# DISCRETE COMPLEX REFLECTION GROUPS

V.L. POPOV

*Lectures delivered at the*
*Mathematical Institute,*
*Rijksuniversiteit Utrecht,*
*October 1980.*

COMMUNICATIONS OF THE MATHEMATICAL INSTITUTE
RIJKSUNIVERSITEIT UTRECHT 15-1982

Table of contents

page

1. Notation and formulation of the problem. 4

1.1. Notation. 4

1.2. Motions and reflections. 5

1.3. Main problem. 7

1.4. Irreducibility. 7

1.5. What is already known: $k = \mathbb{R}$. 10

1.6. What is already known: $k = \mathbb{C}$. 13

2. Formulation of the results. 29

2.1. Complexifications and real forms. 29

2.2. Classification of infinite irreducible complex noncrystallographic r-groups: 30

2.3. Ingredients of the description. 31

2.4. Cohomology. 32

2.5. Description of the group of linear parts: the result. 35

2.6. The list of irreducible infinite crystallographic complex groups. 36

2.7. Equivalence. 45

2.8. The structure of an extension of Tran W by Lin W. 46

2.9. The rings and fields of definition of Lin W. 46

2.10 Further remarks. 49

3. Several auxiliary results and the classification of irreducible infinite noncrystallographic complex r-groups. 51

3.1. The subgroups of translations. 51

3.2. Some auxiliary results. 55

3.3. Semidirect products. 56

3.4. Classification of the irreducible infinite complex non-crystallographic r-groups. 58

page

4. Invariant lattices 59
4.1. Root lattices. 59
4.2. The lattices with a fixed root lattice. 60
4.3. Futher remarks on lattices. 64
4.4. Properties of the operator S. 64
4.5. Root lattices and infinite r-groups. 67
4.6. Description of the groups of linear parts: proof. 68
4.7. Description of root lattices. 71
4.8. Invariant lattices in the case s = n + 1. 79
5. The structure of r-groups in the case s = n + 1. 83
5.1. The cocycle c. 83
5.2. Example. 86
References. 89

## Introduction

These notes essentially record the content of a course of five lectures given at the Mathematical Institute of the Rijksuniversiteit Utrecht in October 1980. The aim of these lectures was to develop the theory of discrete groups generated by affine unitary reflections, and in fact, to provide a classification of these groups. I made an attempt to give an exposition in such a way that our results would be comparable with the classical results of the theory of discrete groups generated by affine reflections in a real euclidian space, which was developed in the former half of this century by Coxeter, Witt, Stiefel and others. Both theories have much in common. However, the classification of complex groups is more complicated (and less geometrical) than the classification over the reals. The problem of developing the theory of discrete groups generated by affine unitary reflections is a comparatively old one; I was informed that it was posed by A. Borel about 15 years ago. My general aim during the lectures was to explain the main ideas and to give proofs only of the theorems that are not of a strictly technical nature therefore, I restricted myself to examples or simply to formulating results in all technical cases (which are, however, not always trivial). A more detailed exposition will appear elsewhere.

I wish to thank the Mathematical Institute of the Rijksuniversiteit Utrecht for its hospitality. I am grateful to Professor T.A. Springer on whose initiative these lectures were given and written up. I am also grateful to A.M. Cohen for the interesting discussions I had with him and for his help on preparing these notes. My thanks go also to the secretaries of the University of Utrecht, the Netherlands for the careful typing of the manuscript.

## 1. Notation and formulation of the problem.

We assume in this chapter that the ground field k be $\mathbb{R}$ or $\mathbb{C}$.

### 1.1. Notation.

Let E be an affine space over k, dim E = n, and V be its space of translations. If $v \in V$, we denote by $\gamma_v$ the corresponding translation of E, i.e.

$$\gamma_v(a) = a + v,\ a \in E.$$

Let A(E) be the group of all affine transformations of E and

$$\text{Tran } A(E) = \{\gamma_v \mid v \in V\}.$$

We denote by GL(V).V the natural semidirect product of GL(V) and V. Its elements are pairs (P,v), where $P \in GL(V)$, $v \in V$, and the group operations are given by formulas

$$(P,v)(Q,w) = (PQ, Pw+v),$$

$$(P,v)^{-1} = (P^{-1}, -P^{-1}v).$$

Let

$$\text{Lin}: A(E) \to GL(V)$$

be the standard homomorphism defined by formula

$$\gamma(a + v) = \gamma(a) + (\text{Lin } \gamma)v,\ \gamma \in A(E),\ a \in E,\ v \in V.$$

If we take a point $a \in E$ as origin, we obtain an isomorphism

$$\kappa_a: A(E) \to GL(V).V$$

given by the formula



$$\kappa_a(\gamma) = (\text{Lin } \gamma, \gamma(a) - a).$$

Identifying A(E) and GL(V).V by means of $\kappa_a$, we obtain the action of GL(V).V on E given by the formula

$$(P,v)q = a + P(q - a) + v,\ q \in E.$$

The dependence on a is given by

$$\kappa_b(\gamma) = \kappa_a(\gamma_{a-b}\gamma\gamma_{b-a}),\ \gamma \in A(E),\ b \in E.$$

For every $\gamma \in A(E)$ and $P \in GL(V)$ we use the notations

$$H_\gamma = \{a \in E \mid \gamma(a) = a\},$$

$$H_P = \{v \in V \mid Pv = v\}.$$

These are subspaces of E and V respectively.

Let $\langle\ |\ \rangle$ be a positive definite inner product on V, i.e. V is an euclidian ($k = \mathbb{R}$), resp. hermitian ($k = \mathbb{C}$) linear space with respect to $\langle\ |\ \rangle$ (linear in the first coordinate). Let also U(V) = $= \{P \in GL(V) \mid P \text{ preserves } \langle\ |\ \rangle\}$; this is a compact group. The space E becomes a euclidean, resp. hermitian affine metric space with respect to the distance given by the formula

$$\rho(a,b) = \sqrt{\langle a - b \mid a - b\rangle},\ a,b \in E.$$

### 1.2. Motions and reflections.

We say that $\gamma \in A(E)$ is a motion of E if $\gamma$ preserves the distance $\rho$. It is easy to see that $\gamma$ is a motion iff $\text{Lin } \gamma \in U(V)$.

Definition. An affine reflection $\gamma \in A(E)$ is an element with the properties:

1) $\gamma$ is a motion,

2) $\gamma$ has finite order,

3) codim $H_\gamma = 1$.

A linear reflection $R \in GL(V)$ is an element with the properties:

1) $R \in U(V)$,

2) $R$ has finite order,

3) codim $H_R = 1$.

The subspaces $H_\gamma$ and $H_R$ are called the mirrors of $\gamma$, resp. $R$.

Sometimes we shall simply say reflection when it is clear what we are talking about.

If $R$ is a linear reflection then the line

$$\ell_R = \{v \in V \mid v \perp H_R\}$$

is called the root line of $R$. If $v \in \ell_R$, $\langle v | v \rangle = 1$, then $Rv = \theta v$, where $\theta \neq 1$ is a primitive root of 1 (if $k = \mathbb{R}$ then $\theta = -1$, if $k = \mathbb{C}$ then $\theta$ may be arbitrary). The pair $(v,\theta)$ determines $R$ completely and every pair $(u,\eta)$, with $\langle u | u \rangle = 1$, $\eta \neq 1$ a primitive root of unity ( $= -1$ if $k = \mathbb{R}$) , can be obtained in such a way from a reflection. We shall write

$$R = R_{v,\theta}$$

Some properties of the reflections are contained in the following

Proposition. Let $\gamma \in A(E)$, $a \in E$ and $\kappa_a(\gamma) = (R,v)$. Then

1) $\gamma$ is a reflection iff $R$ is a reflection and $v \perp H_R$.

2) If $\gamma$ is a reflection and $R = R_{e,\theta}$ then

$$H_\gamma = a + H_R + (1 - \theta)^{-1} v.$$

3) $R_{e,\theta}\, v = v - (1 - \theta) \langle v | e \rangle e$.

4) If $\gamma$ is a reflection and $\delta$ is a motion then $\delta\gamma\delta^{-1}$ is a reflection.

Proof is left to the reader. □



1.3. Main problem.

We shall say that a subgroup $W$ of $A(E)$ is an r-group if it is discrete and generated by affine reflections.

If $E$ and $E'$ are two affine spaces and $G \subseteq A(E)$ and $G' \subseteq A(E')$ are two arbitrary subgroups, then we shall say that $G$ and $G'$ are equivalent if there exists an affine bijection $\phi: E \to E'$ such that

$$W' = \phi W \phi^{-1}.$$

This means that after identifying $E$ and $E'$ by means of an arbitrary fixed isomorphism, the groups $G$ and $G'$, as subgroups of $A(E)$, have to be conjugate in $A(E)$.

We want to emphasize here that even when $E$ and $E'$ are affine metric spaces, $\phi$ need not to be distance preserving.

Our main concern in these lectures will be to classify r-groups up to equivalence.

We shall show now that in solving this problem one can restrict attention to irreducible groups.

1.4. Irreducibility.

Let $W$ be a subgroup of $A(E)$. We shall say that $W$ is reducible if there exist affine metric spaces $E_j$, $j=1,\dots,m, m \geqslant 2$, and subgroups $W_j$ of $A(E_j)$ such that $W$ is equivalent to $W_1 \times \dots \times W_m \subseteq A(E_1 \times \dots \times E_m)$. Otherwise $W$ is called irreducible. Clearly, every group is equivalent to a product of irreducible groups (but its decomposition need not be unique).

Theorem. Let $W \subseteq A(E)$ be a nontrivial subgroup generated by affine reflections (possibly not discrete). Then:

a) $W$ is equivalent to a product $W_1 \times \dots \times W_m$.

where $W_j$, $1 \leqslant j \leqslant m$, are irreducible groups, and $W_j$ is either generated by affine reflections or trivial (hence 1-dimensional), but not all $W_j$ are trivial.

b) W is irreducible iff Lin W is an irreducible linear group (generated by linear reflections).

c) $W_j$, $1 \leq j \leq m$, are uniquely defined up to equivalence and order.

d) Every product of the type described in a) is a group generated by reflections.

Proof: a) The statement follows from the equality

$$H_{(\gamma_1,\dots,\gamma_m)} = H_{\gamma_1} \times \dots \times H_{\gamma_m}$$

(hence $(\gamma_1,\dots,\gamma_m)$ is a reflection iff one and only one of $\gamma_j$, $1 \leq j \leq m$, is a reflection and the others are 1)

b) The "if" part is obvious. Let us prove the "only if" part.

As the group W is generated by reflections, the group Lin W lies in U(V). Therefore, Lin W is a completely reducible linear group. Let

$$V = \bigoplus_{j=1}^{m} V_j.$$

where $V_j$, $1 \leq j \leq m$, are irreducible Lin W-modules. Consider the subspaces

$$E_j = a + V_j, \qquad 1 \leq j \leq m,$$

where $a \in E$ is an origin, and let

$$\pi_j : W \to A(E_j), \quad 1 \leq j \leq m,$$

be the morphism given by the formula

$$\pi_j(\gamma) = \kappa_a^{-1} (\operatorname{Lin} \gamma \big|_{V_j}, p_j(\gamma(a) - a))$$

(here $p_j : V \to V_j$ is the natural projection). Let $W_j = \pi_j(W)$. Then



it is not difficult to check that the map

$$\phi: E \to E_1 \times \dots \times E_m,$$

given by the formula

$$\phi(q) = (a + p_1(q - a), \dots, a + p_m(q - a)), \; q \in E,$$

defines an equivalence of W and $W_1 \times \dots \times W_m$.

c) Suppose that $W \subset A(E)$ and $W' \subset A(E')$ are two equivalent groups generated by affine reflections and let $\phi: E \to E'$ establish the equivalence of these groups. Let $W = W_1 \times \dots \times W_r$ and $W' = W'_1 \times \dots \times W'_s$ be decompositions into products of irreducible groups and let $E = E_1 \times \dots \times E_r$, $E' = E'_1 \times \dots \times E'_s$, $V = V_1 \oplus \dots \oplus V_r$, $V' = V'_1 \oplus \dots \oplus V'_s$, be the corresponding decompositions of the affine spaces and its spaces of translations. We consider $W_j$ and $W'_l$ as subgroups of W and W' resp.

It is clear that Lin $\psi$ yields the equivalence of the linear groups Lin W and Lin W' ( in the usual sense). Hence $(\operatorname{Lin} \psi)V_j$ is a simple Lin W'-submodule of V' for every j.

Let $\gamma \in W$ be a reflection. Then $\gamma \in W_p$ for some p, see a) above. But $\psi\gamma\psi^{-1} = \gamma'$ is also a reflection (inside W'). Therefore $\gamma' \in W'_q$ for some q.

The root line of Lin $\gamma'$ is contained in $V'_q$ and this line is $(\operatorname{Lin} \psi)\ell$, where $\ell$ is a root line of Lin $\gamma$. Clearly, $\ell \subset V_p$. Hence

$$V'_q \cap (\operatorname{Lin} \psi)V_p \neq 0.$$

It follows now from the irreducibility that in fact

$$V'_q = (\operatorname{Lin} \gamma)V_p.$$

Therefore

$$\psi W_p \psi^{-1} \subset W'_q$$

(because $W_p$ is generated by all reflections $\tau$ for which Lin $\tau$ has its root line in $V_p$). The same proof is valid for the inverse inclusion. Hence,

$$\psi W_p \psi^{-1} = W'_q$$

and we can proceed by induction.

d) This is clear (see the equality in a)). □

Therefore we shall from now on only consider the case of irreducible r-groups.

1.5. What is already known: case $k = \mathbb{R}$.

Let W be an irreducible r-group in A(E).

One has two possibilities: either W is finite or W is infinite.

If W is finite then there exists a point in E which is fixed under W. Indeed, let $a \in E$ be an arbitrary point. Then

$$b = a + \frac{1}{|W|} \sum_{\gamma \in W} (\gamma(a) - a)$$

is fixed under W. Therefore $\kappa_b$ gives an isomorphism of W with Lin W, see Section (1.1), i.e. W is a linear group generated by reflections (we identify E and V by choosing the origin b).

We shall consider now the case $k = \mathbb{R}$.

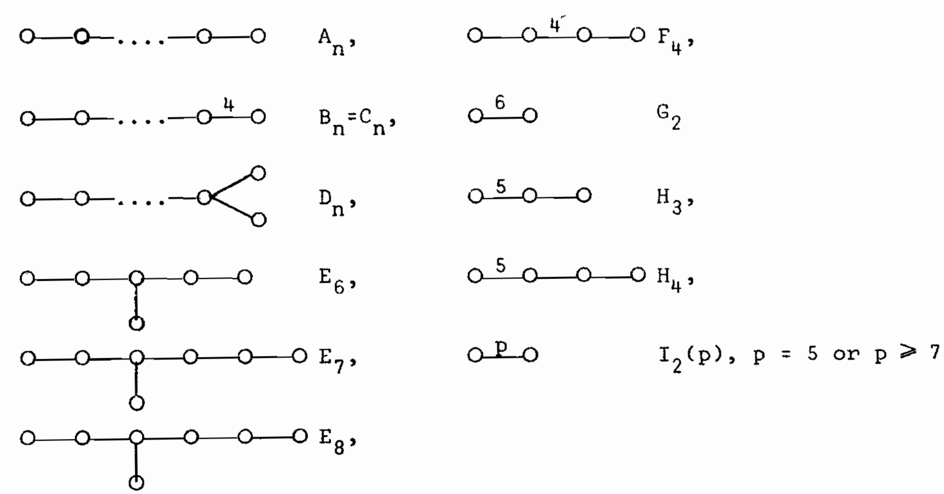
A beautiful classical theory concerning this case was developed by Coxeter, Witt, Stiefel, see [8], the results of which we shall recapitulate here.



1) W is finite.

W is either a Weyl group of an irreducible root system, or a dihedral group, or one of two exceptional groups $H_3$ or $H_4$.

