{Z}} H(M,m)\lambda^m$.

<u>Lemma.</u>

1) There are h polynomials $Q_0,\dots,Q_{h-1} \in \mathbb{Q}[X]$ with rational coefficients such that

$$H(M,mh+i) = Q_i(m)$$

for all $i = 0,\dots,h-1$ and $m >> 0$.

2) There are s polynomials $S_1,\dots,S_s \in \mathbb{Q}[X]$ with rational coefficients and $\alpha_1,\dots,\alpha_s$ s-roots of unity such that

$$H(M,m) = \sum_{i=1}^{s} S_i(m)\alpha_i^m .$$

The proof of this lemma is first obtained for $M = k[X_1,\dots,X_n]$ and then we use Hilbert's syzygies theorem.

<u>Definition.</u>

The index of regularity is the integer number $a(M)$ such that:

$H(M,mh+i) = Q_i(m)$, $\forall m,i$ such that $mh+i > a(M)$,

but $H(M,m_oh+i_o) \neq Q_{i_o}(m_o)$ for $m_oh+i_o = a(M)$.

<u>Proposition.</u>

1) $a(M)$ does not depend on the polynomials Q_i which appear in the lemma. In fact, $a(M)$ is the degree of the rational fraction $F(M,\lambda)$.

2) $a(M)$ is the minimum of the integer numbers such that

$$H(M,m) = \sum_{i=1}^{s} S_i(m)\alpha_i^m \quad \text{for all } m > a(M).$$

3) If $E^i = \mathrm{Ext}_B^i(M, B[-e_1-e_2 \ldots -e_n])$, then:

$$a(M) = -\min\{m \,/\, \text{the coefficient of } \lambda^m \text{ in } \sum_{i=0}^{n}(-1)^i F(E^i,\lambda) \text{ is different from zero}\}.$$

<u>Corollary.</u>

If M is a Cohen Macaulay B-module of dimension d, we note by K_M the canonical module $K_M := \mathrm{Ext}_B^{n-d}(M, B[-e_1 \ldots -e_n])$. In this case we have $a(M) = -\min\{m \,/\, (K_M)_m \neq 0\}$.

<u>Remark.</u> The index of regularity was studied in the Cohen Macaulay homogeneous case by Lazard, Schenzel. Goto-Watanabe too define $a(M)$ by the formula obtained in the corollary.

<u>Lemma.</u>

If $f \in B$ is a homogeneous regular element for M (i. e. a non-zero divisor in M), then

$$a(M/fM) = a(M) + \deg f.$$

In particular, if $f_1, \ldots, f_k$ is a regular sequence in B, then

$$a(B/(f_1,\ldots,f_k)) = \sum_{i=1}^{k} \deg f_i - \sum_{i=1}^{n} \deg X_i .$$

<u>Example.</u> Let $B = k[X]$ with $\deg X = e > 1$, then the Hilbert function of B is given by

$$H(m) = \begin{cases} 1 & \text{if } m \in e\mathbb{N} \\ 0 & \text{otherwise} \end{cases},$$

where the polynomials Q_i are $Q = 1$ or $Q_i = 0$, the Hilbert function coincides with one of these polynomials for $n > -e$.

2. NORMAL GRADED RINGS

Let Z be a normal variety. By $W\,\mathrm{div}(Z,\mathbb{Z})$ we denote the free group generated by subvarieties of Z of codimension one and $W\,\mathrm{div}(Z,\mathbb{Q}) = W\,\mathrm{div}(Z,\mathbb{Z}) \otimes_{\mathbb{Z}} Q$. For $D = \sum r_i W_i$ we set $\lfloor D \rfloor = \sum [r_i] W_i$, where $[r_i]$ is the integer part of r_i. We associate to D the sheaf $\mathcal{O}_Z(D)$ given by

$$\mathcal{O}_Z(D)\big|_U = \{ f \in K(Z) / \mathrm{div}(f) + D\big|_U \geqslant 0 \} .$$

$\mathcal{O}_Z(D)$ is a reflexive rank-one sheaf. We have $\mathcal{O}_Z(D) = \mathcal{O}_Z(\lfloor D \rfloor)$ and $\mathrm{Hom}(\mathcal{O}_Z(D), \mathcal{O}_X) = \mathcal{O}_X(-\lfloor D \rfloor)$.

Now we can give Demazure's theorem ([D]):

Theorem.

