ON THE RIGIDITY PROBLEM FOR THE COMPLEX PROJECTIVE SPACE

by

Thomas PETERNELL

Introduction :

The subject of this paper is the following rigidity problem for the complex projective spaces $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$:

If X is a Moišezon manifold which is topologically $\mathbb{P}_n$, is then $X \simeq \mathbb{P}_n$ analytically ?

A Moišezon manifold is a compact manifold having $\dim X$ algebraically independent meromorphic functions.

In case $n = 2$ the positive solution (even without assuming X Moišezon) has been given by Hirzebruch-Kodaira [H-K] and Yau [Y1]. The case $n = 3$ is treated in [P]. A special case is the so-called deformation problem for $\mathbb{P}_n$, for this we refer to [K] $(n=3)$.

The purpose of this paper is two-fold :

First to prove some results and to discuss open problems in case $n > 3$; second to discuss the basic ideas of the solution in case $n = 3$.

In section 1 we first investigate the "linear" structure of the given Moišezon manifold X : Hodge decomposition etc. In particular we get $\mathrm{Pic}(X) = \mathbb{Z}$ and fix in all what follows the generator $\mathcal{O}_X(1)$ of $\mathrm{Pic}(X)$ for which we have $c_1(\mathcal{O}_X(1))^n > 0$, $n = \dim X$. By Hirzebruch-Kodaira and Yau it is sufficient to prove the projectivity of X . There are basically two obstructions to projectivity :

a) there might be irreducible curves $C \subset X$ with $C \sim 0$, i. e. $(C.c_1(\mathcal{O}_X(1))) = 0$;

b) there might be irreducible curves $C \subset X$ with "negative" homology, i. e. $(C.c_1(\mathcal{O}_X(1))) < 0$ (one can also say that there are non-irreducible effective curves homologous to 0 in X) .

We prove (theorem 1.4) that, assuming b) does not occur (i. e. $\mathcal{O}_X(1)$ is numerically effective), $X \simeq \mathbb{P}_n$. Here we have also to assume that the canonical bundle $\mathcal{K}_X \neq \mathcal{O}_X(n+1)$ it n is even. It is indicated that a generalized Calabi conjecture (for degenerate metrics) would imply the non-existence of this last phenomen. This would be an analogue to Yau's proof that the canonical bundle of a projective manifold which is topologically $\mathbb{P}_n$ cannot be ample. A basic ingredient for 1.4 is the Kawamata-Viehweg vanishing theorem.

If X is topologically $\mathbb{P}_n$ and also a scheme, then case a) cannot appear and we are able to prove $X \simeq \mathbb{P}_n$ (1.6).

Section 2 discusses the general problem, which is now reduced to prove the non-existence of irreducible curves ~ 0 . The method is to make X projective by performing a finite number of blow up's with smooth centers and to study extremal rational curves in the sense of Mori [M] on the resulting manifold $\hat{X}$. Here it turns out quickly that the case $n > 3$ is so complicated that we are forced to restrict ourselves to the case $n = 3$. In order to make the basic ideas transparent we will also assume that X can be made projective by one or two blow up's.

1.- General results (n arbitrary) :

We begin with the elementary

Proposition 1.1 :

Let X be a Moišezon manifold which is topologically isomorphic to $\mathbb{P}_n$. Then :

a) $H^q(X,\mathcal{O}) = 0$ for $q > 0$;

b) $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$;

c) fixing a generator $\mathcal{O}_X(1)$ of $\mathrm{Pic}(X)$ such that $c_1(\mathcal{O}_X(1))^n > 0$, $n = \dim X$, we have $\mathcal{K}_X \simeq \mathcal{O}_X(\pm(n+1))$ (clearly $\mathcal{O}_X(1)$ is uniquely determined). If n is odd, $\mathcal{K}_X \simeq \mathcal{O}_X(-(n+1))$.

Proof :

a) and b) are elementary (see [P]) and follow from the basic fact that Hodge decomposition holds on a Moišezon manifold.

In order to prove c) we proceed as in [H-K]. By a) the holomorphic Euler characteristic

$$\chi(X) = 1 .$$

The Pontrjagin classes

$$p_i \in H^{4i}(X,\mathbb{Z})$$

are topological invariants.

Using Riemann-Roch (for Moišezon manifolds) we have

$$\chi(X) = T(X) ,$$

where $T(X)$ is the Todd genus of X .

Hence $T(X) = 1$.

Since the image of $c_1(X) \in H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{Z})$ depends only on the topological structure of X (see [H-K]) we can write

$$c_1(X) = (2s+n+1) ,$$

identifying $c_1(\mathcal{O}_X(1))$ with 1 in $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$.

By the invariance of the Pontrjagin numbers, the definition of $T(X)$ (by $c_1(X)$ and the p_i) and some formal computations from [H-K] we get $T(X) = \binom{n+s}{n} = 1$, hence (c) .