The description of these groups is usually given by means of their Coxeter graphs. This is done in the following way. It is known that W is generated by n elements $R_j$, $1 \leqslant j \leqslant n$, which are reflections in the faces of a Weyl chamber C. There exists a unique set of vectors $e_j$, $1 \leqslant j \leqslant n$, of unit length with the property: $R_j = R_{e_j,\theta_j}$ (where in fact $\theta_j = -1$) and $C = \{v \in V | \langle e_j | v \rangle > 0,\ 1 \leqslant j \leqslant n\}$. The angle between the mirrors $H_{R_i}$ and $H_{R_j}$ is of the form $\frac{\pi}{m_{ij}}$, $m_{ij} \in \mathbb{Z}$ ; $m_{ij} \geqslant 2$. The nodes of the Coxeter graph of W are in bijective correspondence with the reflections $R_j$, $1 \leqslant j \leqslant n$. Two nodes $R_i$ and $R_j$ are connected by an edge iff $m_{ij} \geqslant 3$. The weight of this edge is equal to $m_{ij}$ (if $m_{ij} = 3$ then the weight is usually omitted). The complete list of finite irreducible r-groups is given by the following table of Coxeter graphs.



Let us consider the numbers

$$c_{ij} = (1 - \theta_i)(1 - \theta_j) < e_i | e_j >< e_j | e_i > = 4 \cos^2 \frac{\pi}{m_{ij}}.$$

One can change the weight $m_{ij}$ to the numbers $c_{ij}$ for all of the edges of the Coxeter graph. In this manner another weighted graph results. Clearly, one graph determined the other. We shall show later the newly obtained graphs can be generalized to the complex

2) W is infinite.

W is an affine Weyl group of an irreducible root system.

This group is a semidirect product of Lin W, which is a (finite) Weyl group of a certain root system R, and the lattice of rank n generated by the dual root system $\check{R}$. The groups Lin W, thus obtained are distinguished from the others of the list above in the following way (see [1], Ch. VI):

Theorem. Let $K \subset GL(V)$ be an irreducible finite r-group. Then the following properties are equivalent:

a) K = Lin W where W is an infinite r-group.

b) There exists a K-invariant lattice in V of rank n.

c) K is defined over $\mathbb{Q}$.

d) K is the Weyl group of a certain irreducible root system, i.e. a group whose Coxeter graph is one of $A_n$, $B_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$.

e) All the numbers $c_{ij}$ lie in $\mathbb{Z}$.

f) The ring with unity, generated over $\mathbb{Z}$ by all of the numbers $c_{ij}$, coincides with $\mathbb{Z}$.



1.6. What is already known: case $k = \mathbb{C}$.

Let $k = \mathbb{C}$ and $W \subset A(E)$ be an irreducible r-group.

1) W is finite.

Shephard and Todd, [2], gave the complete list of such groups. A modern and unified approach was presented by Cohen, [3].

We shall describe this classification in a form that is more convenient for us, i.e. by means of certain graphs (as it was done in the real case).

Let $R_j = R_{e_j,\theta_j}$, $1 \leq j \leq s$, be a generating system of reflections of W. We can assume that

$$\theta_j = e^{\frac{2\pi i}{m_j}}.$$

Therefore this system (and hence W) is uniquely defined by the system of lines $\ell_{R_j}$ in V with the multiplicities $m_j$, $1 \leq j \leq s$.

It is well known that an arbitrary set of vectors in V is uniquely (up to isometry) defined by means of a certain set of numbers (more precisely by the corresponding Gram matrix). Let us show that the same is true for an arbitrary set of lines in V (i.e. points of the corresponding projective space).

Proposition. Let $\{\ell_j\}_{j \in J}$ and $\{\ell'_j\}_{j \in J}$ be two sets of lines in V. and let $e_j \in \ell_j$, $e'_j \in \ell'_j$ be arbitrary vectors with $1 = < e_j | e_j > = < e'_j | e'_j >$. For any finite set of indices $j_1, \ldots, j_m \in J$ let us consider the numbers

$$h_{j_1 \ldots j_m} = < e_{j_1} | e_{j_2} >< e_{j_2} | e_{j_3} > \ldots < e_{j_{m-1}} | e_{j_m} >< e_{j_m} | e_{j_1} >$$

and

$$h'_{j_1 \ldots j_m} = < e'_{j_1} | e'_{j_2} >< e'_{j_2} \; e'_{j_3} > \ldots < e'_{j_{m-1}} | e'_{j_m} >< e'_{j_m} | e'_{j_1} > .$$

Then $h_{j_1..j_m}$, resp. $h'_{j_1..j_m}$, is independent of the choice of the vectors $e_j$, resp. $e'_j$, $j \in J$. Moreover, the systems $\{\ell_j\}_{j\in J}$ and $\{\ell'_j\}_{j\in J}$ are isometric (i.e. $g\ell_j = \ell'_j$, $j \in J$, for a certain $g \in U(V)$) if and only if

$$h_{j_1..j_m} = h'_{j_1..j_m} \qquad (*)$$

for arbitrary $j_1,...j_m \in J$.

Proof. We need only prove that if (*) is fulfilled then for every $i \in J$ there exists a number $\lambda_i \in \mathbb{C}$ such that

$$< e'_j | e'_k > = < \lambda_j e_j | \lambda_k e_k >$$

for every $j,k \in J$, i.e. the systems $\{e'_j\}_{j\in J}$ and $\{\lambda_j e_j\}_{j\in J}$ have the same Gram matrix (all the other statements are evident).

Let us fix an index $t \in J$. It is not difficult to see that one can assume that the system $\{\ell_j\}_{j\in J}$ is "connected", i.e. there exitsts for every $j \in J$ a sequence $j_1,..,j_m \in J$ such that $j_1 = j$, $j_m = t$ and $< e_{j_s} | e_{j_{s+1}} > \neq 0$, $s = 1, ..,m-1$. It is now a matter of straightforward computation to check that one can take

$$\lambda_j = \prod_{s=1}^{m-1} \frac{< e'_{j_{s+1}} | e'_{j_s} >}{< c_{j_{s+1}} | e_{j_s} >} .$$

We have seen above that an r-group is defined by a system $\{\ell_j, m_j\}_{j\in J}$ of lines $\ell_j \in V$ with the multiplicities $m_j \in \mathbb{Z}$. It follows from the proposition that such a system is uniquely (up to isometry) defined by a system of numbers

$$c_{j_1..j_m} = h_{j_1..j_m} \prod_{s=1}^{m} (1 - e^{2\pi i/m_{j_s}})$$

(one can derive from these numbers the multiplicities because $c_j = 1 - e^{2\pi i/m_j}$). It will become clear later on why $h_{j_1..j_m}$ is multiplied by $\prod_{s=1}^{m} (1 - e^{2\pi i/m_{j_s}})$ (and not, say by

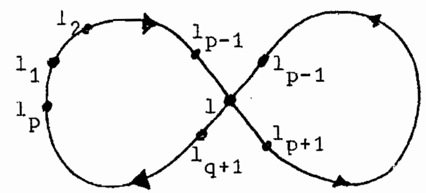


$\prod_{s=1}^{m}(e^{2\pi i/m_{j_s}})$); the numbers $c_{j_1...j_m}$ are of great importance in the whole theory. We call them cyclic products.

So the group W (with a fixed generating system of reflections) is uniquely (up to equivalence) defined by the corresponding set of cyclic products. As a matter of fact one only needs to know the so called simple cyclic products $c_{j_1..j_m}$, i.e. those with all indices $j_1,...j_m$ distinct, because

$$c_{l_1..l_{p-1} l\, l_{p+1}..l_{q-1} l\, l_{q+1}..l_r} =$$

$$= c_{l_1..l_{p-1} l\, l_{q+1}..l_r} \cdot c_{l\, l_{p+1}..l_{q-1}} .$$




We want to point out here several properties of the cyclic products.

a) If $j'_1,...j'_m$ is a cyclic permutation of $j_1,...j_m$ then

$$c_{j_1...j_m} = c_{j'_1...j'_m}.$$

In other words, $c_{j_1..j_m}$ depends only on a cycle $\sigma = (j_1,...,j_m)$. We use therefore the notation

$$c_\sigma = c_{j_1..j_m}.$$

In particular,

$$c_{j_k} = c_{k_j}$$

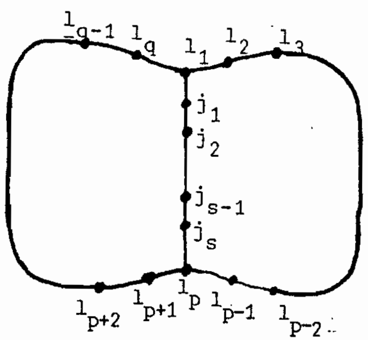
b) If $\sigma = (j_1,\dots j_m)$ then

$$c_\sigma c_{\sigma-1} = c_{j_1 j_2} c_{j_2 j_3} \dots c_{j_{m-1} j_m} c_{j_m j_1}$$

c) One can reconstruct all the simple cyclic products (hence all the cyclic products) only from the "homologically independent" ones. The following formula and drawing may illustrate what we have in mind:

$$c_{l_1 l_2 \dots l_p j_s j_{s-1} \dots j_1} \cdot c_{l_p l_{p+1} \dots l_{q-1} l_q l_1 j_1 j_2 \dots j_s} =$$

$$= c_{l_1 l_2 \dots l_{p-1} l_p l_{p+1} \dots l_{q-1} l_q} \cdot c_{l_1 j_1} \cdot c_{j_1 j_2} \dots c_{j_{s-1} j_s} \cdot c_{j_s l_p}$$

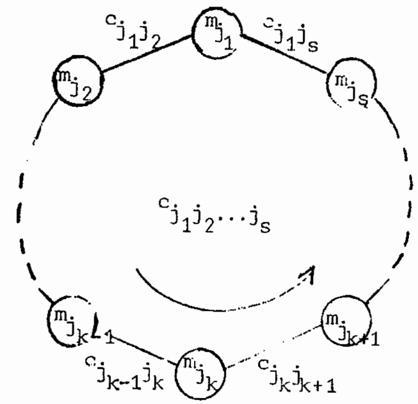



A system of lines $\ell_j \in V$ with the multiplicities $m_j$, $j \in J$, can be described by <u>a graph</u> as follows.

The nodes of the graph are in bijective correspondence with the lines $\ell_j$, $j \in J$. If a node represents the line $\ell_j$ then <u>this node has the weight</u> $m_j$. Two nodes $\ell_j$ and $\ell_k$ are connected by an edge



iff $c_{jk} \neq 0$ and if they are connected then <u>the weight of the edge</u> is equal to $c_{jk}$. <u>It will be convenient to omit the weight of the node, resp. the edge, if it is equal to</u> 2, <u>resp.</u> 1. Moreover, every simple cycle of this graph is supplied with an arbitrary (but fixed) orientation and has weight equal to the corresponding cycle product:




Therefore we now have a way <u>to represent a finite</u> r-<u>group</u> W ⊂ GL(V) <u>with a fixed generating system of reflections by means of a graph corresponding to the system of lines</u> $\ell_{R_j} \subset V$ <u>with the multiciplicities</u> $m_j$, $1 \leqslant j \leqslant s$. (Of course, using another system of generators one obtains <u>another</u> graph which represents <u>the same</u> group. This nonuniqueness in the representations of the group by means of its graph occurs because, contrary to the real case, there is no known canonical method for constructing a generating system of reflections of a finite complex r-group. The problem of finding such a method is still unsolved and seems to be very interesting). It is easy to see that, W being irreducible, the graph is <u>connected</u>. This graph is called <u>the graph of the group W</u>.

(with respect to a fixed generating system of reflections).

The classification of the finite complex irreducible r-groups W was given in [2], [3] by means of generating systems of reflections. It is now a matter of more of less straightforward computation to reformulate the result by means of the graphs. We need several notations to formulate the corresponding theorem:

Notation:

$\omega = e^{2\pi i/3}$

$\eta = e^{2\pi i/5}$ $\qquad\qquad$ $\zeta_m = e^{2\pi i/m}$

$\varepsilon = e^{2\pi i/8}$

It appears a posteriori that all the graphs under consideration are planar and either have no simple cycles or have only one such cycle (of length 3). We assume that this cycle is counter-clockwise oriented.

We also fix a numbering of the nodes of the graph (in an arbitrary fashion). The number of the node is written beside the node (but the weight of the node is written inside).

In the table below the ring with unity generated over $\mathbb{Z}$ by all cyclic products is also given. We need this ring later on; it plays an important role in the theory and does not depend on the choice of the generating system of reflections (and hence on the graph that represents the group).

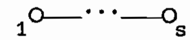
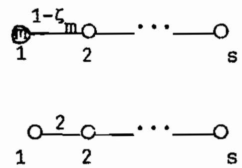
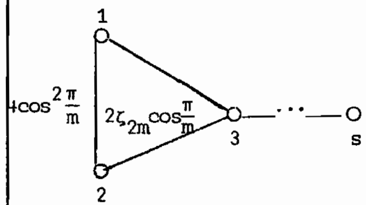
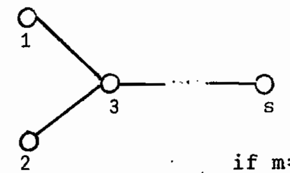
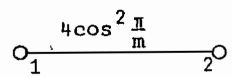
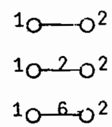
Theorem (classification of irreducible complex finite r-groups).

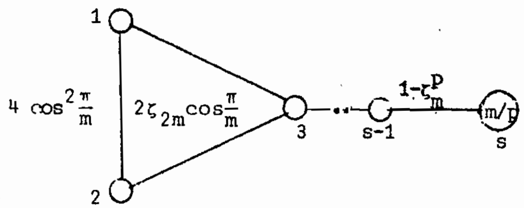
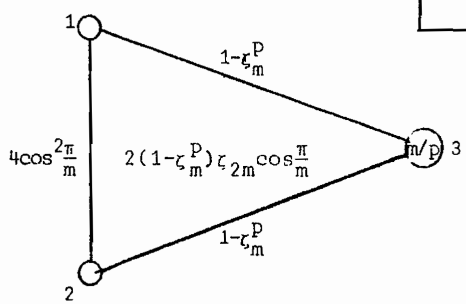
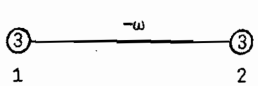
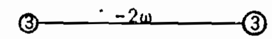
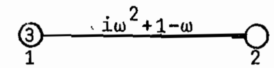
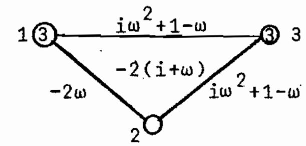
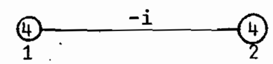
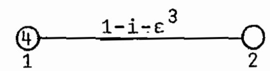
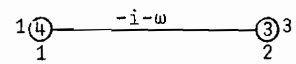
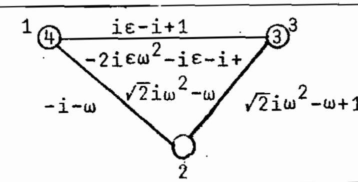
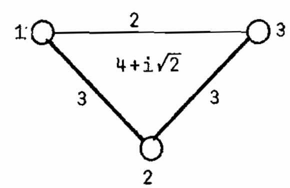
Any irreducible complex finite r-groups (with respect to a fixed generating system of reflections) corresponds to a graph of Table 1 (our numbering of the groups coincides with the one of Shephard and Todd [2]; the notation of types is as in [2],[3], [4]):

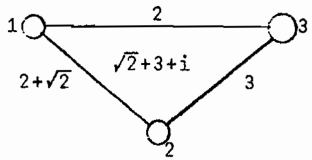
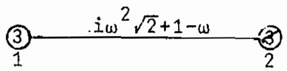
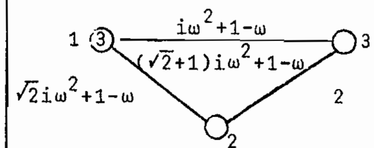
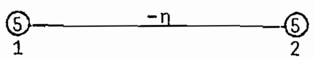
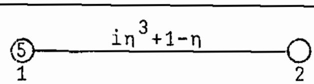
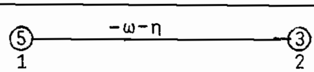
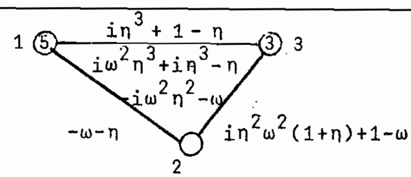
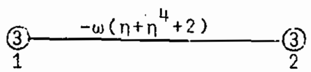


Table 1

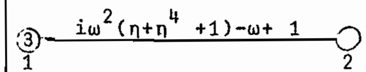
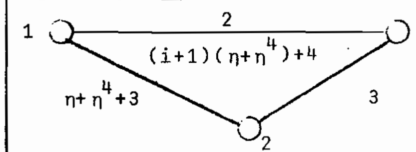
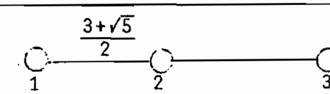
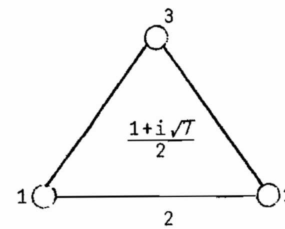
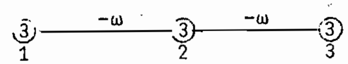
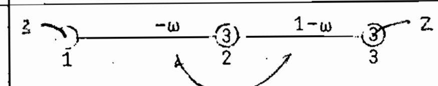
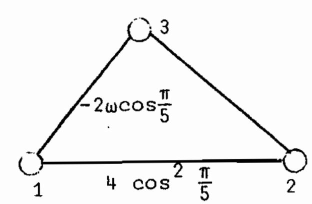
The irreducible complex finite r-groups.