Let $A = \oplus A_n$ be a normal graded algebra of finite type over k, T a homogeneous element of degree 1 in $\mathrm{Fr}(A)$. If we consider the k-normal scheme $X = \mathrm{Proj}(A)$, then there exists one and only one divisor $D \in W\,\mathrm{div}(X,\mathbb{Q})$ such that

$$A_n = H^0(X, \mathcal{O}_X(nD))T^n \quad \text{for} \quad n \geqslant 0$$

in $\mathrm{Frac}(A)$ and

$$\mathcal{O}(n) = \mathcal{O}_X(nD)T^n, \quad \forall n \in \mathbb{Z} .$$

Demazure also gives a modification of the singularity of the cone $\mathrm{Spec}(A)$ in the vertex $\{m\}$, where m is the maximal graded ideal of A. Let

$$C^+ = \mathrm{Spec}(\bigoplus_{n \geqslant 0} \mathcal{O}_X(nD)T^n)$$

$$U(X,D) = \mathrm{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(nD)T^n)$$

and

$$S = C^+ - U(X,D) .$$

Theorem ([D]).

C^+ is normal and we have the following Cartesian diagram with ψ birational.

$$\begin{array}{ccccc} & & U(X,D) & \xrightarrow{\sim} & U = Y-\{m\} \\ & \swarrow^{\pi} & \downarrow & & \downarrow \\ X & \xleftarrow{\pi} & C^+ & \xrightarrow{\psi} & Y = \mathrm{Spec}(A) \\ & \nwarrow^{\pi}_{\sim} & \uparrow & & \uparrow \\ & & S & \longrightarrow & \{m\} \end{array}$$

X is the geometric quotient of C^+ (resp. $U(X,D)$) by the $\mathbb{C}^*$-action, the map π induces an isomorphism between S and X.

Theorem.

Suppose $k = \mathbb{C}$ is the complex field and $Y = \mathrm{Spec}(A)$ has a Cohen-Macaulay isolated singularity in the vertex of the cone Y. Then C^+ and X have only cyclic quotient singularities.

I thank F. Knop, who writes a proof of this theorem for me.

3. PLURIGENERA

In the following let $k = \mathbb{C}$. Let X be a normal variety, $\dim X = n$. We consider X with the usual topology. Let X_{reg} be the set of regular points of X, $\Omega_{X_{reg}}$ the sheaf of n-holomorphic forms on X_{reg} and put $\Omega_X^{[m]} = i_*(\Omega_{X_{reg}}^{\otimes m})$, where i is the inclusion $X_{reg} \hookrightarrow X$. $\Omega_X^{[m]}$ is a reflexive rank-one sheaf.

Let $U \subset X$ be an open, relatively compact set. We define the sheaf of m-uple n-forms which are locally 2/m-integrable by $\omega \in L^{2/m}(U, \Omega_X^{[m]}) \subset \Omega_X^{[m]}(U)$ if $\int_{U-X_{sing}} (\omega \wedge \bar{\omega})^{1/m} < \infty$, where X_{sing} denotes the singular locus of X; $(\omega \wedge \bar{\omega})^{1/m}$ is defined in local coordinates $(z_1,\dots,z_n)$: if $\omega = \Phi(z)(dz_1 \wedge \dots \wedge dz_n)^m$, then

$$(\omega \wedge \bar{\omega})^{1/m} = |\Phi(z)|^{2/m}\left(\frac{i}{2\pi}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$$

and the integral means $\lim\limits_{U'} \int\limits_{U-U'} (\omega \wedge \bar{\omega})^{1/m}$, where U' varies in the neighbourhoods of $U \cap \mathrm{Sing}(X)$. In order to calculate plurigenera we give the following results:

Proposition ([S]).

Let $\pi: \tilde{X} \longrightarrow X$ be a normal crossing resolution of singularities. If E denotes the reduced exceptional divisor, then:

$$L^{2/m}(U, \Omega_X^{[m]}) = \pi_* \mathcal{O}(mK_{\tilde{X}}+(m-1)E)(U) .$$

This is a generalization of Picard's lemma.

Lemma. (Burns, Kimio Watanabe)

Suppose that X has only quotient singularities, i. e. locally X is the quotient of an open ball $B(0, \varepsilon) \subset \mathbb{C}^n$ by a finite group of linear unitary transformations, where no element fixes a hyperplane in $\mathbb{C}^n$, then $\forall U$:

$$L^{2/m}(U, \Omega^{[m]}) = \Omega^{[m]}(U) .$$

Definition.

If $x \in X$ is an isolated singularity in X, the plurigenera δ_m are the integer number

$$\delta_m = \dim_{\mathbb{C}} \Omega^{[m]}(U)/L^{2/m}(U, \Omega^{[m]})$$

for a small neighbourhood U of x.