Notation :

For any Moišezon manifold X with $\mathrm{Pic}(X) = \mathbb{Z}$ we fix $\mathcal{O}_X(1)$ as in 1.1.

Remark : If X is projective in 1.1 Yau has proved ([Y1], [Y2]) that the case $\mathcal{O}_X \simeq \mathcal{K}_X(n+1)$, i. e. $\mathcal{K}_X$ ample, cannot appear. Combining this with [H-K], any projective manifold which is topologically $\mathbb{P}_n$ is analytically $\mathbb{P}_n$.

Proposition 1.2 :

Let X be a Moišezon manifold, $n = \dim X$, $\mathcal{L} \in \mathrm{Pic}(X)$. Assume that $c_1(\mathcal{L})^n > 0$, that $\mathcal{L}$ and $\mathcal{L} \otimes \mathcal{K}_X^{-1}$ are numerically effective (i. e. $(c_1(\mathcal{L}).C) \geq 0$, $(c_1(\mathcal{L} \otimes \mathcal{K}_X^{-1}).C) \geq 0$ for any irreducible curve $C \subset X$). Then there is $m \in \mathbb{N}$ such that $\mathcal{L}^m$ is generated by global sections.

Proof :

If X is a smooth complete scheme of finite type over $\mathbb{C}$, then 1.2 is proved in [Kw 1]. But it turns out that this additional assumption is never used.

Corollary 1.3 :

Let X be a Moišezon manifold with $\mathrm{Pic}(X) = \mathbb{Z}$. Assume that $\mathcal{O}_X(1)$ is numerically effective. Then there is $m \in \mathbb{N}$ such that $\mathcal{O}_X(m)$ is globally generated.

Proof :

$\mathcal{K}_X \simeq \mathcal{O}_X(\lambda)$ for some $\lambda \in \mathbb{Z}$. Put $\mathcal{L} := \mathcal{O}_X(|\lambda| + 1)$ and apply 1.2 to $\mathcal{L}$.

Theorem 1.4 :

Let X be a Moišezon manifold which is topologically $\mathbb{P}_n$. Assume that $\mathcal{O}_X(1)$ is numerically effective and that $\mathcal{K}_X \neq \mathcal{O}_X(n+1)$ if n is even. Then $X \simeq \mathbb{P}_n$.

Proof :

a) First I claim that

$$h^0(X, \mathcal{O}_X(\mu)) = \binom{n+\mu}{n}$$

for all $\mu \in \mathbb{N}$.

Using $c_1(X) = n+1$, the invariance of the Pontrjagin classes and Riemann-Roch we get as in [H-K] :

$$\chi(X, \mathcal{O}_X(\mu)) = \binom{n+\mu}{n}.$$

So it is sufficient to prove

(*) $$H^q(X,\mathcal{O}_X(\mu)) = 0 \quad \text{for} \quad q > 0 \quad \text{and all} \quad \mu \in \mathbb{N} .$$

If X is projective this is exactly the Kawamata-Viehweg vanishing theorem ([Kw 2], [V]) (one gets (*) even for $\mu > -4$) . By the Grauert-Riemen-Schneider vanishing theorem $R^i\pi_*(\mathcal{K}_{\hat{X}}) = 0$, $i > 0$, for a projective desingularization $\pi : \hat{X} \to X$ ([G-R]), this remains true for X Moišezon instead of X projective. So (*) is proved.

b) Now consider the meromorphic map

$$\phi : X \to \mathbb{P}(H^0(X,\mathcal{O}_X(1)) = \mathbb{P}_n$$

given by a basis of $H^0(X,\mathcal{O}_X(1)) \simeq \mathbb{C}^{n+1}$. We proceed similarly as in [H-K]. Let $B := \{x \in X \mid s(x) = 0 \text{ for all } s \in H^0(X,\mathcal{O}_X(1))\}$.

Since for any $s \in H^0(X,\mathcal{O}_X(1))$, $s \neq 0$, the hypersurface $\{s=0\}$ is necessarily irreducible (since $[\{s=0\}] = 1$ in $H^2(X,\mathbb{Z}) = \mathbb{Z}$ and no hypersurface is cohomologous to 0) it follows that codim $B \geq 2$ (otherwise $B = \{s=0\}$ for all $s \in H^0(X,\mathcal{O}_X(1))$, $s \neq 0$, but given $x \in X$ there is $s \in H^0(X,\mathcal{O}_X(1))$ such that $s(x) = 0$, $s \neq 0$ (because $h^0(X,\mathcal{O}_X(1)) \geq 2$) .

Now I claim :

(**) ϕ is bimeromorphic.

Let us first remark :

(***) If $Y \subset X$ is an irreducible analytic set of dimension r , then $[Y] \geq 0$ in $H^{2n-2r}(X,\mathbb{Z}) = \mathbb{Z}$ (where the last identification is made by $c_1(\mathcal{O}_X(1))^{n-r} = 1$) $[Y] \geq 0$ means : $(c_1(\mathcal{O}_X(1))^r.Y) \geq 0$.