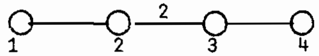
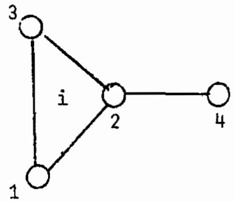
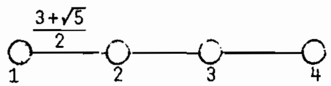
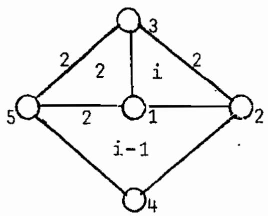
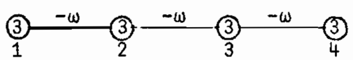
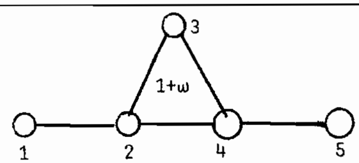
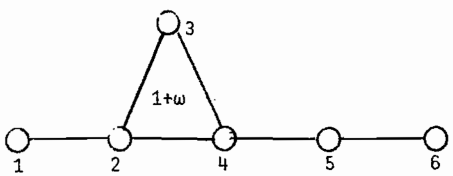
| No | Type | Graph | Ring generated by cyclic products | dim V |
|---|---|---|---|---|
| 1 | $A_s$<br>$s \geq 1$ |  | $\mathbb{Z}$ | s |
| 2 | $G(m,1,s)$<br>$m \geq 2, s \geq 2$,<br>type $B_s = C_s$<br>if $m = 2$ | <br>if m=2 | $\mathbb{Z}[e^{2\pi i/m}]$,<br>$\mathbb{Z}[\omega]$ if m=3,6,<br>$\mathbb{Z}[i]$ if m=4,<br>$\mathbb{Z}$ if m=2 | s |
| 2 | $G(m,m,s)$<br>$m \geq 2, s \geq 3$,<br>type $D_s$<br>if m=2 | <br><br>if m=2 | $\mathbb{Z}[e^{2\pi i/m}]$,<br>$\mathbb{Z}[\omega]$ if m=3,6,<br>$\mathbb{Z}[i]$ if m=4,<br>$\mathbb{Z}$ if m=2 | s |
| 2 | $G(m,m,2) =$<br>$= I_2(m), m \geq 3$,<br>$A_2$ if m=3,<br>$B_2$ if m=4,<br>$G_2$ if m=6 | <br><br>if m=3,<br>if m=4,<br>if m=6 | $\mathbb{Z}[4\cos^2\frac{\pi}{m}]$,<br>$\mathbb{Z}$ if m=3,4,6 | 2 |

| No | Type | Graph | Ring generated by cyclic products | dim V |
|---|---|---|---|---|
| 2 | $G(m,p,s-1)$<br>$m \geq 2$,<br>$s \geq 4$<br>$p \mid m$<br>$p \neq 1,m$ |  | $\mathbb{Z}[e^{2\pi i/m}]$,<br>$\mathbb{Z}[i]$ if $m=4, p=2$,<br>$\mathbb{Z}[\omega]$ if $m=6, p=2$ and $m=6, p=3$ | $s-1$ |
| 2 | $G(m,p,2)$<br>$m \geq 2$,<br>$p \mid m$,<br>$p \neq 1,m$ |  | $\mathbb{Z}[\zeta_m, 2\cos\frac{\pi}{m}, \zeta_m(1-\zeta_m^p)]$,<br>$\mathbb{Z}[2i]$ if $m=4, p=2$,<br>$\mathbb{Z}[\omega]$ if $m=6, p=2$,<br>$\mathbb{Z}[2\omega]$ if $m=6, p=3$ | 2 |
| 3 | $[\ ]^m$ |  | $\mathbb{Z}[e^{2\pi i/m}]$,<br>$\mathbb{Z}[\omega]$ if $m=6,3$,<br>$\mathbb{Z}[i]$ if $m=4$,<br>$\mathbb{Z}$ if $m=2$ | 1 |
| 4 | $3[3]3$ |  | $\mathbb{Z}[\omega]$ | 2 |
| 5 | $3[4]3$ |  | $\mathbb{Z}[\omega]$ | 2 |
| 6 | $3[6]2$ |  | $\mathbb{Z}[e^{2\pi i/12}]$<br>$\alpha\beta = \frac{1+\sqrt{3}}{2\sqrt{3}}$ | 2 |
| 7 | $\langle 3,3,2 \rangle_6$ |  | $\mathbb{Z}[e^{2\pi i/12}]$ | 2 |
| 8 | $4[3]4$ |  | $\mathbb{Z}[i]$<br>$\frac{i}{2}$ | 2 |
| 9 | $4[6]2$ |  | $\mathbb{Z}[\varepsilon]$<br>$\frac{1+\sqrt{2}}{2\sqrt{2}}$ | 2 |
| 10 | $4[4]3$ |  | $\mathbb{Z}[e^{2\pi i/12}]$<br>$\frac{1+\sqrt{3}}{2\sqrt{3}}$ | 2 |
| 11 | $\langle 4,3,2 \rangle_{12}$ |  | $\mathbb{Z}[e^{2\pi i/24}]$ | 2 |
| 12 | $GL(2,3)$ |  | $\mathbb{Z}[i\sqrt{2}]$ | 2 |

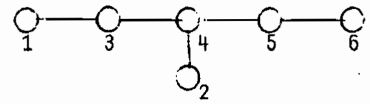
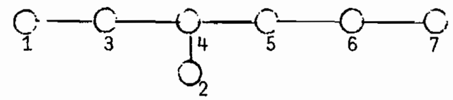
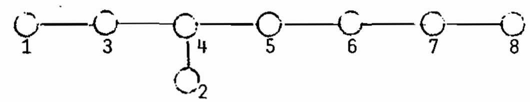
| No | Type | Graph | Ring generated by cyclic products | dim V |
|---|---|---|---|---|
| 13 | $< 4,3,2 >_2$ |   | $\mathbb{Z}[i,\sqrt{2}]$ | 2 |
| 14 | 3[8]2 |   | $\mathbb{Z}[\omega,i\sqrt{2}]$ $\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{3}}$ | 2 |
| 15 | $< 4,3,2 >_6$ |   | $\mathbb{Z}[i,\omega,\sqrt{2}]$ | 2 |
| 16 | 5[3]5 |   | $\mathbb{Z}[\eta]$ $\frac{1}{2(1-\cos\frac{2\pi}{5})} = \frac{1+\sqrt{5}}{2\sqrt{5}}$ | 2 |
| 17 | 5[6]2 |   | $\mathbb{Z}[e^{2\pi i/20}]$ $\frac{1}{2}+\frac{1}{2\sqrt{2-2\cos\frac{2\pi}{5}}}$ | 2 |
| 18 | 5[4]3 |   | $\mathbb{Z}[e^{2\pi i/15}]$ | 2 |
| 19 | $< 5,3,2 >_{30}$ |   | $\mathbb{Z}[e^{2\pi i/60}]$ | 2 |
| 20 | 3[5]3 |   | $\mathbb{Z}[\omega, \frac{1+\sqrt{5}}{2}]$ | 2 |



| No | Type | Graph | Ring generated by cyclic products | dim |
|---|---|---|---|---|
| 21 | 3[10]2 |   | $\mathbb{Z}[\omega,i\frac{1+\sqrt{5}}{2}]$ | 2 |
| 22 | $< 5,3,2 >_2$ |   | $\mathbb{Z}[\frac{\sqrt{5}-1}{2}, i\frac{\sqrt{5}-1}{2}]$ | 2 |
| 23 | $H_3$ |   | $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ | 3 |
| 24 | $J_3(4)$ |   | $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$ | 3 |
| 25 | $L_3$ |   | $\mathbb{Z}[\omega]$ | 3 |
| 26 | $M_3$ |   | $\mathbb{Z}[\omega]$ | 3 |
| 27 | $J_3(5)$ |   | $\mathbb{Z}[\omega, \frac{1+\sqrt{5}}{2}]$ | 3 |

| No | Type | Graph | Ring generated by cyclic products | dim V |
|---|---|---|---|---|
| 28 | $F_4$ |   | $\mathbb{Z}$ | 4 |
| 29 | $[2\ 1;1]^4$ $= N_4$ |   | $\mathbb{Z}[i]$ | 4 |
| 30 | $H_4$ |   | $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ | 4 |
| 31 | $[(\frac{1}{2}\gamma_3^4)^{+1}]$ $= EN_4$ |   | $\mathbb{Z}[i]$ | 4 |
| 32 | $L_4$ |   | $\mathbb{Z}[\omega]$ | 4 |
| 3 | $[2\ 1;2]^3$ $= K_5$ |   | $\mathbb{Z}[\omega]$ | 5 |
| 34 | $[2\ 1;3]^3$ $= K_6$ |   | $\mathbb{Z}[\omega]$ | 6 |



| No | Type | Graph | Ring generated by cyclic products | dim |
|---|---|---|---|---|
| 35 | $E_6$ |   | $\mathbb{Z}$ | 6 |
| 36 | $E_7$ |   | $\mathbb{Z}$ | 7 |
| 37 | $E_8$ |   | $\mathbb{Z}$ | 8 |

Remark. For those groups among them that are of the form Lin W, where W is an infinite irreducible complex r-group, one can obtain an explicit expression for the set of lines $\ell_{r_j}$, $1 \leqslant j \leqslant s$, from Table 2 below (by taking a vector $e_j$ of unit length in each $\ell_{R_r}$, $1 \leqslant j \leqslant s$). The explicit expressions of this kind for other groups may either be found in [3] or be derived directly from the graphs.

2) W is infinite

This case was not investigated earlier and is our main concern in these lectures. The results are formulated in the next section. Here we shall give only several trivial examples, which show first of all, that the groups under consideration do exist and, secondly, that we have here a phenomenon which does not occur in the real case.

Examples.

We consider the case where $n = \dim E = 1$.

Let $a \in E$ be a point. We identify $A(E)$ and $GL(V).V$ by means of $\kappa_a$, see Section 1.1.

Let $\Gamma \neq 0$ be a lattice in $V$. (here and further on a lattice means a discrete subgroup of the additive group of a vector space, not necessarily of maximal rank)

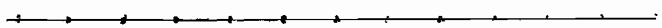
Let us consider a subgroup

$$W = \{(\pm 1,t) \mid t \in \Gamma\}$$

of $GL(V).V$. This subgroup acts on $E$ (see Section 1.1) and it is easy to check that it is an infinite irreducible complex r-group.

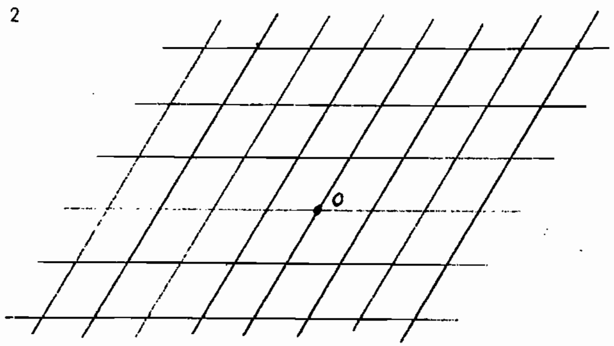
There are two possibilities: either $\mathrm{rk}\,\Gamma = 1$ or $\mathrm{rk}\,\Gamma = 2$.

If $\mathrm{rk}\,\Gamma = 1$




then $E/W$ is not compact, i.e., by definition, $W$ is a noncrystallographic group. In this case, $W$ can be viewed as a real r-group on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$.

If $\mathrm{rk}\,\Gamma = 2$

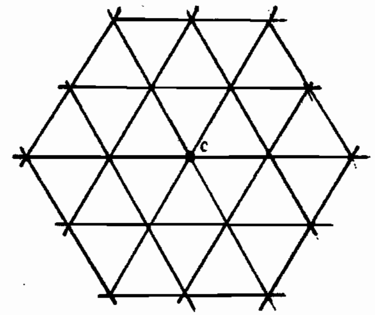



then $E/W$ is compact, i.e., by definition, $W$ is a crystallographic group.

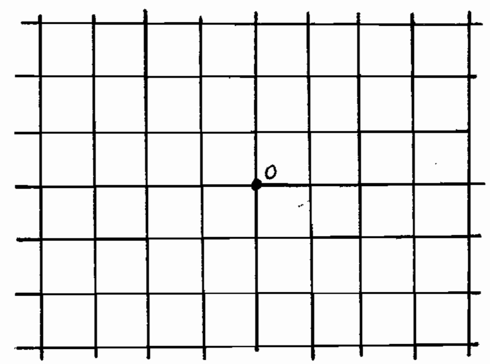


As we know, it is impossible for an infinite real r-group to be noncrystallographic, see Section 1.5.

Denote by $\Lambda$ a lattice of equilateral triangles in $V$ $(= \mathbb{C})$,




and by $L$ a lattice of squares in $V$ $(= \mathbb{C})$




Then it is not difficult to check that the groups

$$\{(\omega^l,t) \mid t \in \Lambda,\ l \in \mathbb{Z}\},$$

$$\{(\pm\omega^l,t) \mid t \in \Lambda,\ l \in \mathbb{Z}\}$$

and

$$\{(i^l,t) \mid t \in L,\ l \in \mathbb{Z}\}$$

are infinite irreducible complex crystallographic r-groups.

Exercise. Prove that all infinite irreducible complex 1-dimensional r-groups are equivalent to subgroups of those described in these examples.



## 2. Formulation of the results

We assume in this chapter that $k = \mathbb{C}$.

Let W be an irreducible infinite r-group, $W \subset A(E)$. As we have seen in the example above, there are two possibilities: either W is noncrystallographic (i.e. E/W is not compact) or W is crystallographic (E/W is compact). First, we shall describe the structure of noncrystallographic groups. To do this we need an auxiliary construction.

### 2.1. Complexifications and real forms.

Let us consider V as a real vector space (of dimension 2n). A linear subspace $V_{\mathbb{R}}$ of this real vector space is called a real form of V if

a) the natural map

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to V$$

is an isomorphism, i.e. some (hence, any) $\mathbb{R}$-basis of $V_{\mathbb{R}}$ is a $\mathbb{C}$-basis of V;

b) the restriction $\langle | \rangle \big|_{V_{\mathbb{R}}}$ of $\langle | \rangle$ to $V_{\mathbb{R}}$ is realvalued (hence $V_{\mathbb{R}}$ is euclidean with respect to $\langle | \rangle \big|_{V_{\mathbb{R}}}$).