Now we can give the following

Theorem.

Let $(Y = \mathrm{Spec}\ A, m)$ be a quasi-homogeneous Gorenstein isolated singularity, then

$$\gamma_m = \dim_{\mathbb{C}}\left(\bigoplus_{i \leq ma(A)} A_i \right),$$

where $a(A)$ is the index of regularity of the Hilbert function of A.

<u>Corollary.</u>

Let $A = k[X_1,\dots,X_n]/(f_1,\dots,f_k)$ be a quasi-homogeneous complete intersection isolated singularity, then the plurigenera γ_m can be calculated explicitly by using the Koszul graded complex

$$0 \to B[-\deg f_1 - \dots - \deg f_k] \to \dots \to \bigoplus_{i=1}^{k} B[-\deg f_i] \to B \to A \to 0 .$$

4. NEWTON'S GENERIC COMPLETE INTERSECTION

<u>Definition.</u>

Let $f \in \mathbb{C}[X_1,\dots,X_n]$, $f(X) = \sum c_\alpha X^\alpha$, we define

a) the Newton polyhedron of f: $\Gamma^+(f) =$ Convex hull in $\mathbb{R}^n$ of $\bigcup_{c_\alpha \neq 0} \{\alpha\} + \mathbb{R}^n_+$;

b) the Newton boundary of f: $\Gamma(f)$ the union of compact faces of $\Gamma^+(f)$;

c) we say f is "commode" if $\forall i \in [1\dots n]$ there exists an m_i such that $c_{m_i} X_i^{m_i}$ appears in f.

<u>Definition.</u>

Let $f_\ell \in \mathbb{C}[X_1,\dots,X_n]$, $\ell = 1,\dots,k$ be a sequence of "commode" polynomials, Γ_ℓ^+ their Newton polyhedra, for $a \in (\mathbb{R}_+)^d$ we

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$$m^\ell(a) = \{\min \langle a,\alpha\rangle / \alpha \in \Gamma_\ell^+\}$$

and

$$f_\ell^a = \sum_{\substack{\alpha \in \Gamma_\ell^+ \\ \langle a,\alpha\rangle = m^\ell(a)}} A_{\ell,\alpha} X^\alpha .$$

The sequence $\{f_1,\dots,f_k\}$ is said to be non-degenerate (with respect to the Newton polyhedron) if for all $1 \le j \le k$ and all $a \in (\mathbb{R}_+ - \{0\})^n$ the following condition is satisfied:
In each point $q \in (\mathbb{C} - \{0\})^n$ such that

$$f_1^a(q) = \dots = f_j^a(q) = 0$$

the differentials $df_1^a(q),\dots,df_j^a(q)$ are linearly independent in the tangent space of $\mathbb{C}^n$ in q.
In this case we also say that the germ H_k defined near the origin by

$$H_k = \{x \in \mathbb{C}^n / f_1(x) = \dots = f_k(x) = 0\}$$

is non-degenerate with respect to $\Gamma_1^+,\dots,\Gamma_k^+$.

<u>Remark.</u> 1) This is a generic notion in the sense of the Zariski topology;
2) We have only a finite number of conditions in $a \in (\mathbb{R}_+ - \{0\})^n$. In fact, the set $\alpha \in \Gamma_\ell^+ / \langle a,\alpha\rangle = m^\ell(a)$ will have only one element in general and a monomial is never zero for $q \in (\mathbb{C} - \{0\})^n$.

<u>Theorem.</u>

Let $f_1,\dots,f_k \in \mathbb{C}[X_1,\dots,X_n]$ be a non-degenerate sequence (with respect to their Newton polyhedra), let $H_k = \{x \in \mathbb{C}^k / f_1(x) = \dots = f_k(x) = 0\}$ be the isolated singularity defined near the origin by $f_1,\dots,f_k$, then the plurigenera are:

$$\mathcal{g}_m(H_k) = R(m(\Gamma_1^+ + \dots + \Gamma_k^+)) - \sum_{i=1}^{k} R(m(\Gamma_1^+ + \dots + \hat{\Gamma}_i + \dots + \Gamma_k)) + \dots + (-1)^{k-1} \sum_{i=1}^{k} R(m\,\Gamma_i^+)$$

where $R(\Delta)$ denotes the number of integer points with all coordinates being strictly positive and not in the interior of Δ, $\hat{\Gamma}_i^+$ means that the term Γ_i^+ is not contained in the sum.

The proof of this theorem requires the theorem on the resolution of singularities [M2].

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