The proof of (***) is just a theorem of Kleiman (see [H1]).

Now let $V := \overline{\phi(X \setminus B)} \subset \mathbb{P}_n$. Then (**) is equivalent to $\dim V = n$.

Let $r := \dim V$. Let $\pi : \hat{X} \to X$ be a finite sequence of blow up's with smooth centers such that $\hat{X}$ is projective. Let $S \subset X$ and $\hat{S} = \pi^{-1}(S)$ be the degeneracy sets of π . Then codim $S \geq 2$.

For all irreducible curves $C \not\subset S$ we have :

(+) $$(c_1(\mathcal{O}_X(1)).C) > 0 .$$

For the proof see [P1], prop. 2.1, in case $n = 3$; the general case being analogous. More generally :

$$(c_1(\mathcal{O}_X(1))^r.Y) > 0$$

for all irreducible r-dimensional subvarieties $Y \not\subset S$.

(One can also argue as follows. Performing a modification if necessary, we may assume Y projective smooth. Clearly $\varkappa(Y,\mathcal{O}_X(1)) = r$ (since $Y \not\subset S$). Then use [Kw2], lemma 3).

Choose $x \in X \setminus (B \cup S)$ such that V is smooth in $\phi(x)$.

Let $T_0 \subset \mathbb{P}_n$ be the projective closure of the tangent space $T := T_{\phi(x)}V$ of V in $\phi(x)$. Then T_0 is linear and of dimension r.

We prove :

(++) $$\phi(X \setminus (B \cup S) \subset T_0 .$$

In fact, let $y \in X \setminus (B \cup S)$ and assume that $\phi(y) \not\in T_0$.

Then we find $L \subset \mathbb{P}_n$ linear and transversal to T_0 with $\dim L = n - r$ and $\phi(x)$, $\phi(y) \in L$.

Let $W := \overline{\phi^{-1}(L)} \subset X$. Clearly $x, y \in W$.

We can write $W = \{s_1 = 0\} \cap \ldots \cap \{s_r = 0\}$ for some $s_i \in H^0(X,\mathcal{O}_X(1))$. Thus $[W] = 1$ in $H^{2n-2s}(X,\mathbb{Z})$.

Consequently $W = W_0 \cup Y_1 \cup \ldots \cup Y_q$, where $\dim W_0 = s$, $W_0 = 1$, W_0 irreducible, and the Y_i are irreducible analytic sets in S which are cohomologous to 0. This follows easily trom (+) using induction on r.

Hence $W \setminus (B \cup S) = W_0 \setminus (B \cup S)$.

Now $W_0 \setminus (B \cup S)$ is an open non-empty connected analytic subvariety of W_0, hence $\phi(W_0 \setminus (B \cup S))$ is connected and thus contains a path connecting $\phi(x)$ and $\phi(y)$. Since $\phi(W_0 \setminus (B \cup S)) \subset L \cap V$, L cannot be transversal to T_0, contradiction. So (++) is true.

As in [H-K] we conclude

$$\phi(X \setminus (B \cup S)) \subset T_0$$

and hence $V = T_0$, so $r = n$. So ϕ is bimeromorphic proving (**).

Let $\nu \in \mathbb{N}$. A base of $H^0(X,\mathcal{O}_X(\nu))$ defines a canonical embedding $\mathbb{P}_n \hookrightarrow \mathbb{P}_n$, $N = \binom{n+\nu}{n} - 1$, by substituting the sections of $H^0(\mathbb{P}_n,\mathcal{O}_{\mathbb{P}_n}(1))$ into a base of the homogeneous polynomials of degree ν in $n+1$ variables.

We get the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathbb{P}_n \\ & \searrow^{\psi} & \updownarrow \\ & & \mathbb{P}_N \end{array} ,$$

ψ being defined by $N+1$ sections t_j of $H^0(X,\mathcal{O}_X(\nu))$ in the same way as described above.

ϕ being bimeromorphically, the sections t_j are linearly independent. Since $h^0(X,\mathcal{O}_X(\nu)) = \binom{n+\nu}{n}$ by a), they are a base of $H^0(X,\mathcal{O}_X(\nu))$.

Choosing ν such that $H^0(X,\mathcal{O}_X(\nu))$ generates $\mathcal{O}_X(\nu)$ (1.2 !), we conclude $B = \emptyset$.

Furthermore if $\nu >> 0$, the set where ϕ is not biholomorphic, is contained in S (take a very ample line bundle $\mathcal{L}$ on $\hat{X}$, then $\pi_*(\mathcal{L})^{**} \in \text{Pic}(X)$, hence of the form $\mathcal{O}_X(\nu)$, and this ν works, if it is also a multiple of the ν chosen above).

