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On equimultiplicity of the normal cone

INTRODUCTION. The starting point for this short note is a result on families of varieties due to Teissier in the complex-analytic case ([10]) and to Lipman in the algebraic case ([4]). Let the family be given by a morphism $f : X \longrightarrow Y$ and assume that X contains a subvariety Z on which f induces a finite projection. For any $y \in Y$ one can define a multiplicity $e(y)$ of the fibre X_y at $X_y \cap Z$, and the above mentioned result is concerned with the local constancy of the number $e(y)$. Namely, if X is locally given by a local ring R and Z by an ideal I of R, and if f is flat, then $e(y)$ is locally constant if and only if $\mathrm{ht}(I) = \ell(I)$, where I denotes the analytic spread of I (see [3] for definition).

Now the condition $\mathrm{ht}(I) = \ell(I)$ given by Lipman and Teissier does not involve the base space Y of the family, and therefore the question arises: Given just $Z \subset X$, is there a "natural" construction of a family $f : X \longrightarrow Y$ such that $f|Z$ is finite? Since there is no obvious such construction, we replace X by the normal cone $C_Z(X)$ which approximates X near to Z. Then we may take $Y = Z$ since we have natural morphisms $Z \hookrightarrow C_Z(X) \twoheadrightarrow Z$. By applying some techniques which we have developed in earlier Papers, we will derive some results on the equimultiplicity of the fibres of the morphism $C_Z(X) \longrightarrow Z$ along $Z \subset C_Z(X)$.

We will now introduce some notation in order to state the results. This notation (together with the assumptions stated below) will be kept fixed for the rest of the paper.

We fix a quasi-unmixed local ring R with maximal ideal $\mathfrak{m}$ and with an infinite residue field. P will denote a prime ideal of R such that R/P is regular. We put

$X = \operatorname{Spec} R$, $Z = \operatorname{Spec} R/P$,

$C = C_Z(X) = \operatorname{Spec} gr_P(R)$

$Bl_P(R) = \operatorname{Proj}(\bigoplus_{n\geq 0} P^n)$ = the blowing up of X with center Z

$E = \operatorname{Proj}(gr_P(R)) \subset Bl_P(R)$ the exceptional divisor

C_0 = the fibre of $C \longrightarrow Z$ at $\mathfrak{m}/P = \operatorname{Spec}(gr_P(R) \otimes_R R/\mathfrak{m})$

E_0 = the fibre of $E \longrightarrow Z$ at $\mathfrak{m}/P = \operatorname{Proj}(gr_P(R) \otimes_R R/\mathfrak{m})$.

We will assume that $\operatorname{codim}_X(Z) \geq 1$. We note that the fibres of $C \longrightarrow Z$ are cones, and the intersection of a fibre with $Z \subset C$ is the vertex of that cone. Therefore we will call $C \longrightarrow Z$ an equimultiple family of vertices if the fibres are equimultiple along Z . Now the main results can be stated in the following way:

PROPOSITION 1. If X is equimultiple along Z , then the following conditions are equivalent:

(i) $C \longrightarrow Z$ is an equimultiple family of vertices.

(ii) E is generically Cohen-Macaulay along E_0 .

One consequence of Proposition 1 will be that if $E \longrightarrow Z$ is flat (and in particular if X is normally flat along Z) then $C \longrightarrow Y$ is an equimultiple family of vertices (see Corollary 1).

PROPOSITION 2. Assume that

1) X is equimultiple along Z ,

2) $C \longrightarrow Z$ is an equimultiple family of vertices,

3) C_0 has no embedded flat components.

Then X is normally flat along Z .

We point out that without the assumption of X being equimultiple along Z not much can be said about $C \longrightarrow Z$ (compare example 2 below). This corresponds to the fact that this equimultiplicity is essential in order to get some control over the ring $gr_P(R)$, as one also can see from the results of [1].

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for suggesting the problem.

1. We start by recalling some algebraic results which will be needed in the proofs. From Ratliff's papers on quasi-unmixed rings one can deduce the following facts:

(1.1) From R being quasi-unmixed it follows that any localisation of $gr_P(R)$ is quasi-unmixed again. Let R_1 be the local ring of a point of $Bl_P(R)$. Then R_1 is quasi-unmixed, and moreover if R_1 corresponds to a closed point of $Bl_P(R)$ then $\dim R_1 = \dim R$.