If $V_{\mathbb{R}}$ is a real form of V then V is the complexification of $V_{\mathbb{R}}$.

Let $a \in E$ be a point. We can consider E as a real affine space of dimension 2n. The affine subspace

$$E_{\mathbb{R}} = a + V_{\mathbb{R}}$$

of this affine space is called a real form of E and E is called the complexification of $E_{\mathbb{R}}$.

It is clear that every real euclidean linear, resp. affine space is isomorphic to a real form of a certain complex hermitian linear, resp. affine space.

Proposition. One has the following properties:

1) $U(V)$ acts transitively on the set of real forms of $V$.

2) The group of motions of $E$ acts transitively on the set of real forms of $E$.

3) Every motion $\gamma$ of a euclidean affine space $E_{\mathbb{R}}$ can be extended in a unique way to a motion $\gamma_{\mathbb{C}}$ of $E$. This motion $\gamma_{\mathbb{C}}$ is called the complexification of $\gamma$ (and $\gamma$ is called the real form of $\gamma_{\mathbb{C}}$).

4) $\dim_{\mathbb{R}} H_{\gamma} = \dim_{\mathbb{C}} H_{\gamma_{\mathbb{C}}}$. Specifically, $\gamma$ is a reflection iff $\gamma_{\mathbb{C}}$ is a reflection.

Proof is left to the reader. □

This proposition gives a method for constructing noncrystallographic infinite r-groups. Indeed, let $G \subset A(E_{\mathbb{R}})$ be an infinite (real) r-group. Then it is easy to see that

$$G_{\mathbb{C}} = \{\gamma_{\mathbb{C}} \mid \gamma \in G\} \subset A(E)$$

is an infinite complex noncrystallographic r-group (and $G_{\mathbb{C}}$ is irreducible if and only if $G$ is).

2.2. Classification of infinite irreducible complex noncrystallographic r-groups: the result.

It appears that the construction above leads to any such group. More precisely, one has the following theorem (see also Section 1.5,2)):

Theorem. Let $W$ be an infinite irreducible complex r-group. Then $W$ is noncrystallographic if and only if it is equivalent to the complexification of an irreducible affine Weyl group.

Proof is given in Section 3.4.

The description of crystallographic groups is much more complicated. In order to give this description we need some preparations and extra notation.



2.3. Ingredients of the description.

The subgroup of translations in $W$ will be denoted by Tran $W$.

$$\text{Tran } W = W \cap \text{Tran } A(E),$$

cf. Section 1.1. It is clear that Tran $W \lhd W$ and

$$W/\text{Tran } W \cong \text{Lin } W.$$

We usually identify Tran $W$ with a subgroup of the additive group of $V$ by means of the map $\gamma_v \mapsto v$. Clearly this subgroup is a Lin $W$-invariant lattice in $V$.

It will be proven in Section 3.1 that Tran $W$ is a lattice of full rank (i.e. of rank $2n$) and Lin $W$ is a finite group (hence, Lin $W$ is a finite irreducible complex linear r-group, see 1.4). Therefore, to describe $W$, one needs to point out a group Lin $W \subset GL(V)$ from the Shephard and Todd list (i.e. from the theorem in 1.6), a Lin $W$-invariant lattice Tran $W \subset V$ of rank $2n$ and the way Lin $W$ and Tran $W$ are "glued" together. This is done below as follows:

1) Lin $W$ is given by its graph as in Section 1.6.

2) Tran $W$ is described explicitly by linear combinations of vectors $e_j$, $1 \leq j \leq s$, that generate Tran $W$. Here $R_j = R_{e_j,\theta_j}$, $1 \leq j \leq s$, is a fixed generating system of reflections of Lin $W$ which is related to the graph of Lin $W$ given in 1) as described in 1.6. To point out the vectors $e_j$, $1 \leq j \leq s$, explicitly, we assume that $V$ is a subspace of a standard hermitian infinitedimensional coordinate space $\mathbb{C}^{\infty}$ i.e. the space, whose elements are the sequences $(a_1,a_2,\ldots)$ with only a finite number of nonzero elements $a_j$, and a scalar product defined by the formula

$$< (a_1,a_2,\ldots) \mid (b_1,b_2,\ldots) > = \sum_{j=1}^{\infty} a_j\bar{b}_j .$$

The vectors $e_j$, $1 \leq j \leq s$, are given by their coordinates on a standard basis $\varepsilon_1, \varepsilon_2, \ldots$ of $\mathbb{C}^\infty$, where

$$\varepsilon_j = (0,\ldots,0,1,0,\ldots).$$

3) The problem how to describe the "glueing" of Lin W and Tran W comes down to the determination of an extension of Tran W by Lin W,

$$0 \to \text{Tran } W \to W \to \text{Lin } W \to 1.$$

Therefore it is done by means of cohomology. Let us show how it can be done.

2.4. Cohomology.

Let G be a subgroup of A(E) and write T = Tran G, K = Lin G. Choose a point $a \in E$. Take $P \in K$ and let $\gamma \in G$ be such that Lin $\gamma = P$. We have

$$\kappa_a(\gamma) = (P, s(P)),\ s(P) \in V.$$

It is easy to see that the map

$$\bar{s}\colon K \to V/T,\ \bar{s}(P) = s(P) + T,$$

is well defined and is in fact a 1-cocycle, i.e.

$$\bar{s}(PQ) = \bar{s}(P) + P\,\bar{s}(Q),\ P,Q \in K.$$

(here K acts on V/T in the natural way).

Vice versa, if

$$\bar{r}\colon K \to V/T$$

is an arbitrary 1-cocycle, let us consider an arbitrary map

$$r\colon K \to V$$



such that

$$\bar{r}(P) = r(P) + T,\ P \in K.$$

Then the set

$$\{(P, r(P) + t) \mid t \in T,\ P \in K\}$$

is a subgroup H of A(E) with Lin H = K and Tran H = T.

If we replace a by an other point $b \in E$, then (see 1.1).

$$\kappa_b(\gamma) = \kappa_a(\gamma_{a-b}\,\gamma\,\gamma_{b-a}) = (P, s(P) + \underbrace{v - Pv}_{\text{1-coboundary}}),\ \text{where } v = a-b.$$

Therefore we have a bijection between the set of Tran A(E)-conjugacy classes of subgroups G of A(E) with Lin G = K, Tran G = T and the group $H^1(K,V/T)$.

However we have to consider subgroups of A(E) up to equivalence, i.e. up to A(E)-conjugation (and not just up to Tran A(E)-conjugation)! This can be done as follows by means of an extra relation on $H^1(K,V/T)$. Let

$$N(K,T) = \{Q \in GL(V) \mid QKQ^{-1} = K,\ QT = T\}.$$

If $Q \in N(K,T)$ and $\bar{s}\colon K \to V/T$ is a 1-cocycle, resp. 1-coboundary, then it is easy to check that the map

$$Q(\bar{s})\colon K \to V/T$$

given by the formula

$$Q(\bar{s})(P) = Q\bar{s}(Q^{-1}PQ),\ P \in K,$$

is again a 1-cocycle, resp. 1-coboundary (here Q acts on V/T in the natural way). Therefore we have an action of N(K,T) on $H^1(K,V/T)$ (clearly, by means of automorphisms).

Let $\delta \in A(E)$ be such that $\text{Lin } \delta G \delta^{-1} = K$, $\text{Tran } \delta G \delta^{-1} = T$. We want to calculate the cocycle that corresponds to $\delta G \delta^{-1}$. Changing $\delta$ to $\delta \gamma_v$, where $v = (\text{Lin } \delta^{-1})(a - \delta(a))$, we can assume that $\kappa_a(\delta) = (Q,0)$, $Q \in N(K,T)$. Let $P \in K$ and $\lambda \in G$ be such that $\kappa_a(\lambda) = (Q^{-1}PQ, s(Q^{-1}PQ))$. Then $\kappa_a(\delta\lambda\delta^{-1}) = (P, Qs(Q^{-1}PQ))$. Therefore the cocycle corresponding to $\delta G \delta^{-1}$ is $Q(\bar{s})$ where $\bar{s}$ is the cocycle corresponding to $G$.

We see now that there is a bijection between the set of classes of equivalent subgroups $G \subset A(E)$ with $\text{Lin } G = K$, $\text{Tran } G = T$ and the set of $N(K,T)$-orbits in $H^1(K,V/T)$.

With all these facts in mind, we determine the extension $W$ (of $\text{Tran } W$ by $\text{Lin } W$) by pointing out a 1-cocycle which represents the corresponding element of $H^1(\text{Lin } W, V/\text{Tran } W)$ (in fact, the whole $N(\text{Lin } W, \text{Tran } W)$-orbit in $H^1(\text{Lin } W, V/\text{Tran } W)$). In order to do so, we need only give the values of this 1-cocycle on the elements of a generating system of reflections of $\text{Lin } W$. Technically it is more convenient to realize it as follows.

Let $\widetilde{\text{Lin } W}$ be a free group with generators $r_j$, $1 \leqslant j \leqslant s$. We have an epimorphism $\phi: \widetilde{\text{Lin } W} \to \text{Lin } W$, $\phi(r_j) = R_j$, $1 \leqslant j \leqslant s$. The kernel of $\phi$ is the subgroup of "relations" of $\text{Lin } W$. This epimorphism leads in a natural way to an action of $\widetilde{\text{Lin } W}$ on $V$. A 1-cocycle $c$ of $\widetilde{\text{Lin } W}$ with values in $V$ is given by its values on the generators $r_j$,

$$c(r_j), \quad 1 \leqslant j \leqslant s,$$

and these values may be arbitrary (because $\widetilde{\text{Lin } W}$ is free). It is easy to see that the formula

$$R_j \to c(r_j) + \text{Tran } W, \quad 1 \leqslant j \leqslant s,$$

defines a 1-cocycle of $\text{Lin } W$ with values in $V/\text{Tran } W$ iff $c(F) \in \text{Tran } W$ for every $F \in \text{Ker } \phi$. It is also clear that every



1-cocycle of $\text{Lin } W$ with values in $V/\text{Tran } W$ is obtained in such a way.

We shall give the extension $W$ (of $\text{Tran } W$ by $\text{Lin } W$) by writing down the vectors $c(r_j)$, $1 \leqslant j \leqslant s$.

We are now ready to formulate the results of the classification of infinite irreducible crystallographic r-groups.

Denote by $K_b$ the finite linear irreducible r-group which has the number $b$ in the list of Shephard and Todd (i.e. in the first column of Table 1. This in spite of the slight confusion with Cohen's notation $K_5, K_6$).

2.5. Description of the group of linear parts: the result. First of all, there is an analogue of the theorem of Section 1.5.

Theorem. Let $K \subset GL(V)$ be an irreducible finite r-group. Then the following properties are equivalent:

a) There exists a nonzero $K$-invariant lattice in $V$.

b) There exists a $K$-invariant lattice of rank $2n$ in $V$.

c) $K = \text{Lin } W$ where $W$ is an infinite crystallographic r-group.

d) The ring with unity, generated over $\mathbb{Z}$ by all cyclic products of a graph of $K$, lies in the ring of algebraic integers of a purely imaginary quadratic extension of $\mathbb{Q}$.

e) $K$ is defined over a purely imaginary quadratic extension of $\mathbb{Q}$.

f) $K$ is one of the groups:

$K_1$; $K_2$ $(m=2,3,4,6)$; $K_3$ $(m=2,3,4,6)$;

$K_4$; $K_5$; $K_8$; $K_{12}$; $K_{24}$; $K_{25}$; $K_{26}$; $K_{28}$; $K_{29}$;

$K_{31}$; $K_{32}$; $K_{33}$; $K_{34}$; $K_{35}$; $K_{36}$; $K_{37}$.

Proof is given in Section 4.6.

Now we shall describe the crystallographic groups themselves.

2.6. The list of irreducible infinite crystallographic complex groups.

This list is given in the following theorem (we use the notation: $\Omega = \{z \in \mathbb{C} \mid -\frac{1}{2} \leqslant \mathrm{Re}\, z < \frac{1}{2},\ |z| \geqslant 1$ if $\mathrm{Re}\, z \leqslant 0$ and $|z| > 1$ if $\mathrm{Re}\, z > 0\}$ - this is the "modular strip"; $[\alpha,\beta] = \{a\alpha + b\beta \mid a,b \in \mathbb{Z}\}$ for arbitrary $\alpha,\beta \in \mathbb{C}$).

Theorem. The following list is the complete list of irreducible infinite crystallographic complex r-groups W (considered up to equivalence).

The proof is given in the subsequent chapters.



The irreducible infinite crystallographic complex r-groups.

Table 2

| Notation of W | n = dim W | Lin W | Tran W | $e_1,\dots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[A_s]^{\alpha}$, $s \geqslant 1$ | s | $K_1$, type $A_s$, $s \geqslant 1$ | $[1,\alpha]e_1+\dots+[1,\alpha]e_s$, $\alpha \in \Omega$ | $e_j=(\varepsilon_j-\varepsilon_{j+1})/\sqrt{2}$, $j = 1,\dots,s$ | |
| $[G(2,1,s)]_1^{\alpha}$, $s \geqslant 3$ | s | $K_2$, type $G(2,1,s)$, $s \geqslant 3$ | $[1,\alpha]e_1+[1,\alpha]\sqrt{2}e_2+\dots+[1,\alpha]\sqrt{2}e_s$, $\alpha \in \Omega$ | $e_1 = \varepsilon_1$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s$ | $c = 0$ |
| $[G(2,1,s)]_2^{\beta}$, $s \geqslant 3$ | | | $[1,\beta]e_1+[1,\frac{1+\beta}{2}]\sqrt{2}e_2+\dots+[1,\frac{1+\beta}{2}]\sqrt{2}e_s$, $\beta \in \Omega$ | | |
| $[G(2,1,s)]_3^{\gamma}$, $s \geqslant 3$ | | | $[1,\gamma]e_1+[\frac{1}{2},\gamma]\sqrt{2}e_2+\dots+[\frac{1}{2},\gamma]\sqrt{2}e_s$, $\gamma \in \Omega$ | | |
| $[G(2,1,s)]_4^{\delta}$, $s \geqslant 3$ | | | $[1,\delta]e_1+[1,\frac{\delta}{2}]\sqrt{2}e_2+\dots+[1,\frac{\delta}{2}]\sqrt{2}e_s$, $\delta \in \Omega$ | | |
| $[G(2,1,s)]_5^{\lambda}$, $s \geqslant 3$ | | | $[1,\lambda]e_1+[\frac{1}{2},\frac{\lambda}{2}]\sqrt{2}e_2+\dots+[\frac{1}{2},\frac{\lambda}{2}]\sqrt{2}e_s$, $\lambda \in \Omega$ | | |
| $[G(3,1,s)]_1$, $s \geqslant 2$ | s | $K_2$, type $G(3,1,s)$, $s \geqslant 2$ | $[1,\omega]e_1+[1,\omega]\sqrt{2}e_2+\dots+[1,\omega]\sqrt{2}e_s$ $= \mathbb{Z}[Ge_1]$ | | |
| $[G(3,1,s)]_2$, $s \geqslant 2$ | | | $[1,\omega]e_1+[1,\omega]i\sqrt{\frac{2}{3}}e_2+\dots+[1,\omega]i\sqrt{\frac{2}{3}}e_s$ $\simeq \mathbb{Z}[G(ke_2)]$ | | |