We conclude that $\phi : X \to \mathbb{P}_n$ is a modification with degeneracy set $D \subset S$.

The smoothness of $\mathbb{P}_n$ implies by the Grauert-Remmert purity - of - branch theorem that $\text{codim}\, D = 1$, if $D \neq \emptyset$. Since $\text{codim}\, S \geq 2$, we have $D = \emptyset$, hence ϕ is biholomorphic.

Remark : If we don't assume that $\mathcal{O}_X(1)$ is numerically effective in 1.4, then there are serious obstacles to carry over the proof. First it is unclear whether the vanishing $H^q(X,\mathcal{O}_X(\mu)) = 0$ holds. If we assume the vanishing then it is not at all clear whether ϕ is bimeromorphic since there are subvarietes with "negative homology", so we cannot conclude $W = W_0 \cup Y_1 \cup \ldots \cup Y_q$ as in the proof of 1.4. And last, even if ϕ is bimeromorphic, we only know that X is rational and don't know $B = \emptyset$.

Next we want to discuss the assumption "$c_1(X) \neq -(n+1)$ if n is even" in 1.4. As already mentioned, in the projective case Yau has shown that this phenomen cannot occur. We will discuss a possible generalization of his methods.

So assume that $\mathcal{K}_X \simeq \mathcal{O}_X(n+1)$ in the situation of 1.4. Then X is of general type. By 1.2 we find m such that $\mathcal{O}_X(m)$ is globally generated. Let $\phi : X \to V$ be the Stein factorization of the morphism $X \to \mathbb{P}(H^0(X,\mathcal{O}_X(m))$.

If $m >> 0$, then ϕ is a modification. V is projective, normal, $\dim V = \dim X$ and $\phi_*(\mathcal{O}_X) = \mathcal{O}_V$. If $C \subset X$ is an irreducible curve, then $\dim \phi(C) = 0$ if and only if $(C.c_1(\mathcal{O}_X(1))) = 0$, i. e. $C \sim 0$ (cp. [Kw3], theorem 2). Let $\mathcal{L}$ be an ample line bundle on V . Then $\phi^*(\mathcal{L}) \simeq \mathcal{O}_X(\mu)$ for some $\mu \in \mathbb{N}$. We can choose a Hermitian metric on L (= line bundle associated in the classical sense to $\mathcal{L}$) such that the induced metric h on $\phi^*(L)$ has the following property : the associated curvature form ω is positive definite on $X \setminus S$ ($\pi : \hat{X} \to X$, S as in the proof of 1.4). But ω has zeros on S .

Call ω a degenerate Kähler form, coming from a degenerate Kähler metric g on X . Then we can formulate a

Generalized Calabi Conjecture :

There exists a degenerate Kähler form ω' on X such $\mathrm{Pic}(\omega') = \omega'$ (so there exists a "degenerate Kähler-Einstein metric" on X) .

Remember that (a part of) the Calabi conjecture solved by Yau says that on a compact Kähler manifold X with $c_1(X) < 0$, i. e. $\mathcal{K}_X$ ample, there exists a Kähler-Einstein metric.

A form of the generalized Calabi conjecture (namely if X is Kähler) has been proved in [Y3].

Now assume that X has a degenerate Kähler-Einstein metric ω' . Then one can prove :

$$(-1)^n \int_X 2(n+1)\, c_2(X)\, c_1(X)^{n-2} \geq (-1)^n \int_X n\, c_1(X)^n ,$$

where $c_i(X)$ are differential forms computed by the Ricci curvature of X with respect to the degenerate Kähler-Einstein metric. This is the same formula as in the non-degenerate case proved by Yau. Furthermore equality holds if and only if the holomorphic sectional curvature of $X \setminus S$ is constant negative. These facts hold because they hold in the Kähler case and since the essential computations are local (see [K-W]).

One expects that then X is covered by a non-compact manifold (by the ball in the non-degenerate case), so the fact that X is simply connected would give a contradiction.

Now we investigate the obstruction in the general case.

Proposition 1.5 :

Let X be a Moišezon manifold without effective curves ($\Sigma n_i C_i$, $n_i \in \mathbb{N}$, C_i irreducible curves) homologous to 0. Assume that X is topologically $\mathbb{P}_n$. Then X is projective (and hence $X \simeq \mathbb{P}_n$).

Proof :

We already noticed that we can find $m \in \mathbb{N}$ such that the Stein factorization $\phi : X \to V$ of $X \to \mathbb{P}(H^0(X,\mathcal{O}_X(m))$ is a modification of X onto a normal projective variety and that for any irreducible curve $C \subset X$: $\dim \phi(C) = 0$ iff $C \sim 0$ (this follows also from the fact that for some $\mu \in \mathbb{N}$, $\mathcal{O}_X(\mu)$ is induced by a line bundle on V). So by our assumption, ϕ is finite. But then X is projective.

Remark : If X is a scheme 1.5 also follows immediately from [H].