The condition that X is equmultiple along Z means that $e(R) = e(R_P)$, and in [2] we have shown:

(1.2) If $e(R) = e(R_P)$ then $ht(P) = \ell(P)$, where $\ell(P) = \dim C_\circ$.

Now since R/P is regular by assumption, we can choose $\underline{x} = (x_1,\ldots,x_r)$ such that $r = \dim R/P$ and $P + \underline{x}R = \mathfrak{m}$. Assume $e(R) = e(R_P)$ and let R_1 be a local ring of $Bl_P(R)$ corresponding to a closed point above $\mathfrak{m}$. Then (1.1) and (1.2) imply

$$\dim(R_1/\mathfrak{m}R_1) = \ell(P)-1 = ht(P)-1 = \dim R-(r+1) = \dim R_1-(r+1) .$$

Even if R_1 corresponds to a non-closed point above $\mathfrak{m}$, then by localising and using the fact that the rings of $Bl_P(R)$ are quasi-unmixed, we still have $\dim(R_1/\mathfrak{m}R_1) = \dim R_1-(r+1)$. Therefore we deduce:

(1.3) If R_1 is a local ring of $Bl_P(R)$ such that $\mathfrak{m}R_1 \neq R_1$ and if $e(R) = e(R_P)$, then $ht(\mathfrak{m}R_1) = r+1$ and $x_1,\ldots,x_r$ are part of a system of parameters of R_1/PR_1.

From [6] we take the following result:

(1.4) Let $\underline{y} = (\underline{y}_1,\ldots,\underline{y}_s)$ be part of a system of parameters of R. Then $e(R) = e(R/\underline{y}R)$ if and only if $\underline{y}$ can be extended to a minimal reduction of $\mathfrak{m}$ and R_Q is Cohen-Macaulay for every minimal prime ideal Q of $\underline{y}R$.

Now let us write $\mathfrak{m} = P + \underline{x}R$ again, where $\underline{x} = (x_1, \ldots, x_r)$ and $r = \dim R/P$. Then there is a canonical epimorphism of graded rings

$$\varphi : gr_P(R) \otimes_R R/\mathfrak{m}[X_1, \ldots, X_r] \longrightarrow gr_{\mathfrak{m}}(R)$$

which is induced by the inclusion $P \subset \mathfrak{m}$ on $gr_P(R) \otimes_R R/\mathfrak{m}$ and which maps X_i to $in_{\mathfrak{m}}\, x_i$. In [4] one can find the following result, originally due to Schickhoff:

(1.5) $e(R) = e(R_P)$ if and only if φ has a nilpotent kernel.

For any graded ring $A = \underset{n\geq 0}{\oplus} A_n$ for which $A_\circ$ is local, we will use the notation A^V for the localisation of A at its unique maximal homogeneous ideal.

In $gr_P(R)^V$ consider the ideal P^* generated by $\underset{n\geq 1}{\oplus} P^n/P^{n+1}$. Then from $\dim gr_P(R)^V = \dim R$ and $gr_P(R)^V/P^* \simeq R/P$ we see that $ht(P^*) = ht(P)$. Since the associated graded ring of $gr_P(R)^V$ with respect to P^* is simply $gr_P(R)$ itself, it is also clear that $\ell(P^*) = \ell(P)$ and therefore:

(1.6) $ht(P) = \ell(P) \Longleftrightarrow ht(P^*) = \ell(P^*)$ (compare [5]).

Let R_1 be a local ring of $B\ell_P(R)$ such that $PR_1 \neq R_1$. Then there is a unique homogeneous prime ideal Q of $gr_P(R)$ such that $R_1/PR_1 \simeq gr_P(R)_{(Q)}$. Furthermore there is some $t \in P/P^2$ which is not contained in Q, and for any such t there is an isomorphism

$$gr_P(R) \simeq gr_P(R)_{(Q)}[T]_{N[T]},$$

mapping t to T, where N denotes the maximal ideal of $gr_P(R)_{(Q)}$. From this isomorphism we deduce:

(1.7) $gr_P(R)_Q$ is Cohen-Macaulay if and only if $gr_P(R)_{(Q)}$ is Cohen-Macaulay.

2. We will now turn to the proof of Proposition 1, and to avoid heavy notation we put $F_P(R) = gr_P(R) \otimes_R R/\mathfrak{m}$. The symbol $e(-)$ will denote the multiplicity of a local ring with respect to the maximal ideal.