| Notation of W | n = dim W | Lin W | Tran W | $e_1,\dots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[G(4,1,s)]_1$ $s \geq 2$ | s | $K_2$ type $G(4,1,s)$ $s \geq 2$ | $[1,i]e_1+[1,i]\sqrt{2}e_2+\dots+[1,i]\sqrt{2}e_s$ $\mathbb{Z}[Ge_1]$ | $e_1 = \varepsilon_1$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s$ | c = 0 |
| $[G(4,1,s)]_2$ $s \geq 2$ | | | $[1,i]e_1+[1,i]\varepsilon e_2+\dots+[1,i]\varepsilon e_s$ $\mathbb{Z}[G(\varepsilon e_2)]$ | | |
| $[G(6,1,s)]$ $s \geq 2$ | s | $K_2$ type $G(6,1,s)$ $s \geq 2$ | $[1,\omega]e_1+[1,\omega]\sqrt{2}e_2+\dots+[1,\omega]\sqrt{2}e_s$ | | |
| $[G(2,2,s)]^\alpha$ $s \geq 3$ | s | $K_2$ $G(2,2,s)$ $s \geq 3$ | $[1,\alpha]e_1+\dots+[1,\alpha]e_s$, $\alpha \in \Omega$ | $e_1=-(\varepsilon_1+\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s$ | |
| $[G(3,3,s)]$ $s \geq 3$ | s | $K_2$, type $G(3,3,s)$ $s \geq 3$ | $[1,\omega]e_1+\dots+[1,\omega]e_s$ | $e_1=\omega\varepsilon_1-\varepsilon_2$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s$ | |
| $[G(4,4,s)]$ $s \geq 3$ | s | $K_2$ type $G(4,4,s)$ $s \geq 3$ | $[1,i]e_1+\dots+[1,i]e_s$ | $e_1=(i\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s$ | |



| Notation of W | n = dim W | Lin W | Tran W | $e_1,\dots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[G(6,6,s)]$ $s \geq 3$ | s | $K_2$, type $G(4,4,s)$ $s \geq 3$ | $[1,\omega]e_1+\dots+[1,\omega]e_s$ | $e_1=((1+\omega)\varepsilon_1-\varepsilon_2)/\sqrt{2}$ $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$ $j=2,\dots,s$ | c = 0 |
| $[G(2,1,2)]_1^\alpha$ | 2 | $K_2$, type $G(2,1,2)$ = type $G(4,4,2)$ | $[1,\alpha]e_1+[1,\alpha]\sqrt{2}e_2$, $\alpha \in \Omega$ | $e_1=\varepsilon_1$ $e_2=(\varepsilon_1-\varepsilon_2)/\sqrt{2}$ | |
| $[G(2,1,2)]_2^\beta$ | | | $[1,\beta]e_1+[1,\frac{\beta}{2}]\sqrt{2}e_2$, $\beta \in \Omega$ | | |
| $[G(2,1,2)]_3^\gamma$ | | | $[1,\gamma]e_1+[1,\frac{1+\gamma}{2}]\sqrt{2}e_2$, $\gamma \in \Omega$ | | |
| $[G(6,6,2)]_1^\alpha$ | 2 | $K_2$, type $G(6,6,2)$ | $[1,\alpha]e_1+[1,\alpha](2+\omega)e_2$, $\alpha \in \Omega$ | $e_1=((1+\omega)\varepsilon_1-\varepsilon_2)/\sqrt{2}$ $e_2=(\varepsilon_1-\varepsilon_2)/\sqrt{2}$ | |
| $[G(6,6,2)]_2^\beta$ | | | $[1,\beta]e_1+[1,\frac{\beta}{3}](2+\omega)e_2$, $\beta \in \Omega$ | | |
| $[G(6,6,2)]_3^\gamma$ | | | $[1,\gamma]e_1+[1,\frac{1+\gamma}{3}](2+\omega)e_2$, $\gamma \in \Omega$ | | |
| $[G(6,6,2)]_4^\delta$ | | | $[1,\delta]e_1+[1,\frac{2+\delta}{3}](2+\omega)e_2$, $\delta \in \Omega$ | | |

| Notation of W | n = dim W | Lin W | Tran W | $e_1,\dots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[G(4,2,s-1)]_1$, $s \geq 3$ | s-1 | $K_2$, type $G(4,2,s-1)$, $s \geq 3$ | $T = [1,i]e_1+\dots+[1,i]e_{s-1}$ | $e_1=(i\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$ | c = 0 |
| $[G(4,2,s-1)]^*_1$, $s \geq 3$ | s-1 | $K_2$, type $G(4,2,s-1)$, $s \geq 3$ | $T = [1,i]e_1+\dots+[1,i]e_{s-1}$ | $e_1=(i\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$ | $c(r_j)=0$, $j=1,\dots,s-1$, $c(r_s)=e_s/\sqrt{2}$ |
| $[G(4,2,s-1)]_2$, $s \geq 3$ | s-1 | $K_2$, type $G(4,2,s-1)$, $s \geq 3$ | $T \cup (T + \frac{1+i}{2}(e_1 + e_2)) = [1,i]e_1+\dots+[1,i]e_{s-1}+\frac{1}{\sqrt{2}}[1,i]e_s$ | $e_1=(i\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$ | c = 0 |
| $[G(4,2,2)]_3$ | 2 | $K_2$, type $G(4,2,2)$ | $[1,i]e_1 + [1,i](1+i)e_2$ | $e_1=(i\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$ | c = 0 |
| $[G(6,2,s-1)]_1$, $s \geq 3$ | s-1 | $K_2$, type $G(6,2,s-1)$, $s \geq 3$ | $[1,\omega]e_1+\dots+[1,\omega]e_{s-1}$ | $e_1=((1+\omega)\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$. | c = 0 |
| $[G(6,2,2)]_2$ | 2 | $K_2$, type $G(6,2,2)$ | $[1,\omega]e_1+[1,\omega](2+\omega)e_2$ | $e_1=((1+\omega)\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$. | c = 0 |



| Notation of W | n = dim W | Lin W | Tran W | $e_1,\dots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[G(6,3,s-1)]_1$, $s \geq 3$ | s-1 | $K_2$, type $G(6,3,s-1)$, $s \geq 3$ | $[1,\omega]e_1+\dots+[1,\omega]e_{s-1}$ | $e_1=((1+\omega)\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$ | c=0 |
| $[G(6,3,2)]_2$ | 2 | $K_2$, type $G(6,3,2)$ | $[1,2\omega]e_1+[2,\omega]e_2$ | $e_1=((1+\omega)\varepsilon_1-\varepsilon_2)/\sqrt{2}$, $e_j=(\varepsilon_{j-1}-\varepsilon_j)/\sqrt{2}$, $j=2,\dots,s-1$, $e_s=\varepsilon_{s-1}$ | c=0 |
| $[K_3(3)]$ $J_6/\mathbb{Z}_3\times\mathbb{Z}_3$ $\omega_1=\omega_2$ | 1 | $K_3$, m=3 | $[1,\omega]e_1$ | $e_1=\varepsilon_1$ | c=0 |
| $[K_3(4)]$ $X_8/\mathbb{Z}_4\times\mathbb{Z}_2$ | 1 | $K_3$, m=4 | $[1,i]e_1$ | $e_1=\varepsilon_1$ | c=0 |
| $[K_3(6)]$ $J_6/\mathbb{Z}_6$ $A_2^{(6)}$ | 1 | $K_3$, m=6 | $[1,\omega]e_1$ | $e_1=\varepsilon_1$ | c=0 |
| $[K_4]$ | 2 | $K_4$ | $[1,\omega]e_1+[1,\omega]e_2$ | $e_1=\varepsilon_1$, $e_2=\frac{1-\omega}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ | c=0 |
| $[K_5]$ | 2 | $K_5$ | $[1,\omega]e_1+[1,\omega]\sqrt{2}e_2$ | $e_1=\varepsilon_1$, $e_2=\frac{1-\omega}{3}(\sqrt{2}\varepsilon_1+\varepsilon_2)$ | c=0 |

| Notation of W | n = dim W | Lin W | Tran W | $e_1,\ldots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[K_8]$ $A_3^{(4)}$ | 2 | $K_8$ | $[1,i]e_1+[1,i]e_2$ | $e_1=\varepsilon_1$, $e_2=\frac{1-i}{2}(\varepsilon_1-\varepsilon_2)$ | c=0 |
| $[K_{12}]$ | 2 | $K_{12}$ | $[1,i\sqrt{2}]e_1+[1,i\sqrt{2}]e_2$ | $e_1=\frac{1}{\sqrt{2}}\varepsilon_1+\frac{1+i}{2}\varepsilon_2$, $e_2=\frac{\sqrt{2}+(\sqrt{2}-2)i}{4}\varepsilon_1+\frac{2+\sqrt{2}-\sqrt{2}i}{4}\varepsilon_2$, $e_3=\frac{1}{\sqrt{2}}\varepsilon_1+\frac{1-i}{2}\varepsilon_2$ | c=0 |
| $[K_{12}]^*$ | | | | | $c(r_1)=c(r_2)=0$, $c(r_3)=\frac{1+i}{2}e_3$ |
| $[K_{24}]$ | 3 | $K_{24}$ | $[1,\frac{1+i\sqrt{7}}{2}]\,{}_1+[1,\frac{1+i\sqrt{7}}{2}]e_2+$ $+[1,\frac{1+i\sqrt{7}}{2}]e_3$ | $e_1=\varepsilon_2$, $e_2=(1-i\sqrt{7})(\varepsilon_2+\varepsilon_3)/4$, $e_3=(-\varepsilon_1-\varepsilon_2+\frac{1+i\sqrt{7}}{2}\varepsilon_3)/2$ | c=0 |
| $[K_{25}]$ | | $K_{25}$ | $[1,\omega]e_1+[1,\omega]e_2+[1,\omega]e_3$ | $e_1=\varepsilon_3$, $e_2=\frac{1-\omega}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$, $e_3=-\omega\varepsilon_2$ | |
| $[K_{26}]_1$ | | $K_{26}$ | $[1,\omega]e_1+[1,\omega]e_2+[1,\omega]\sqrt{2}e_3$ | $e_1=\frac{1-\omega^2}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$, $e_2=\varepsilon_3$, $e_3=\frac{1}{\sqrt{2}}(\varepsilon_2-\varepsilon_3)$ | |
| $[K_{26}]_2$ | | | $[1,\omega]e_1+[1,\omega]e_2+[1,\omega]i\sqrt{\frac{2}{3}}e_3$ | | |



| Notation of W | n = dim W | Lin W | Tran W | $e_1,\ldots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[F_4]_1^\alpha$ | 4 | $K_{28}$ | $[1,\alpha]e_1+[1,\alpha]e_2+[1,\alpha]\sqrt{2}e_3+[1,\alpha]\sqrt{2}e_4$, $\alpha\in\Omega$ | $e_j=(\varepsilon_{j+1}-\varepsilon_{j+2})/\sqrt{2}$, $j=1,2$, $e_3=\varepsilon_4$, $e_4=(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$ | c=0 |
| $[F_4]_2^\beta$ | | $F_4$ | $[1,\beta]e_1+[1,\frac{\beta}{2}]e_2+[1,\beta]\sqrt{2}e_3+[1,\frac{\beta}{2}]\sqrt{2}e_4$, $\beta\in\Omega$ | | |
| $[F_4]_3^\gamma$ | | | $[1,\gamma]e_1+[1,\gamma]e_2+[1,\frac{1+\gamma}{2}]\sqrt{2}e_3+$ $+[1,\frac{1+\gamma}{2}]\sqrt{2}e_4$, $\gamma\in\Omega$ | | |
| $[K_{29}]$ | | $K_{29}$ | $[1,i]e_1+[1,i]e_2+[1,i]e_3+[1,i]e_4$ | $e_1=\frac{1}{\sqrt{2}}(\varepsilon_2-\varepsilon_4)$, $e_2=\frac{1}{\sqrt{2}}(-i\varepsilon_2+\varepsilon_3)$, $e_3=\frac{1}{\sqrt{2}}(-\varepsilon_3+\varepsilon_4)$, $e_4=\frac{-1+i}{2\sqrt{2}}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$ | |
| $[K_{31}]$ | | $K_{31}$ | $[1,i]e_1+[1,i]e_2+[1,i]e_3+[1,i]e_4$ | $e_1=\frac{1}{\sqrt{2}}(\varepsilon_2-\varepsilon_4)$, $e_2=\frac{1}{\sqrt{2}}(-i\varepsilon_2+\varepsilon_3)$, $e_3=\frac{1}{\sqrt{2}}(-\varepsilon_3+\varepsilon_4)$, $e_4=\frac{-1+i}{2\sqrt{2}}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$, $e_5=\frac{1-i}{\sqrt{2}}\varepsilon_4$ | |
| $[K_{31}]^*$ | | | | | $c(r_j)=0$, $j=1,2,3,4$, $c(r_5)=\frac{1+i}{2}e_5$ |

| Notation of W | n = dim W | Lin W | Tran W | $e_1,\dots,e_s$ | cocycle c |
|---|---|---|---|---|---|
| $[K_{32}]$ $A_5^{(3)}$ | 4 | $K_{32}$ | $[1,\omega]e_1+[1,\omega]e_2+[1,\omega]e_3+[1,\omega]e_4$ | $e_1=\varepsilon_3$, $e_2=\frac{1-\omega}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$, $e_3=-\omega\varepsilon_2$, $e_4=\frac{\omega^2-\omega}{3}(-\varepsilon_1+\varepsilon_2+\varepsilon_4)$ | |
| $[K_{33}]$ | 5 | $K_{33}$ | $[1,\omega]e_1+\dots+[1,\omega]e_n$ | $e_1=\frac{\omega}{\sqrt{2}}(\varepsilon_5+\varepsilon_6)$, $e_2=-\frac{\omega}{2\sqrt{2}}(-\varepsilon_1+(1+2\omega)\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6)$, $e_j=\frac{1}{\sqrt{2}}(\varepsilon_{j-2}-\varepsilon_{j-1})$, $j=3,4,\dots,n$ | |
| $[K_{34}]$ | 6 | $K_{34}$ | | | |
| $[E_6]^\alpha$ | | $K_{35}$, $E_6$ | $[1,\alpha]e_1+\dots+[1,\alpha]e_n$, $\alpha\in\Omega$ | $e_1=(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)/2\sqrt{2}$, $e_2=(\varepsilon_1+\varepsilon_2)/\sqrt{2}$, $e_j=(-\varepsilon_{j-2}+\varepsilon_{j-1})/\sqrt{2}$, $j=3,\dots,n$. | c=0 |
| $[E_7]^\alpha$ | 7 | $K_{36}$, $E_7$ | | | |
| $[E_8]^\alpha$ | 8 | $K_{37}$, $E_8$ | | | |



## 2.7. Equivalence.

**Theorem.** The following list is the complete list of groups W and W', W ≠ W', from Table 2 which are equivalent:

Table 3

| Pairs of equivalent irreducible infinite crystallographic complex r-groups | | |
|---|---|---|
| W | W' | condition |
| $[G(2,1,s)]_2^{1+\omega}$, $s \geq 3$ | $[G(2,1,s)]_3^{1+\omega}$, $s \geq 3$ | -- |
| $[G(2,1,s)]_2^{1+\omega}$, $s \geq 3$ | $[G(2,1,s)]_4^{1+\omega}$, $s \geq 3$ | -- |
| $[G(2,1,s)]_3^{i}$, $s \geq 3$ | $[G(2,1,s)]_4^{i}$, $s \geq 3$ | -- |
| $[G(2,1,2)]_2^{\beta}$ | $[G(2,1,2)]_2^{-2/\beta}$ | $-2/\beta \in \Omega$ |
| $[G(2,1,2)]_2^{1+\omega}$ | $[G(2,1,2)]_3^{1+\omega}$ | -- |
| $[G(2,1,2)]_2^{\beta}$ | $[G(2,1,2)]_3^{1-2/\beta}$ | $1-2/\beta \in \Omega$ |
| $[G(2,1,2)]_2^{\gamma}$ | $[G(2,1,2)]_3^{(\gamma-1)/(\gamma+1)}$ | $(\gamma-1)/(\gamma+1) \in \Omega$ |
| $[G(2,1,2)]_2^{\beta}$ | $[G(2,1,2)]_3^{-1-2/\beta}$ | $-1-2/\beta \in \Omega$ |
| $[G(6,6,2)]_2^{\beta}$ | $[G(6,6,2)]_2^{-3/\beta}$ | $-3/\beta \in \Omega$ |
| $[G(6,6,2)]_3^{\gamma}$ | $[G(6,6,2)]^{(2\gamma-1)/(\gamma+1)}$ | $(2\gamma-1)/(\gamma+1) \in \Omega$ |
| $[G(6,6,2)]_2^{\beta}$ | $[G(6,6,2)]^{-1+3/\beta}$ | $-1+3/\beta \in \Omega$ |
| $[G(6,6,2)]_2^{\beta}$ | $[G(6,6,2)]_3^{2-3/\beta}$ | $2-3/\beta \in \Omega$ |
| $[F_4]_2^{\beta}$ | $[F_4]_2^{-2/\beta}$ | $-2/\beta \in \Omega$ |
| $[F_4]_2^{1+\omega}$ | $[F_4]_3^{1+\omega}$ | -- |
| $[F_4]_2^{\beta}$ | $[F_4]_3^{1-2/\beta}$ | $1-2/\beta \in \Omega$ |
| $[F_4]_3^{\gamma}$ | $[F_4]_3^{(\gamma-1)/(\gamma+1)}$ | $(\gamma-1)/(\gamma+1) \in \Omega$ |
| $[F_4]_2^{\beta}$ | $[F_4]_3^{-1-2/\beta}$ | $-1-2/\beta \in \Omega$ |

Proof of this theorem is rather technical and will not be given here.