In case X is a scheme the general problem can be solved completely.

Theorem 1.6 :

Let X be a Moišezon manifold which is topologically $\mathbb{P}_n$. Assume that X is a scheme (strictly speaking, X is the complex space associated to a complete scheme of finite type over $\mathbb{C}$). Then $X \simeq \mathbb{P}_n$.

Proof :

Since there are no irreducible curves ~ 0 in a smooth complete scheme, we must only show there is no irreducible curve $C \subset X$ such that

$$[C] = (C.c_1(\mathcal{O}_X(1))) < 0 .$$

Then we can apply 1.5 or the remark.

Assume $[C] < 0$. Then for any irreducible hypersurface $Y \subset X$ (= 1-codimensional subvariety) :

$$(C.Y) < 0 ,$$

since $[Y] = c_1(\mathcal{O}_X(\mu))$ in $H^2(X,\mathbb{Z}) = \mathbb{Z}$, $\mu > 0$.

Thus $C \subset Y$.

Take $x \in C$. Let U be an affine neighborhood of x in the scheme X. Then any irreducible component Y of $X \setminus U$ has codimension 1 (X is irreducible). Since $x \notin Y$, we have $C \not\subset Y$, contradiction.

Let us finish this section with a reformulation of proposition 1.5.

Let $Z_1(X)$ be the free group of 1-cycles on X (so $\alpha \in Z_1(X)$ has the form $\alpha = \Sigma\, n_i\, C_i$, $n_i \in \mathbb{Z}$, $C_i \subset X$ irreducible curves).

Let $A_1(X) := Z_1(X)$ / rational equivalence (see e. g. [F]).

Proposition 1.7 :

Let X be a Moišezon manifold which is topologically $\mathbb{P}_n$. If $A_1(X) \simeq \mathbb{Z}$, then $X \simeq \mathbb{P}_n$.

Proof :

Consider the "cycle map"

$$cl : A_1(X) \to H_2(X,\mathbb{Z}) \simeq \mathbb{Z} ,$$

associating to each $\alpha \in A_1(X)$ its homology class.

Clearly $cl \neq 0$ (because not all curves ~ 0) , so cl is injective (since $A_1(X) \simeq \mathbb{Z}$) [1]. By 1.5 it is sufficient to show that there are no effective curves $\Sigma\, n_i\, C_i \sim 0$. Assuming $\Sigma\, n_i\, C_i \sim 0$, $\Sigma\, n_i\, C_i$ must also be rationally equivalent to 0 . But this is impossible on a Moišezon manifold as one deduces from the definition of rational equivalence. (One can also use the Grothendieck group $K_0(X)$) .

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1) cl is a homomorphism of $\mathbb{Z}$-modules !

2.- The general case :

In this section we attack the general problem : if X is Moišezon and topologically $\mathbb{P}_n$, is $X \simeq \mathbb{P}_n$?

The method is the following : take a finite sequence of blow up's with smooth centers

$$\hat{X} \xrightarrow{\pi_\nu} X_{r-1} \xrightarrow{\pi_{\nu-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$$

such that $\hat{X}$ is projective and that r is minimal. Let $\pi := \pi_1 \cdot \dots \cdot \pi_\nu$.

Then the canonical bundle $\mathcal{K}_{\hat{X}}$ is not numerically effective, i. e. there is a curve $C \subset \hat{X}$ such that $\mathcal{K}_{\hat{X}}|C$ is negative. Hence the results of Mori [M] are applicable : we find an extremal rational curve associated to an extremal ray on $\hat{X}$ (for the notations see [M], [Kw 1], [Kw 3]), which we can contract : there exists a normal projective variety Y and a surjective morphism $\phi : \hat{X} \to Y$ with $\phi_*(\mathcal{O}_{\hat{X}}) = \mathcal{O}_Y$ and the following property : $\dim \phi(C) = 0$ iff $[C] \in R$ for any irreducible curve $C \subset \hat{X}$.

For the proof of the contraction theorem see [M] in case $\dim X = 3$, the general analogous case follows from [Kw 1].

Now one has to distinguish two different cases :

a) $\dim Y \leq n-1$ $(n = \dim X)$

b) $\dim Y = n$.

First it can be checked rather easily (see [P] for $n = 3$) :

Proposition 2.1 :

The case $\dim Y < n-1$ *is impossible.*

Proof :

One selects carefully two irreducible curves $C_1 , C_2 \subset \hat{X}$ with $\dim \phi(C_1) = \dim \phi(C_2) = 0$, so $C_1 \sim C_2$ with some $\lambda > 0$. But one can choose the C_i such that

$$(\pi(C_i).c_1(\mathcal{O}_X(1))) < 0 ;$$

on the other hand $\pi(C_1) \sim \lambda' \pi(C_2)$, $\lambda' \geq 0$, yielding a contradiction.

It is more difficult to verify (see [P]).