PROOF of Proposition 1. Since $e(R) = e(R_P)$ by assumption, we know from (1.2) that $ht(P) = \ell(P)$, which means that all the fibres of $C \longrightarrow Z$ have the same dimension. By semicontinuity of the multiplicity of the fibre (see [4], Proposition (3.1)), we see that (i) is equivalent to

$$\text{(i')} \qquad e((gr_P(R) \otimes_R R_P)^V) = e(F_P(R)^V) .$$

Choosing $\underline{x} = (x_1,\dots,x_r)$ such that $r = \dim R/P$ and $P + \underline{x}R = \mathfrak{m}$, we have

$$F_P(R)^V \cong gr_P(R)^V/\underline{x}^* \, gr_P(R)^V,$$

where $x_i^* \in R/P$ is the initial form of x_i with respect to P and $\underline{x}^* = (x_1^*,\dots,x_r^*)$. By (1.6) we know that $ht(P^*) = \ell(P^*) = \dim gr_P(R)^V - r$. Therefore if we choose $\underline{y}^* = (y_1^*,\dots,y_s^*)$ in $gr_P(R)^V$ generating a minimal reduction of P^*, then $\underline{x}^*,\underline{y}^*$ generate a minimal reduction of the maximal ideal $P^* + \underline{x}^* gr_P(R)^V$ of $gr_P(R)^V$. Using (1.4) (applied to $gr_P(R)^V$) we see that

$$(1) \qquad e(gr_P(R)^V) = e(F_P(R)^V) \Longleftrightarrow \begin{cases} \text{for any minimal prime ideal } Q \text{ of} \\ \underline{x}^* gr_P(R) \text{, the local ring } gr_P(R)_Q \\ \text{is Cohen-Macaulay.} \end{cases}$$

Now using the isomorphism $(gr_P(R) \otimes_R R_P)^V \cong gr_P(R)^V_{P^*}$ and $ht(P^*) = \ell(P^*)$ again we obtain

$$(2) \qquad e((gr_P(R) \otimes_R R_P)^V) = e(gr_P(R)^V_{P^*}) = e(gr_P(R)^V)$$

by (1.2). Combining (1) and (2) we conclude that (i) and (i') are equivalent to (i") For any minimal prime ideal Q of $\underline{x}^*gr_P(R)$, the local ring $gr_P(R)_Q$ is Cohen-Macanlay.

Since we assumed $\operatorname{codim}_X(Z) = ht(P) \geq 1$, the minimal primes Q of $\underline{x}^*gr_P(R)$ are in one-to-one correspondence with the generic points of the subscheme $E_\circ$ of E . Now applying (1.7) we finally get that (i") is equvalent to (ii).

COROLLARY 1. If $E \longrightarrow Z$ is flat then $C \longrightarrow Z$ is an equimultiple family of vertices.

PROOF. Flatness of $E \longrightarrow Z$, which has been called projective normal flatness in [7], implies $e(R) = e(R_P)$ (see [3], Thm. 1). and we can apply Proposition 1. Condition (ii) there is certainly equvalent to the following one: If R_1 is a local ring of $B\ell_P(R)$ such that the corresponding homomorphism $R \longrightarrow R_1$ is local, and if Q is any minimal prime ideal of $\mathfrak{m}\cdot R_1$, then $(R_1)_Q/P\cdot(R_1)_Q$ is Cohen-Macaulay. Choosing $\underline{x} = (x_1,\dots,x_r)$ as above, we know from [7], Cor. (1.7), that $\underline{x}$ is a regular sequence on R_1/PR_1 , which proves the assertion.

COROLLARY 2. Let $r = \dim Z$ and assume X equimultiple along Z . If $B\ell_P(R)$ satisfies the Serre condition S_{r+1} then $C \longrightarrow Z$ is an equimultiple family of vertices. This holds in particular if $B\ell_P(R)$ is Cohen-Macaulay.

PROOF. This follows directly from Proposition 1 together with (1.3).

For the proof of Proposition 2 we need the following

LEMMA. Let A be a quasi-unmixed local ring having no embedded components and let $\mathfrak{n} \subset A$ be a nilpotent ideal. If $e(A) = e(A/\mathfrak{n})$, then $\mathfrak{n} = (0)$.