2.8. The structure of an extension of Tran W by Lin W.

As we have seen in Section 1.5, if $k = \mathbb{R}$ then the structure of an infinite irreducible r-group W as an extension of Tran W by Lin W is very simple: it is always a semidirect product. The situation is more complicated when $k = \mathbb{C}$, because there exist infinite irreducible complex crystallographic r-groups W which are not semidirect products of Tran W and Lin W.

Theorem. The groups W from Table 2 which are not semidirect products of Tran W and Lin W are

$$[G(4,2,s)]_1^*, \ [K_{12}]^* \text{ and } [K_{31}]^*.$$

Theorem. Let $K \subset GL(V)$ be a finite irreducible r-group and let $T \subset V$ be a K-invariant lattice. Assume that there exists a crystallographic r-group W with Lin W = K, Tran W = T. Then the set of those elements of $H^1(K,V/T)$ which correspond to such subgroups W is in fact a subgroup of $H^1(K,V/T)$ and the order of this subgroup is $\leqslant 2$.

2.9. The rings and fields of definition of Lin W.

As we have seen in Section 1.5, if $k = \mathbb{R}$ then the group Lin W for an infinite irreducible r-group W is defined over $\mathbb{Q}$. If $k = \mathbb{C}$ then Lin W for an infinite irreducible crystallographic r-group W is defined over a certain purely imaginary quadratic extension of $\mathbb{Q}$, see the theorem in Section 2.5. We can describe this extension precisely.

Theorem. Let $K \subset GL(V)$ be a finite irreducible complex r-group. Then the ring with unity generated over $\mathbb{Z}$ by the set of all cyclic products related to an arbitrary fixed generating system of reflections of K



coincides with the ring $\mathbb{Z}[\mathrm{Tr}K]$ generated over $\mathbb{Z}$ by the set of traces of all elements of K. The ring $\mathbb{Z}[\mathrm{Tr}K]$ is the minimal ring of definition of K. This ring is equal to $\mathbb{Z}$ iff K is the complexification of the Weyl group of an irreducible root system.

Proof is given in the Section 4.6.

It is easily seen from Table 1 and the theorem above that for the groups K = Lin W, where W is an infinite irreducible crystallographic r-group, one has the following table:

Table 4

**Linear parts of irreducible infinite crystallographic complex r-groups**

| $\mathbb{Z}[\mathrm{Tr}K]$ | $\mathbb{Z}$ | $\mathbb{Z}[i]$ | $\mathbb{Z}[2i]$ | $\mathbb{Z}[i\sqrt{2}]$ | $\mathbb{Z}[\omega]$ | $\mathbb{Z}[2\omega]$ | $\mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$ |
|---|---|---|---|---|---|---|---|
| K | $K_1=A_s, s\geq 1$; | $G(4,1,s), s\geq 2$; | $G(4,2,2)$ | $K_{12}$ | $G(3,1,s), s\geq 2$; | $G(6,3,2)$ | $K_{24}$ |
| | $G(2,1,s)=B_s, s\geq 2$; | $G(4,4,s), s\geq 3$; | | | $G(6,1,s), s\geq 2$; | | |
| | $G(2,2,s)=D_s, s\geq 3$; | $G(4,2,s), s\geq 3$; | | | $G(3,3,s), s\geq 3$; | | |
| | $G(6,6,2)=G_2$; | $K_3(m=4)$; | | | $G(6,6,s), s\geq 3$; | | |
| | $K_{28}=F_4$; | $K_8$; ✓ | | | $G(6,2,s), s\geq 2$; | | |
| | $K_{35}=E_6$; | $K_{29}$; | | | $G(6,3,s), s\geq 3$; | | |
| | $K_{36}=E_7$; | $K_{31}$. | | | $K_3(m=3,6)$; | | |
| | $K_{37}=E_8$. | | | | $K_4$; ✓ | | |
| | | | | | $K_5$; ✓ | | |
| | | | | | $K_{25}$; ✓ | | |
| | | | | | $K_{26}$; ✓ | | |
| | | | | | $K_{32}$; ✓ | | |
| | | | | | $K_{33}$; | | |
| | | | | | $K_{34}$. | | |
| fraction field of $\mathbb{Z}[\mathrm{Tr}K]$ | $\mathbb{Q}$ | $\mathbb{Q}(\sqrt{-1})$ | $\mathbb{Q}(\sqrt{-1})$ | $\mathbb{Q}(\sqrt{-2})$ | $\mathbb{Q}(\sqrt{-3})$ | $\mathbb{Q}(\sqrt{-3})$ | $\mathbb{Q}(\sqrt{-7})$ |



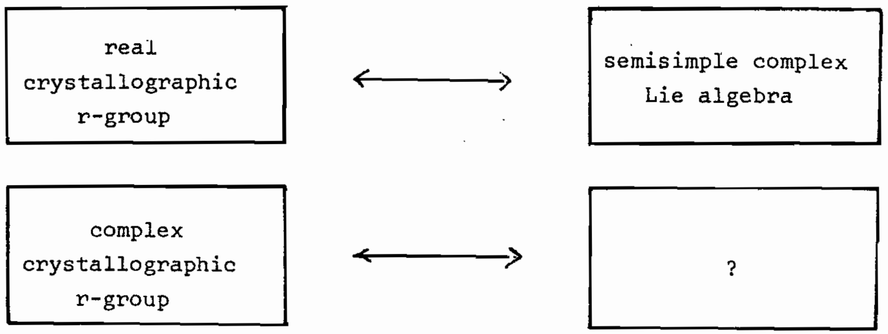
2.10. Further remarks.

a) In contrast to the real case, there exist 1-parameter families of inequivalent irreducible complex infinite crystallographic r-groups W with a fixed linear part Lin W (i.e. the groups with a fixed linear part may have moduli). We shall see below that an irreducible crystallographic r-group W with Lin W = K has moduli iff $\mathbb{Z}[\mathrm{Tr}K] = \mathbb{Z}$, i.e. iff K is the complexification of the Weyl group of an irreducible root system.

b) It follows from Table 4 (and from a known result in algebraic number theory) that the ring $\mathbb{Z}[\mathrm{Tr\ Lin\ } W]$, where W is an infinite irreducible crystallographic r-group, is always a unique factorisation domain. It would be interesting to have an a priori proof of this fact.

c) If k = $\mathbb{R}$ then it is known (and was a priori proved in 1948 - 51 by Chevalley and Harish-Chandra) that there exists a bijective correspondence between the set of classes of equivalent infinite (hence crystallographic) r-groups (= affine Weyl groups) and the set of classes of isomorphic complex semisimple Lie algebras.

Question: is it possible to attach to an infinite complex crystallographic r-group a sort of "global object" (like a semi-simple Lie algebra in the real case) in a such way that the correspondence between these r-groups and "global objects" will be bijective?




We do not know whether such an object exists or not. It is funny that we can calculate (see 4.4) the group which, by analogy with the real case, might be "the center" of this hypothetical object.



## 3. Several auxiliary results and the classification of irreducible infinite noncrystallographic complex r-groups.

Now we move on to proofs. The greater part of these proofs relies heavily on the fact that for an infinite r-group W the subgroup Tran W is sufficiently "massive".

### 3.1. The subgroup of translations.

In order to clarify the last statement we need the following principal fact:

Theorem. Let $W \subset A(E)$ be an infinite r-group. Then Tran $W \neq 0$.

Proof. Assume that Tran W = 0. Then Lin: $W \to U(V)$ is a monomorphism.

It follows from the discreteness of W that $(\overline{\text{Lin } W})^0$ is a torus (here bar denotes the closure and 0 denotes the connected component of unity). (See [5], Ch. III §4, ex 13a.)

Set $S = \text{Lin } W \cap (\overline{\text{Lin } W})^0$.

Then Lin $W \triangleright S$ and $[\text{Lin } W : S] < \infty$. Let also

$$V_0 = \{v \in V \mid (\overline{\text{Lin } W})^0 v = v\}$$

and let $V_1$ be the subspace of V determined by

$$V = V_0 \oplus V_1, \; V_0 \perp V_1.$$

The subspaces $V_0$ and $V_1$ are S-invariant. Definitely, $V_1 \neq 0$ (indeed, if not, then $S = 1$, and hence $|\text{Lin } W| < \infty$ which is a contradiction with $|\text{Lin } W| = |W|$ ).

The set

$$\{P \in (\overline{\text{Lin } W})^0 \mid H_p = V_0\}$$

is dense in $(\overline{\text{Lin } W})^0$. Therefore $\{P \in S \mid H_p = V_0\}$ is dense in $(\overline{\text{Lin } W})^0$.

Hence

$\{P \in S | H_P = V_0\} \neq 0$

(because $S$ is dense in $(\overline{\text{Lin } W})^0$).

<u>We claim that there exists a point</u> $a \in E$ <u>with</u> $\gamma(a) - a \in V_0$ <u>for every</u> $\gamma \in W$.

Consider an arbitrary point $b \in E$ and an operator $P_0 \in S$ such that $H_{P_0} = V_0$. Take $\gamma_0 \in W$ with Lin $\gamma_0 = P_0$. We have $\gamma_0(b)-b=t_0+t_1$ where $t_0 \in V_0$, $t_1 \in V_1$. But the restriction of $1-P_0$ to $V_1$ is non-degenerate. Hence there exists a vector $t \in V_1$ such that $(1-P_0)t = t_1$. So, we have $\gamma_0(b+t)-(b+t) = (\gamma_0(b)+P_0t)-(b+t) = (\gamma_0(b)-b) + (P_0-1)t = t_0+t_1-t_1 = t_0 \in V_0$. Put $a = b+t$. We have $\gamma_0(a)-a \in V_0$. We shall prove that $a$ is a point as wanted.

Let us first check that $\gamma(a)-a \in V_0$ if $\gamma \in W$, Lin $\gamma \in S$. Write $P = \text{Lin } \gamma$, $v_0 = \gamma_0(a)-a \in V_0$ and $v = \gamma(a)-a$. We need to prove that $v \in V_0$. But $S$ is commutative, so $PP_0 = P_0P$. Hence $\gamma\gamma_0 = \gamma_0\gamma$. We have now:

$$\kappa_a(\gamma_0) = (P_0,v_0) = \kappa_a(\gamma\gamma_0\gamma^{-1}) = (P,v)(P_0,v_0)(P^{-1},-P^{-1}v) =$$

$$= (PP_0P^{-1},-PP_0P^{-1}v+Pv_0+v) = (P_0,-P_0v+Pv_0+v).$$

So, $-P_0v+Pv_0+v = v_0$, i.e. $(1-P_0)v = (1-P)v_0$. But $P \in S$, hence $(1-P)v_0 = 0$. It follows from $\text{Ker}(1-P_0) = V_0$ that $v \in V_0$. This establishes the claim.

Now we can prove $\gamma(a)-a \in V_0$ for arbitrary $\gamma \in W$. Indeed, write $P = \text{Lin } \gamma$, $v = \gamma(a)-a$, as before. We have:

$$\kappa_a(\gamma\gamma_0\gamma^{-1}) = (PP_0P^{-1},-PP_0P^{-1}v+Pv_0+v).$$



It follows from $S \triangleleft \text{Lin } W$ that $V_0$ is $P$-invariant. Therefore we have $Pv_0 \in V_0$. Moreover, if $\gamma' = \gamma\gamma_0\gamma^{-1}$, then Lin $\gamma' \in S$ and, by the claim, $\gamma'(a)-a = -PP_0P^{-1}v+Pv_0+v \in V_0$. Therefore, $-PP_0P^{-1}v+v \in V_0$ and hence $-P_0P^{-1}v+P^{-1}v \in P^{-1}V_0 = V_0$, i.e. $(1-P_0)P^{-1}v \in V_0$. But the image of $1-P_0$ is $V_1$, hence $(1-P_0)P^{-1}v = 0$. Therefore $P^{-1}v \in V_0$, i.e. $v \in PV_0 = V_0$ and we are done.

<u>Now we consider two subgroups of</u> $W$: $W'$, <u>resp</u>. $W''$ <u>is the subgroup generated by those reflections</u> $\gamma$ <u>for which</u> $H_\gamma \ni a$, <u>resp</u>. $H_\gamma \not\ni a$.

The subgroup $W'$ is <u>finite</u>. Indeed, identifying $A(E)$ and $GL(V)V$ by means of $\kappa_a$, we have $\kappa_a(\gamma) = (R,0) \in U(V)$ for any reflection $\gamma \in W'$ and hence for any $\gamma \in W'$. So $\kappa_a(W')$ is a discrete subgroup of a compact group $U(V)$, hence finite.

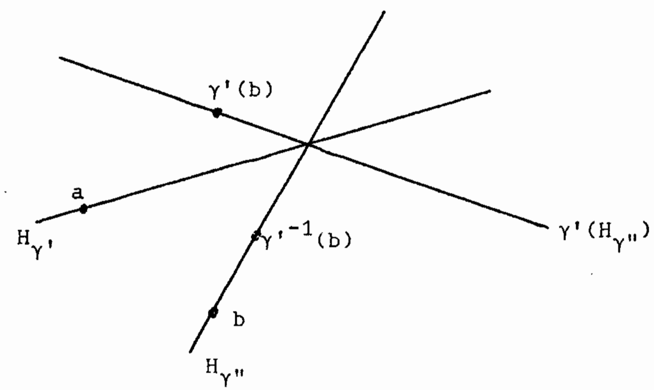
We claim now that $W''$ is <u>infinite</u>. Before proving this, we shall show how to finish the proof of the theorem if the statement holds. Thus, assume $W''$ is an infinite r-group.

Note that Tran $W'' = 0$. Using the above arguments and constructions we obtain a torus $(\overline{\text{Lin } W''})^0$, a subgroup $S'' = \text{Lin } W'' \cap (\overline{\text{Lin } W''})^0$ and a decomposition $V = V_0'' \oplus V_1''$, $V_0'' = \{v \in V | (\overline{\text{Lin } W''})^0 v = v\}$, $V_0'' \perp V_1''$. Also, we have $V_1'' \neq 0$ and $\{P \in S'' | H_P = V_0''\} \neq \phi$. But $W'' \subset W$, therefore Lin $W'' \subset$ Lin $W$, whence $(\overline{\text{Lin } W''})^0 \subset (\overline{\text{Lin } W})^0$. So $S'' \subset S$ and $V_0'' \supset V_0$. It follows now that $\gamma(a)-a \in V_0''$ and $\gamma(a)-a \neq 0$ for every $\gamma \in W''$.