Proposition 2.2 :

If $n = 3$, *also the case* $\dim Y = n-1$ *is impossible*.

The method to prove 2.2 is the following. First one shows $r = 1$ or $r = 2$; let us concentrate on the case $r = 1$. Then $Y \simeq \mathbb{P}$, furthermore $\phi : \hat{X} \to Y$ is a so-called conic bundle (in any case). In [P] it is proved that ϕ is a $\mathbb{P}_1$-bundle. This depends on the fact that X has no exceptional curves C which are not homologous to 0 for this consider the blow-down $\sigma : X \to Z$ of such a C . In [P] a line bundle $\neq \mathcal{O}_Z$ was supposed to exist on Z . This should be proved. It is better to argue in the following way.

Let $A_1(\hat{X})$ be as in 1.7. Then one proves :

$$A_1(\hat{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^2$$

(using $A_1(\mathbb{P}_2) = \mathbb{Z}$) .

Hence the rational analogue of 1.7 gives the projectivity of X , since $A_1(\hat{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^2$ implies $A_1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

(One can also use results of Beauville [B]. Namely, the intermediate Jacobian $J(\hat{X})$ vanishes since $H^3(\hat{X},\mathbb{C}) = 0$. $J(\hat{X})$ being isomorphic to the Prym variety $P(\hat{X})$ associated to the conic bundle $\phi : \hat{X} \to Y$ ([B]), we have $P(\hat{X}) = 0$.

Then it follows from [B] that $A_1(\hat{X}) \simeq \mathbb{Z}^2 \oplus$ torsion - at least in the case that the "discriminant locus" of ϕ is smooth).

(Remark also that $A_1(\hat{X}) \simeq A_1(Y) \oplus A_0(Y)$ in case ϕ is a $\mathbb{P}_1$-bundle (see e. g. [F]) and that by [B] the discriminant locus of ϕ in $\mathbb{P}_2$ must be rational if ϕ is not a $\mathbb{P}_1$-bundle).

It should also be possible to show directly that ϕ is a $\mathbb{P}_1$-bundle by disproving equation (4) in [P], p. 410 (by some messy calculations), in case $c_1(\mathcal{N}_{S|X}) < 0$.

The case $r = 2$ is done in an analogous manner.

So we have to concentrate on the case $\dim Y = n = 3$. In this case ϕ is a modification. There is an irreducible divisor $D \subset \hat{X}$ such that $\phi|\hat{X} \setminus D \to Y \setminus \phi(D)$ is an isomorphism, $\dim \phi(D) \leq 1$ and ϕ is the blow up of $\phi(D)$ in Y (see [M]). One can classify ([M]) the divisors occuring in those situations :

D is a geometrically ruled surface, $\mathbb{P}_2$ or an irreducible reduced singular quadric $Q \subset \mathbb{P}_3$. Of course, in dimension $n > 3$ one cannot expect such a precise classification. This is one reason why we concentrate on $n = 3$.

Now there are two cases to distinguish :

a) $\dim \pi(D) = 2$;

b) $\dim \pi(D) \leq 1$.

b) means that D is an irreducible component of $\hat{S} = \pi^{-1}(S)$.

We have :

<u>Proposition 2.3</u> :

<u>Case</u> a) <u>cannot appear</u> (<u>if</u> $n = 3$) .

The reason is the following. If $\dim \pi(D) = 2$, then any irreducible component S_0 of S , the degeneracy locus of π (i. e. $\hat{X} \setminus \pi^{-1}(S) \to X \setminus S$ is biholomorphic and S minimal), with

$$(c_1(\mathcal{O}_X(1)).S_0) < 0$$

must be in $\pi(D)$ (remember that such an S_0 must exist, otherwise $X \simeq \mathbb{P}$ by 1.4). Namely, if $S_0 \not\subset \pi(D)$, we would have $(S_0.\pi(D)) \geq 0$, but $[\pi(D)] = c_1(\mathcal{O}_X(\mu))$, $\mu > 0$, in $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$.

If $D = \mathbb{P}_2$ or $D = Q$, any curves C_1 , $C_2 \subset \pi(D)$ are homologous : there is a $\mu > 0$ such that $C_1 \sim \mu C_2$. So for all curves $C \subset \pi(D)$ we would have :

$$(C.c_1(\mathcal{O}_X(1))) < 0 .$$

But this inequality is possible for only finitely many curves ; hence D must be ruled.

Then one checks ([P]) that S_0 is the image of a certain distinguish irreducible curve in D (a special section of D) : first one shows that for the invariant $e = e(D)$ in the sense of Hartshorne [H'] one has $e > 0$ (this depends on the exact knowledge of ample divisors on the ruled surface D , see [H']) and then this special section C of D is the uniquely determined section with $C^2 = -e$. Then one proves that $f := \pi|D$ maps $D \setminus C$ biholomorphically on $\pi(D) \setminus S_0$.