PROOF. Put $\bar{A} = A/\mathfrak{n}$ and let $\pi : A \longrightarrow \bar{A}$ be the canonical homomorphism. By standard techniques we may assume that A has an infinite residue field, so we may choose a system of parameters

$y_1,\ldots,y_d$ of A such that

$$e(\underline{y}A, A) = e(A) \quad , \quad \underline{y} = (y_1,\ldots,y_d) \ .$$

Put $\pi(y_i) = \overline{y}_i$ etc. Then since $\underline{y}A$ is nothing but a minimal reduction of the maximal ideal of A, also $\underline{y}\cdot\overline{A}$ is a minimal reduction of the maximal ideal of $\overline{A}$, and consequently

$$e(\underline{y}\overline{A}, \overline{A}) = e(\overline{A}) \ .$$

By the associativity law for multiplicities we know that

$$e(\underline{y}A, A) = \sum_{\mathfrak{p}\in \mathrm{Assh}(0)} e(\underline{y};A/\mathfrak{p})\cdot \ell(A_{\mathfrak{p}}) \ ,$$

where $\mathrm{Assh}(0) = \{\mathfrak{p}\in \mathrm{Spec}\, A \mid \dim A/\mathfrak{p} = \dim A\}$, and similarly

$$e(\underline{y}\overline{A}, \overline{A}) = \sum_{\mathfrak{p}\in \mathrm{Assh}(0)} e(\underline{y};A/\mathfrak{p}) \quad (\overline{A}_{\overline{\mathfrak{p}}}) \ .$$

Since $\ell(\overline{A}_{\overline{\mathfrak{p}}}) \leq \ell(A_{\mathfrak{p}})$ and $e(\underline{y};A/\mathfrak{p}) \neq 0$ for all $\mathfrak{p}\in \mathrm{Assh}(0)$, we conclude from our assumption that

$$\ell(\overline{A}_{\overline{\mathfrak{p}}}) = \ell(A_{\mathfrak{p}}/\mathfrak{n}A_{\mathfrak{p}}) = \ell(A_{\mathfrak{p}}) \quad \text{for all} \quad \mathfrak{p}\in \mathrm{Assh}(0)$$

and therefore

$$\mathfrak{n}\cdot A_{\mathfrak{p}} = 0 \quad \text{for all} \quad \mathfrak{p}\in \mathrm{Assh}(0) \ .$$

Since A is quasi-unmixed and has no embedded components, we know that

$$\mathrm{Assh}(0) = \mathrm{Min}(0) = \mathrm{Ass}(A) \ .$$

So we have shown that $\pi\otimes A_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p}\in \mathrm{Ass}(A)$, and therefore π is an isomorphism.

<u>PROOF</u> of Proposition 2. We use the notation of (1.5). Since we assume $e(R) = e(R_P)$, we know that

$$\varphi : gr_P(R) \otimes_R R/\mathfrak{m}[X_1,\ldots X_r] \longrightarrow gr_{\mathfrak{m}}(R)$$

has a nilpotent kernel. Clearly

$$(3) \qquad e(gr_{\mathfrak{m}}(R)^V) = e(R)$$

and

$$(4) \qquad e((gr_P(R) \otimes_R R/\mathfrak{m}[X_1,\ldots X_r])^V) = e((gr_P(R) \otimes_R R/\mathfrak{m})^V)$$
$$= e((gr_P(R) \otimes_R R_P)^V) = e(R_P)$$

by assumption 2) and the isomorphism $gr_P(R) \otimes_R R_P \simeq gr_{PR_P}(R_P)$. The associated primes of $gr_P(R) \otimes_R R/\mathfrak{m}[X_1,\ldots X_r]$ are the extensions of the associated primes of $gr_P(R) \otimes_R R/\mathfrak{m}$. Since we assumed that this ring has no embedded components, the same holds for the polynomial ring above. Now from (3), (4) and the Lemma we conclude that φ is an isomorphism, and this is equivalent to R being normally flat along P ([9], Thm (1.8)).

REMARK. Combining Proposition 2 with Corollary 1 we see that if X is projectively normally flat along Z and if C_0 has no embedded components, then X is normally flat along Z . This conclusion was proved in [7] under the weaker assumption $\operatorname{depth}(C_0) > 0$ on C_0 .

COROLLARY 3. Assume that

1) X is equimultiple along Z ,
2) $B\ell_P(R)$ is Cohen-Macaulay,
3) $\operatorname{depth}(C_0) > 0$.

Then X is normally flat along Z .