Let $\gamma \in W''$ be a reflection. Then $\kappa_a(\gamma) = (\text{Lin } \gamma, \gamma(a)-a)$, and in view of $\gamma(a)-a \neq 0$, we have: (root line of Lin $\gamma$) $= \mathbb{C}(\gamma(a)-a) \subset V_0''$. Therefore $H_{\text{Lin } \gamma} \supset V_1$ and Lin $\gamma$ acts on $V_1$ trivially. But $W''$ is generated by reflections; therefore Lin $W''$ acts trivially on $V_1$ which contradicts the existence of $P \in S'' \subset \text{Lin } W''$ such that $H_P = V_0''$ and $V_1'' \neq 0$.

It remains to show that $|W''| = \infty$. Let us assume that this is not the case. Then $W''$ has a fixed point $b \in E$. By construction, $b \neq a$. We shall show that $W'$ and $W''$ commute. This then yields that $W = W'W''$ is a finite group (for $W'$ and $W''$ clearly generate $W$), which is a contradiction.

Let $\gamma' \in W'$ and $\gamma'' \in W''$ be reflections. We have $a \in H_{\gamma'}$, $b \in H_{\gamma''}$.




But $\gamma'\gamma''\gamma'^{-1}$ is a reflection with mirror $\gamma'(H_{\gamma''})$. Hence $\gamma'\gamma''\gamma'^{-1}$ is either in $W'$ or in $W''$. If this element is in $W'$, then $\gamma'(H_{\gamma''}) \ni a$, i.e., $H_{\gamma''} \ni \gamma'^{-1}(a) = a$, which is a contradiction. Therefore, $\gamma'(H_{\gamma''}) \ni b$ and $H_{\gamma''} \ni \gamma'^{-1}(b)$. We also have $b \in H_{\gamma''}$ and $b \neq \gamma'^{-1}(b)$ (for otherwise $b \in H_{\gamma'}$, which is absurd).

Consequently, there is a unique line on $b$ and $\gamma'^{-1}(b)$. This line lies in $H_{\gamma''}$ and is orthogonal to $H_{\gamma'^{-1}} = H_{\gamma'}$. Hence $H_{\gamma''} \perp H_{\gamma'}$, i.e. $\gamma'\gamma'' = \gamma''\gamma'$. □

It follows from this theorem that Tran $W$ is "big enough":

Theorem. Let $W$ be an infinite irreducible $r$-group. Then $T$ = Tran $W$ is a lattice of rank $n$ if $k = \mathbb{R}$ and of rank $n$ or $2n$ if $k = \mathbb{C}$ (here $n = \dim_k E$).



Proof. Let $k = \mathbb{R}$. Then $\mathbb{R}T$ is an invariant subspace of the Lin $W$-module $W$. This subspace is nontrivial according to the previous theorem. Therefore $\mathbb{R}T = V$, because of irreducibility (see the theorem in 1.4), and the assertion follows from the equality rk $T = \dim \mathbb{R}T$.

Let $k = \mathbb{C}$. As above, we have $0 \neq \mathbb{C}T = \mathbb{R}T + i\,\mathbb{R}T = V$. Hence $2n = \dim_{\mathbb{R}} V \leq \dim_{\mathbb{R}} \mathbb{R}T + \dim_{\mathbb{R}} i\,\mathbb{R}T = 2\,\mathrm{rk}\,T$. Therefore rk $T \geq n$. But $\mathbb{R}T \cap i\,\mathbb{R}T$ is a Lin $W$-invariant complex subspace of $V$ and hence either $\mathbb{R}T \cap i\,\mathbb{R}T = 0$ or $\mathbb{R}T \cap i\,\mathbb{R}T = V$, i.e. $\mathbb{R}T = V$. If rk $T = \dim \mathbb{R}T > n$ then $\mathbb{R}T \cap i\,\mathbb{R}T \neq 0$, hence $\mathbb{R}T = V$, i.e. rk $T = 2n$. □

Corollary. 1) $|\mathrm{Lin}\ W| < \infty$; 2) if $k = \mathbb{C}$ then $W$ is a crystallographic group iff rk $T = 2n$.

Proof. 1) follows from the fact that Lin $W$ is contained in a compact group and has an invariant lattice of a full rank. □

3.2. Some auxiliary results.

Let $K \subset GL(V)$ be a finite $r$-group and $\mathcal{H}$ be the set of mirrors of all reflections from $K$. Let $H \in \mathcal{H}$. Then it is easy to see that the subgroup of $K$ generated by all reflections $R \in K$ with $H_R = H$, is a cyclic group. Let $m(H)$ be the order of this group.

Theorem. Let $\{R_j\}_{j\in J}$ be a generating system of reflections of $K$ such that the order of $R_j$ is equal to $m(H_{R_j})$ for every $j \in J$. Then each reflection $R \in K$ is conjugate to $R_j^{l_j}$ for certain $j$ and $l_j$.

Proof. Let $\mathcal{O}$ be the $K$-orbit of $H_R$ in $\mathcal{H}$ (it follows from $PH_R = H_{PRP^{-1}}$ that $K$ acts on $\mathcal{H}$). Let $\chi_{\mathcal{O}}$ be the product of linear equations of the mirrors in $O$, normalized by the condition $\chi_O(1) = 1$. Then $\chi_O$ is a character of K, i.e. a homomorphism of K into the multiplicative group of k. The group $\chi_O(K)$ is generated by $\chi_O(R_j)$, $j \in J$. We have also $\chi_O(R) \neq 1$. Therefore there exists a $j \in J$ with $\chi_O(R_j) \neq 1$. But $\chi_O(R_j) \neq 1$ iff $O$ is the orbit of $H_{R_j}$, see [6]. Therefore $gH_R = H_{R_j}$ for a certain $g \in K$ and the statement follows. □

Theorem. Let W be an r-group. Then for any reflection $R \in \text{Lin } W$ there exists a reflection $\gamma \in W$ with $\text{Lin } \gamma = R$.

Proof. Let $\{\rho_j\}_{j \in J}$ be the set of all reflections of W. Then $\{R_j = \text{Lin } \rho_j\}_{j \in J}$ is a generating system of reflections of Lin W. Let $O$ be the Lin W-orbit of $H_R$ in $H$. Then there exists a number l with $H_{R_l} \in O$ and $P_l \in \text{Lin } W$ such that $P_l H_{R_l} = H_{P_l R_l P_l^{-1}} = H_R$. Let $\pi_l \in W$ be such that $\text{Lin } \pi_l = P_l$. We have:
$\text{Lin } \pi_l \rho_l \pi_l^{-1} = P_l R_l P_l^{-1}$ and $\pi_l \rho_l \pi_l^{-1}$ is a reflection, too. Therefore $\chi_O(\text{Lin } W)$ is generated by $\chi_O(R_j)$ where $j \in J' = \{j \in J \mid H_{R_j} = H_R\}$. We have $\chi_O(R) = \chi_O(R_{j_1}) \dots \chi_O(R_{j_s}) = \chi_O(R_{j_1} \dots R_{j_s})$ for certain $j_1, \dots, j_s \in J'$. But $R_{j_1}, \dots, R_{j_s}$ are the reflections whose mirrors are $H_R$. It follows now (see [6]) that $R = R_{j_1} \dots R_{j_s}$. Let us consider the element $\rho = \rho_{j_1} \dots \rho_{j_s}$. We have $\text{Lin } \rho = R$ and it easily follows from the parallelity of the mirrors $H_{\rho_{j_1}}, \dots, H_{\rho_{j_s}}$ that $\rho$ is a reflection. □

## 3.3. Semidirect products.

Let W be a subgroup of A(E). We have

$$0 \to \text{Tran } W \hookrightarrow W \to \text{Lin } W \to 1.$$

When is W a semidirect product of Lin W and Tran W? We have the following criterion:



Theorem. Let $|\text{Lin } W| < \infty$. Then:

a) W is a semidirect product iff there exists a point $a \in E$ such that Lin induces an isomorphism of the stabilizer $W_a$ of a with Lin W.

b) For every finite group $K \subset U(V)$ and every K-invariant subgroup T of V there exists a unique group $W \subset A(E)$ (up to equivalence) such that $\text{Lin } W = K$, $\text{Tran } W = T$ and W is a semidirect product of Lin W and Tran W.

Proof is left to the reader. □

The point a from part a) of this theorem is called a special point of W', see [1].

Using this theorem we can clarify the structure of infinite r-groups in a number of important cases:

Theorem. Let $W \subset A(E)$ be a group generated by reflections. Assume that $|\text{Lin } W| < \infty$ and that Lin W is an essential group (i.e. $\{v \in V \mid (\text{Lin } W)v = v\} = \{0\}$) generated by $n = \dim_k E$ reflections. Then W is a semidirect product of Lin W and Tran W.

Proof. Let $R_1, \dots, R_n$ be a generating system of reflections of Lin W. Then there exists a reflection $\gamma_j \in W$ such that $\text{Lin } \gamma_j = R_j$, $j = 1, \dots, n$, see Section 3.2. We have $\bigcap_{j=1}^{n} H_{R_j} = 0$ because Lin W is essential. Therefore $\bigcap_{j=1}^{n} H_{\gamma_j} \neq \emptyset$; more precisely, this intersection is a single point $a \in E$.
We have now that $\text{Lin}: W_a \to \text{Lin } W$ is a surjective map, hence an isomorphism. Thus we are done in view of the theorem above. □

The conditions of this theorem are always fulfilled if $k = \mathbb{R}$ and W is an infinite irreducible r-group. In general this is not the case if $k = \mathbb{C}$, though "in most cases", it is.

3.4. Classification of the irreducible infinite complex non-crystallographic r-groups.

Let $k = \mathbb{C}$ and let W be a group as in the title. Set $T = \text{Tran } W$. Then $\text{rk } T = n = \dim_{\mathbb{C}} E$, by 3.1.

$\mathbb{R}T$ is a Lin W-invariant $\mathbb{R}$-submodule of V. Let us consider the restriction of $\langle | \rangle$ to $\mathbb{R}T$. We claim that this restriction has only real values. Indeed, Re $\langle | \rangle$ defines an euclidean structure on $\mathbb{R}T$ such that Lin W is orthogonal with respect to this structure. But $\mathbb{C}(\mathbb{R}T) = V$ because of irreducibility. Hence, there exists a canonical extension of Re $\langle | \rangle$, determined up to a hermitian Lin W-invariant scalar product, say $( | )$, on V. Thus we have two Lin-invariant hermitian structures, $\langle | \rangle$ and $( | )$, on V. They are proportional because of irreducibility of Lin W: $\langle | \rangle = \lambda ( | )$, $\lambda \in \mathbb{C}$. Taking restrictions to $\mathbb{R}T$, we have: $\lambda = 1$. In other words, $\mathbb{R}T$ is a real form of V and the restriction of the action of Lin W to $\mathbb{R}T$ gives an irreducible real finite r-group (and Lin W itself is the complexification of this group). It follows from the classification that this group is generated by n reflections, see 1.5. Therefore Lin W is generated by n reflections, too. It follows from the theorem above (section 3.3) that W is a semidirect product. Let $a \in E$ be a special point of W. Then $a+\mathbb{R}T$ is a real form of E. It is clear that $a+\mathbb{R}T$ is W-invariant. The restriction of W to $a+\mathbb{R}T$ is a real form of W. Therefore, this restriction is an affine Weyl group and W is its complexification. This completes the proof of the theorem in ection 2.2. □



4. Invariant lattices.

As we have seen in Section 2.3, one of the ingredients of the description of infinite complex crystallographic r-groups W is the lattice Tran W. We shall show in this chapter how one can find all the invariant lattices of full rank for an arbitrary fixed finite r-group.

We use the following notation:

$K \subset GL(V)$ a finite essential r-group, $n = \dim_k V$,

$\Gamma \subset V$ a K-invariant lattice,

$R_j = R_{e_j,\mu_j}$, $1 \leq j \leq s$, a fixed generating system of reflections of K,

$L$ the set of all root lines of K,

$\Gamma_l = \ell \cap \Gamma$, $\ell \in L$,

$\Gamma_j = \Gamma_{\ell_j}$, $j = 1,\dots,s$.

4.1. Root lattices.

Definition. $\Gamma^0 = \sum_{\ell \in L} \Gamma_\ell$ is called the root lattice associated with $\Gamma$.

If $\Gamma = \Gamma^0$ then $\Gamma$ is called a root lattice.

Theorem. $\Gamma^0$ is a K-invariant lattice and $\text{rk } \Gamma = \text{rk } \Gamma^0$.

Proof. It is clear that $\Gamma^0$ is K-invariant, so let us prove the assertion about ranks.

We can assume that

$$V = \bigoplus_{k=1}^{n} \ell_k$$

because K is essential. Put

$$S = (1-R_{e_1,\mu_1}) + \dots + (1-R_{e_n,\mu_n}).$$

If $v \in \operatorname{Ker} S$, then $Sv = 0 = (1-\mu_1) < v \mid e_1 > e_1 + \ldots + (1-\mu_n) < v \mid e_n > e_n$ therefore $(1-\mu_j) < v \mid e_j > e_j = 0$, because $e_j \in \ell_j$, $<e_j \mid e_j> = 1$, $1 \leq j \leq s$. Hence $<v \mid e_j> = 0$, $1 \leq j \leq s$, i.e. $v = 0$. So, $S$ is non-singular. But $S\Gamma \subset \Gamma_{\ell_1} \oplus \ldots \oplus \Gamma_{\ell_n} \subset \Gamma^0 \subset \Gamma$ and $\operatorname{rk} S\Gamma = \operatorname{rk} \Gamma$. Therefore $\operatorname{rk} \Gamma^0 = \operatorname{rk} \Gamma$. □

Corollary. If $\operatorname{rk} \Gamma = 2n$ (hence $k = \mathbb{C}$) then $\operatorname{rk} \Gamma_\ell = 2$ for every $\ell \in L$. If $\operatorname{rk} \Gamma = n$ then $\operatorname{rk} \Gamma_\ell = 1$ for every $\ell \in L$.

Proof. We may take $\ell_1 = \ell$ in the previous proof. As this proof shows that $\operatorname{rk}(\Gamma_{\ell_1} \oplus \ldots \oplus \Gamma_{\ell_n}) = \operatorname{rk} \Gamma_{\ell_1} + \ldots \operatorname{rk} \Gamma_{\ell_n} = 2n$, the first assertion follows from the evident inequality $\operatorname{rk} \Gamma_\ell \leq 2$. The second assertion can be proved similarly by means of reduction to the real form. □

It appears that one can reconstruct $\Gamma^0$ from the $\Gamma_j$, $1 \leq j \leq s$.

Theorem. $\Gamma^0 = \Gamma_1 + \ldots + \Gamma_s$.

Proof. Let $\ell \in L$ and $u \in \Gamma_\ell$; then there exists $g \in K$ such that $gu \in \Gamma_j$ for certain $j$ (because every reflection in $K$ is conjugate to the a power of some $R_j$, $1 \leq j \leq s$, see Section 3.2). Let $\Gamma' = \Gamma_1 + \ldots + \Gamma_s$. It is easy to see that $\Gamma'$ is invariant, hence $u \in \Gamma'$ and $\Gamma = \Gamma'$. □

The problem of finding of all K-invariant lattices can be solved in two steps: 1) description of all K-invariant root lattices; 2) description of all K-invariant lattices with a fixed associated root lattice. We shall first show how to solve the second problem.

4.2. The lattices with a fixed root lattice.

Definition. $\Gamma^* = \{v \in V \mid (1-P)\, v \in \Gamma \text{ for every } P \in K\}$.

Clearly, $\Gamma^*$ is a subgroup of $V$. It is more convenient to use another description of $\Gamma^*$.



Let

$$\pi : V \to V/\Gamma$$

be a natural map and $(V/\Gamma)^K$ be the set of points fixed by $K$. Then $\Gamma^* = \pi^{-1}((V/\Gamma)^K)$. Therefore,

$$\Gamma^* = \{v \in V \mid (1-R_j)v \in \Gamma_j,\ 1 \leq j \leq s\}.$$

It is clear that $\Gamma^*$ is K-invariant and $\Gamma \subset \Gamma^*$.