Now one distinguishes two cases :

a) f is an isomorphism. Then it is easy to determine $\mathcal{O}_X(1)|D$ intermes of sections and fibers of D (numerically) and to determine the class $[D]$ in X . Using well-known facts on ruled surfaces, we get $\omega_D \simeq \mathcal{O}_X(-2)$ D , $[D] = 2$ and $S_0 \simeq \mathbb{P}_1$. Then we compute some intersections of D in $\hat{X}$ and D in X and by comparison we get a contradiction ;

b) f is not an isomorphism. Then we can relate ω_D and $f^*(\omega_{\pi(D)})$ (dualizing sheaves) and can determine $f^*(\mathcal{O}_X(1))|D$. Now cohomological considerations give a contradiction. For proofs we refer to Part 2 of [P].

Resuming all we have proved we can state :

<u>Theorem 2.4</u> :

<u>In case</u> $\dim X = 3$, <u>any extremal ray on</u> $\hat{X}$ <u>is not numerically effective</u> (<u>in the notation of Mori</u> [M] ; <u>i. e.</u> $\dim Y = 3$, <u>where</u> $\phi : \hat{X} \to Y$ <u>is as above</u>). <u>Furthermore the associated divisor satisfies</u> $\dim \pi(D) \leq 1$.

Now the main result is

<u>Theorem 2.5</u> :

<u>If</u> X <u>is a Moišezon</u> 3-<u>fold which is topologically</u> $\mathbb{P}_3$, <u>then</u> $X \simeq \mathbb{P}_3$.

We indicate the proof in the cases $r = 1$ and $r = 2$.

First assume $r = 1$ (and X not projective). Then S is a smooth curve in X . Take an extremal ray on $\hat{X}$ and let D be the associated divisor. Necessarily $D = \hat{S} := \pi^{-1}(S)$, so D is ruled. If $\phi : \hat{X} \to Y$ is the blow-down of D , then $\phi \neq \pi$ and $X \neq Y$ because Y is projective. So D must have two rulings whence $D = \mathbb{P}_1 \times \mathbb{P}_1$, $S = \mathbb{P}_1$. One computes easily that the normal bundle $\mathcal{N}$ of S in X is of the form $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, so $c_1(\quad) = -2$. But the adjunction formula

$$\mathcal{K}_S \simeq \mathcal{K}_X|S \otimes \det(\mathcal{N})$$

and

$$\mathcal{K}_S \simeq \mathcal{O}_{\mathbb{P}_1}(-2)$$

give

$$\mathcal{K}_X|S \simeq \mathcal{O}_S \quad , \quad \text{so } S \sim 0 .$$

Hence 1.4 gives a contradiction.

Now assume $r = 2$. The center of π_ν is denoted by S_ν .

a) First assume that S is irreducible. Then we distinguish furthermore the two cases

a_1) S is not smooth,

a_2) S is smooth.

a_1) We still have $\hat{S}_2 = \pi_2^{-1}(S_2) = D = \hat{\mathbb{P}}_1 \approx \mathbb{P}_1$ as in the case $r = 1$ (observe that S_1 must be a point and that the strict transform of $\pi_1^{-1}(S_1) \simeq \mathbb{P}_2$ in $\hat{X}$ cannot be ruled).

We compute easily for the normal bundle of S_2 in X_1 :

$$\mathcal{N}_{S_2|X_1} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) .$$

Then again the adjunction formula gives :

$$\mathcal{K}_{X_1}|S_2 \simeq \mathcal{O}_{S_2} .$$

Since π_1 is the blow up of a smooth point, we have

$$c_1(\mathcal{K}_X|S) < c_1(\mathcal{K}_{X_1}|S_2) .$$

So $\mathcal{K}|_X S$ is negative. Since by 1.1, $\mathcal{K}_X \simeq \mathcal{O}_X(-1)$, we conclude that $\mathcal{O}_X(1)|S$ is positive, contradiction (use 1.5).

a_2) S is smooth. Since S is irreducible, $S_2 \subset \check{S}_1 := \pi_1^{-1}(S_1)$.

First one remarks that necessarily $D = \hat{S}_2$ (see [P]). Then as in a_1) :

$$S_2 \simeq S \simeq \mathbb{P}_1 \quad \text{and} \quad \mathcal{N}_{S_2|X_1} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) .$$

S_1 must be a curve (otherwise we could omitt π_1) .

Then S_2 must be the uniquely determined section of $\check{S}_1$ with $S_2^2 = -e$, where $S_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ with $e \geq 0$. (The reason is the following. Write $S_2 \sim a\tilde{S} + bF$ for numerical equivalence on $\check{S}_1$, where $b \geq ae$, $a > 0$ and $\tilde{S}$ is the uniquely determined section of $\check{S}_1$ with $\tilde{S}^2 = -e$ (see [H']), F a fiber of $\pi_1|\check{S}_1$. Thus $(S_2.c_1(\pi_1^*(\mathcal{O}_X(1))) < 0$ implies $(\tilde{S}.c_1(\pi_1^*(\mathcal{O}_X(1))) < 0$, hence $\pi_1(\tilde{S}) \subset S$, i. e. $S_2 = \tilde{S}$) .