PROOF. By Corollary 2, $C \longrightarrow Z$ is an equimultiple family of vertices, and therefore the result will follow from Proposition 2 if we can show that C_0 has no embedded components. Using our earlier notation we know that E is Cohen-Macaulay and E_0 is defined in E by $x_1,\dots x_r$ which are part of a system of parameters on E by (1.3), hence they are a regular sequence. It follows that $E_\circ$ has no embedded components, and so the only embedded component of $C_\circ$ could possibly be its vertex. But this is excluded by assumption 3).

3. We will now turn to some examples, for which we fix an infinite ground field k.

<u>EXAMPLE 1</u> (see [8]). We take

$$R = k\,[\![x^4,x^{10},x^5y,y^2]\!] \subset k\,[\![x,y]\!]$$

and

$$P = (x^4,x^{10},x^5y)\quad R \quad .$$

Then

$$R \simeq k\,[\![U,V,W,Z]\!]\ /(U^5-V^2,W^2-VZ)$$

where U,V,W,Z correspond to x^4,x^{10},x^5y,y^2 respectively. In [8] it is shown that

$$gr_{\mathfrak{m}}(R) \simeq k[U,V,W,Z]/(V^2,W^2-VZ)$$

and

$$gr_P(R) \simeq k\,[\![Z]\!]\ [U,V,W]/(V^2,ZV,VW^2,W^4)\ .$$

Moreover $e(R) = e(R_P)$ and R/P is regular. Obviously we have $e(R) = 4$ and therefore also $e((gr_P(R) \otimes_R R_P)^V) = 4$. Since

$$gr_P(R) \otimes_R R/\mathfrak{m} \simeq k[U,V,W]/(V^2,VW^2,W^4)$$

we see that

$$e((gr_P(R) \otimes_R R/\mathfrak{m})^V) = \ell(k[V,W]/(V^2,VW^2,W^4)) = 6\ .$$

Therefore $C \longrightarrow Z$ is not an equimultiple family of vertices and $B\ell_P(R)$ is not Cohen-Macaulay by Corollary 2. Note that $B\ell_{\mathfrak{m}}(R)$ is Cohen-Macaulay since $gr_{\mathfrak{m}}(R)$ is a complete intersection.

By Corollary 1 we know that if X is equimultiple along Z and $C \longrightarrow Z$ is not an equimultiple family of vertices, then X is not projectively normally flat along Z. In the above example, this can also be checked by direct computation using the form of $gr_P(R)$ given above. We point out that finding the equations for $gr_P(R)$ is the most difficult part in this example.

<u>EXAMPLE 2</u>. Here we will show that $C \longrightarrow Z$ may be an equimultiple family of vertices without X being equimultiple along Z. Probably the simplest such example is

$$R = k[[x^2, xy, y^2]]$$

and

$$P = (xy, y^2)R .$$

Clearly $e(R) = 2$ and $e(R_P) = 1$. Since

$$R \simeq k[[U,V,W]] / (UW-V^2) ,$$

we see that

$$gr_P(R) \otimes_R R_P \simeq k((U))[V,W]/(UW)$$

and

$$gr_P(R) \otimes_R R/\mathfrak{m} \simeq k[V,W] ,$$

so both fibres are regular. This example also indicates that nothing can be concluded from the equimultiplicity of the family $C \longrightarrow Z$ if X is not equimultiple along Z. As we have pointed out in the proof of Proposition 1, equimultiplicity of X along Z means that all fibres of $C \longrightarrow Z$ have the same dimension, and without this assumption there is no reasonable way of comparing the multiplicities of these fibres.

4. We add some concluding remarks. The papers [1], [2], [3], [5], [6] and [7] contain some techniques by which one could generalise our results to the case where R/P is not regular. We don't give these more general results here because not only the proofs but also the statements have to be modified, and then they become less geometric.

Next we note that one might also like to consider the question if the fibres of $C \longrightarrow Z$ have constant Hilbert functions at the vertices. By results of Bennett and Hironaka, it is well known that this holds if and only if X is normally flat along Z. This allows to reformulate Proposition 2 and Corollary 3 in the following way: If the assumptions of either one are satisfied, then the fibres of $C \longrightarrow Z$ not only have the same multiplicities at their vertices, but they even have the same Hilbert function.

Finally we raise the following two questions:

<u>QUESTION 1</u>. Which conditions allow to deduce normal flatness of X along Z from projective normal flatness of X along Z ? In particular, is it sufficient to assume X Cohen-Macaulay?

<u>QUESTION 2</u>. Are the same results as above still true if X and Z are complex-analytic spaces?

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