Then the theory of ruled surfaces gives :

$$e = 2 \quad \text{and} \quad \mathcal{N}_{S|X} \simeq \mathcal{O} \oplus \mathcal{O}(-2) .$$

Hence $S \sim 0$ in X , contradiction (by 1.5).

b) Now assume that S is not irreducible but connected (otherwise we come down to the case $r = 1$) .

First assume

b_1) $[\pi_1(S_2)] < 0$.

We proceed as in a_2) : we must have $D = S_2 = \mathbb{P}_1 \times \mathbb{P}_1$, since the strict transform of $\pi_1^{-1}(S_1)$ in $\hat{X}$ cannot be ruled, $\mathbb{P}_2$ or Q . Hence we get as above :

$$(c_1(\mathcal{K}_{X_1}).S_2) = 0$$

and consequently :

$$(c_1(\mathcal{K}_X).\pi_1(S_2)) < 0$$

which gives a contradiction as in a_1).

b_2) $[S_1] < 0$. Again $D = \hat{S}_2 = \mathbb{P}_1 \times \mathbb{P}_1$, Clearly $S_2 \not\sim 0$ (since $(S_2.\check{S}_1) \neq 0$) , thus $s_0 \times \mathbb{P}_1 \sim \mathbb{P}_1 \approx t_0$ for all s_0 , $t_0 \in \mathbb{P}_1$, hence $\dim \phi(\hat{S}_2) = 1$ and Y is smooth by [M]. Since Y is projective and $\mathcal{K}_Y$ is not numerically effective, there is an extremal ray R' on Y . As before we show that R' is not numerically effective. Hence the contraction $\phi' : Y \to Y'$ is a modification ; let $\hat{D}'$ be the associated divisor in Y . Let D' be the strict transform of D' in $\hat{X}$. Since the Picard number $\rho(Y) = 2$, one has $\rho(Y') = 1$. Thus the line bundle

$$\mathcal{G} := (\phi' \circ \phi)_* \, (\pi^*(\mathcal{O}_X(1)))^{**}$$

must be ample (if Y' is smooth, $\mathcal{G}$ is clearly locally free ; if Y' is singular, the isolated singularity is well-known ; its local divisor class group is 0 ([M], the one singularity with local divisor class group $Z/2\mathbb{Z}$ clearly cannot appear)), because $\mathcal{G}$ cannot be trivial or negative.

We can write :

$$(*) \qquad \pi^*(\mathcal{O}_X(\mu)) \simeq (\phi' \circ \phi)^* \, (\mathcal{G}) \times \lambda_1 \, [D'] \times \lambda_2 \, [\hat{S}_2] \, , \, \lambda_i \in \mathbb{Z} \, ,$$

for some $\mu \in \mathbb{N}$.

Take a fiber F of $\pi_1|\pi_1^{-1}(S_1)$ such that $F \cap S_2 = \emptyset$. Let $\hat{F}$ be the strict transform of F in $\hat{X}$. Intersecting with $\pi^*(\mathcal{O}_X(1))$ and using (*) we get :

$$0 = (c_1(\pi^*(\mathcal{O}_X(1)).\hat{F}) = (c_1((\phi' \circ \phi)^* \, (\mathcal{G}))).\hat{F}) + \lambda_1 \, (\hat{D}'.\hat{F}) \, .$$

Since $[S_1] < 0$ implies $S_1 \subset \pi^*(\hat{D}')$, we conclude $(\hat{D}'.\hat{F}) > 0$ or $\hat{F} \subset \hat{D}'$.

If F is general, $(\hat{D}'.\hat{F}) > 0$ or $\hat{D}' = \hat{S}_1$.

Assume $(\hat{D}'.\hat{F}) > 0$. Since

$$(c_1((\phi' \circ \phi)^* (\mathcal{G})).\hat{F}) \geq 0 ,$$

we have $\lambda_1 = 0$ and $\dim (\phi' \circ \phi)(\hat{F}) = 0$, whence $\hat{D}' = \hat{S}_1$.

So in any case $D' = \phi(\hat{S}_1)$.

Now it is clear that $\phi(\hat{S}_1)$ cannot be ruled, $\mathbb{P}_2$ or Q , contradiction.[1)]

This proves 2.5 in case $r = 2$.

Of course the general case $r > 3$ is still technically more difficult but most of the basic ideas are already imminent in the cases $r \leq 2$.

At present the cases $n > 3$ seem somewhat hopeless (at least dealing with these methods).

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1) It might also be possible to show b_2) by proving that $[\pi_1(S_2)] < 0$.

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