Theorem. $\Gamma$ is a lattice and $\operatorname{rk} \Gamma^* = \operatorname{rk} \Gamma$.

Proof. Let us first show that $\Gamma^*$ is a lattice. If it is not a lattice, then there exists a vector $v \in \Gamma^*$ such that $\alpha v \in \Gamma^*$ for every $\alpha \in \mathbb{R}$ (because $\Gamma^*$ is a closed subgroup of $V$). Then $(1-P)\ \alpha v \in \Gamma$ for every $\alpha \in \mathbb{R}$ and $P \in K$. Therefore $(1-P)\ v = 0$, as $\Gamma$ is a lattice and, hence, $v = 0$, because $K$ is an essential group. Therefore $\Gamma^*$ is a lattice.

$V$ is a euclidean space with respect to $\operatorname{Re} <\mid>$. Let $\mathbb{R}\Gamma^\perp$ be the orthogonal complement of $\mathbb{R}\Gamma$ in this space. Let $v \in \Gamma^*$ and $v = u + w$, $u \in \mathbb{R}\Gamma$, $w \in \mathbb{R}\Gamma^\perp$. Then $(1-P)v = (1-P)u + (1-P)w \subset \Gamma \subset \mathbb{R}\Gamma$ for every $P \in K$. Therefore $(1-P)w = 0$, hence $w = 0$ (again because $K$ is essential). Thus $\Gamma^* \subset \mathbb{R}\Gamma$. This completes the proof. □

Cohomological meaning of $\Gamma^*$.

We have an exact sequence of groups

$$0 \to \Gamma \to V \to V/\Gamma \to 0.$$

It gives the exact cohomological sequence

$$H^0(K,V) \to H^0(K,V/\Gamma) \to H^1(K,\Gamma) \to H^1(K,V) \to \ldots$$

But $H^0(K,V) = 0$ because $K$ is essential, $H^0(K,V/\Gamma) = (V/\Gamma)^K = \Gamma^*/\Gamma$ and $H^1(K,V) = 0$ because $V$ is divisible.

Therefore

$$\Gamma^*/\Gamma \simeq H^1(K,\Gamma).$$

Now we can explain how to find all K-invariant lattices with a fixed K-invariant root lattice.

Theorem. Let $\Lambda$ be a fixed K-invariant root lattice in V. Then for every lattice $\Gamma$ in V the following properties are equivalent:

a) $\Gamma$ is a K-invariant lattice and $\Gamma^0 = \Lambda$.

b) $\Lambda \subset \Gamma \subset \Lambda^*$ and $\Gamma_j = \Lambda_j$, $1 \leqslant j \leqslant s$.

There exist only a finite number of lattices $\Gamma$ with the properties a) and b).

Proof. a) $\Rightarrow$ b). Let $v \in \Gamma$. Then $(1-R_j)\, v =$
$= (1-\mu_j) \langle v | e_j \rangle e_j \in \Gamma \cap \ell_j = \Gamma_j \subset \Gamma^0 = \Lambda$. Therefore $v \in \Lambda^*$ and $\Gamma \subset \Lambda^*$.

b) $\Rightarrow$ a). Let $\Lambda \subset \Gamma \subset \Lambda^*$. Then $\Gamma = \pi^{-1}\,\pi(\Gamma)$, where $\pi : V \to V/\Lambda$ is the natural map,and $\pi(\Gamma) \subset \pi(\Lambda^*) = (V/\Lambda)^K$. Hence $\pi(\Gamma)$ is K-invariant and therefore $\Gamma$ is also K-invariant. If $\Gamma_j = \Lambda_j$ then $\Gamma^0 = \Lambda$ (because $\Lambda = \Lambda^0 = \Lambda_1 + \ldots + \Lambda_s$). □

We have already seen that for irreducible K one can take $s = n$, if $k = \mathbb{R}$, and $s = n$ or $n+1$, if $k = \mathbb{C}$. It appears that if $s = n$ then there exists a good constructive way to find $\Lambda^*$ by means of $\Lambda$.

Theorem. Let $s = n$ and $\Lambda$ be a K-invariant lattice in V. Let also

$$S = (1-R_1) + \ldots + (1-R_n).$$

(Recall from Section 4.1 that S is nonsingular). Then:

a) $\Lambda^* \subset S^{-1}\,\Lambda$.

b) If $\Lambda^0 = \Lambda$ then $\Lambda^* = S^{-1}\,\Lambda$.



Proof. a) Let $a \in \Lambda^*$, then, by definition, $(1-R_j)a \in \Lambda$ for all $1 \leqslant j \leqslant n$, hence $Sa \in \Lambda$ and $a \in S^{-1}\Lambda$.

b) Let $\Lambda = \Lambda^0$ and $u \in S^{-1}\,\Lambda$. Then $Su = (1-R_1)u + \ldots$
$\ldots + (1-R_n)u \in \Lambda = \Lambda_1 \oplus \ldots \oplus \Lambda_n$. But $(1-R_j)u \in \mathbb{C}e_j$, and $(1-R_j)u \in \Lambda_j \subset \Lambda$, because of linear independence of $e_1,\ldots,e_n$. Therefore, by definition of $\Lambda^*$, we have $u \in \Lambda^*$. □

This theorem is useful in practice because one can explicitly describe the operator S : its matrix with respect to the basis $e_1,\ldots,e_n$ is

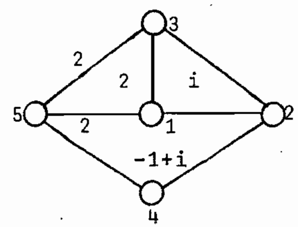
$$\begin{pmatrix} (1-\mu_1)\langle e_1|e_1\rangle & \ldots & (1-\mu_1)\langle e_n|e_1\rangle \\ & \ldots & \\ (1-\mu_n)\langle e_1|e_n\rangle & \ldots & (1-\mu_n)\langle e_n|e_n\rangle \end{pmatrix}$$

What to do if $k = \mathbb{C}$ and $s = n+1$?

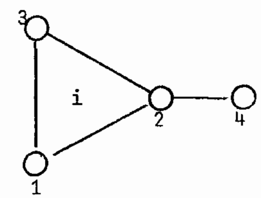
It is not difficult to see that one can take $R_1,\ldots,R_{n+1}$ in such a way that $R_1,\ldots,R_n$ generate a subgroup K' of K, which is also irreducible (the numbering in Table 1 has this property).

Example:

$K = K_{31}$




$K' = K_{29}$




Therefore the problem may be solved as follows: find all K'-invariant lattices and select those lattices among them which are invariant under $R_{n+1}$.

We shall show how this can be done in Section 4.8, after we have explained (in Section 4.7) how to describe invariant root lattices.

By now we want to discuss several general properties of invariant lattices and explain the significance of root lattices in the theory of infinite r-groups.

4.3. Further remarks on lattices.

Theorem. Let $K \subset GL(V)$ be a finite linear irreducible group and $\Gamma$ be a nonzero K-invariant lattice in V. Then $\mathrm{rk}\,\Gamma = n = \dim_k V$ if $k = \mathbb{R}$, and $\mathrm{rk}\,\Gamma = n$ or $2n$, if $k = \mathbb{C}$. Moreover, if K is an r-group, $k = \mathbb{C}$ and $\mathrm{rk}\,\Gamma = n$ then K is the complexification of the Weyl group of a certain irreducible root system.

Proof. as in Section 3.1 and 3.4. □

Corollary. If $k = \mathbb{C}$ and K has an invariant lattice of rank n then K has an invariant lattice of rank 2n.

Proof. K is the complexicification of a Weyl group, let $\Gamma$ be a lattice of rank n in the corresponding real form of V, which is invariant under this Weyl group. Then, for every $z \in \mathbb{C} - \mathbb{R}$ a lattice $\Gamma + z\Gamma$ is K-invariant and has rank 2n. □

It should be noted that if $k = \mathbb{R}$ and K is a Weyl group then a root lattice of rank n is (up to similarity) a lattice $\Lambda$ of radical weights and $\Lambda^*$ is a lattice of weights, see [1]. The matrix of S with respect to a basis of simple roots, is, in this case, the Cartan matrix.

4.4. Properties of the operator S.

Theorem. Let K be irreducible, $s = n$, $\Lambda$ be a nonzero K-invariant lattice and $S = (1-R_1) + \dots + (1-R_n)$. Then:

a) $|\det S|$ and $\mathrm{Tr}\,S \in \mathbb{Z}$ if $\mathrm{rk}\,\Lambda = n$,
$|\det S|^2$ and $2\,\mathrm{Re}\,\mathrm{Tr}\,S \in \mathbb{Z}$ if $\mathrm{rk}\,\Lambda = 2n$.

b) $|\det S|$ depends only on K but not on the choice of the generating system of reflections.



c) $|\det S|$ is divisible by $[\Lambda:\Lambda^0]$ if $\mathrm{rk}\,\Lambda = n$,
$|\det S|^2$ is divisible by $[\Lambda:\Lambda^0]$ if $\mathrm{rk}\,\Lambda = 2n$.

d) $\Lambda = \Lambda^0 \Rightarrow [\Lambda^*:\Lambda] = |\det S|$ if $\mathrm{rk}\,\Lambda = n$ and
$= |\det S|^2$ if $\mathrm{rk}\,\Lambda = 2n$.

e) If $\Lambda = \Lambda^0$ and $d_1, \dots, d_r$ are the invariant factors of a matrix of the endomorphism of $\Lambda$, defined by S, $d_j > 0$, $d_1 | \dots | d_r$, then,

$$H^1(K,\Lambda) \simeq \Lambda^*/\Lambda \simeq \mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_r\mathbb{Z}.$$

Specifically, $d_1, \dots, d_r$ do not depend on the choice of the generating system of reflections.

f) $\Lambda = \Lambda^0 \Rightarrow |H^1(K,\Lambda)| = |\det S|$ if $\mathrm{rk}\,\Lambda = n$ and
$= |\det S|^2$ if $\mathrm{rk}\,\Lambda = 2n$.

Proof. If $\mathrm{rk}\,\Lambda = n$, then, by considering the corresponding real form, we reduce the problem to the case $k = \mathbb{R}$. If $\mathrm{rk}\,\Lambda = 2n$ then $k = \mathbb{C}$. It is known that in this case for every $P \in GL(V)$ one has $\det P_{\mathbb{C}|\mathbb{R}} = |\det P|^2$ and $\mathrm{Tr}\,P_{\mathbb{C}|\mathbb{R}} = 2\,\mathrm{Re}\,\mathrm{Tr}\,P$ (here $P_{\mathbb{C}|\mathbb{R}}$ is P, considered as a linear operator of a 2n-dimensional real vector space V).

Now, a) follows from the fact that a basis of $\Lambda$ is an $\mathbb{R}$-basis of V. We have also:

$$\Lambda^0 \subset \Lambda \subset (\Lambda^0)^* = S^{-1}\Lambda^0, \quad \text{so } [\Lambda:\Lambda^0] \mid [S^{-1}\Lambda^0:\Lambda^0].$$

But $S^{-1}\Lambda^0/\Lambda^0 \simeq \Lambda^0/S\Lambda^0$, whence $[S^{-1}\Lambda^0:\Lambda^0] =$
$= [(\Lambda^0)^*:\Lambda^0] = |\det S|$ if $k = \mathbb{R}$ or $= |\det S_{\mathbb{C}|\mathbb{R}}|$ if $k = \mathbb{C}$.
The assertions b),c),d),e),f) follow from these equations. □

Corollary. a) If $|\det S| = 1$ then $\Lambda = \Lambda^0$, i.e. any invariant lattice is a root lattice.

b) Let $k = \mathbb{C}$ and suppose $|\det S|^2$ is a prime number. Then $\Lambda = \Lambda^0$ or $\Lambda = (\Lambda^0)^*$.

Remark. One can calculate det S directly from the graph of K, i.e.

$$\det S = \sum_{\sigma} \operatorname{sgn} \sigma .\, a_{\sigma}$$

where $\sigma$ runs through the permutations of degree n and $a_{\sigma} = c_{\alpha} c_{\beta} \ldots c_{\gamma}$ for $\sigma = \alpha\, \beta \ldots \gamma$ a decomposition of $\sigma$ as a product of cycles.

Examples.

a) $K = K_1$, type $A_n$. Write $S = S(A_n)$ for the graph

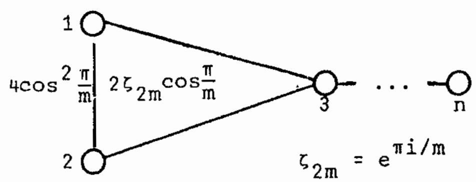
O— ... —O
1 n

Simple cyclic products$\neq 0$ are $c_{j,j+1} = 1$, $1 \leq j \leq n - 1$, and $c_j = 2$, $1 \leq j \leq n$.

We have $\det S(A_1) = 2$ : Assume, by induction, that $\det S(A_k) = k + 1$ if $k < n$. Then,

$\det S(A_n) = c_1 \det S(A_{n-1}) - c_{12} . \det S(A_{n-2}) = 2n - (n-1) = n + 1.$

b) $K = K_2$, type G(m,m,n). Let us consider the generating system of reflections given by the praph




The list of all nonzero simple cyclic products is

$c_j = 2$, $1 \leq j \leq n$;

$c_{13} = c_{23} = c_{34} = c_{45} = \ldots = c_{n-1,n} = 1$

$c_{12} = 4 \cos^2 \frac{\pi}{m}$

$c_{123} = 2 \cos \frac{\pi}{m} e^{\ i/m}$, $c_{132} = 2 \cos \frac{\pi}{m} e^{-\pi i/m}$.



It is easy to see that $\det S = c_1 \det S(A_{n-1}) - c_{12} \det S(A_{n-2}) -$

$- c_{13} . c_2 \det S(A_{n-3}) + c_{123} \det S(A_{n-3}) + c_{132} \det S(A_{n-3}) =$

$= 2n - 4\cos^2 \frac{\pi}{m} (n-1) - 1.2.(n-2) + 2 \cos \frac{\pi}{m} e^{\pi i/m} (n-2) +$

$+ 2 \cos \frac{\pi}{m} e^{-\pi i/m} (n-2) = 4 - 4\cos^2 \frac{\pi}{m} (n-1) + 4 \cos^2 \frac{\pi}{m} (n-2) =$

$= 4 \sin^2 \frac{\pi}{m}$.

Remark. If $k = \mathbb{R}$ and $\Lambda$ is a lattice of radical weights with Weyl group K then $d_1, \ldots, d_r$ are invariant factors of a Cartan matrix and $H^1(K,\Lambda)$ is isomorphic to the centre of a simple simply connected Lie group corresponding to K.

4.5. Root lattices and infinite r-groups.

We shall now show that if there exists a nonzero K-invariant lattice $\Lambda$ then there also exists an infinite r-group W with linW = K and TranW = $\Lambda$.

Theorem. Let $K \subset Gl(V)$ be a finite linear r-group and $\Lambda$ be a nonzero K-invariant lattice in V. Then the semidirect product of K and $\Lambda$ is an r-group iff $\Lambda = \Lambda^0$.

Proof. Let W be the semidirect product of K and $\Lambda$. Let a be a special point. We identify W and $\kappa_a(W) = K.\Lambda \subset GL(V)V$. Then the set of all reflections in W is $\{(R,v) | R$ is a reflection of K and $v \in l_R \cap \Lambda\}$. Let $W^0$ be a subgroup of W generated by this set. Then $K \subset W^0$. Also $\Lambda^0 \subset W^0$, because $\Lambda^0 = \sum_{l \in L} \Lambda_l$. Moreover, if $v \in \Lambda_l$ and l is the root line of a reflection R, then $(R^{-1},0)(R,v) = (0,v)$. Therefore we have $K.\Lambda^0 \subset W^0$. But every reflection in W lies in $K.\Lambda^0$ (because it has the form (R,v), where R is a reflection and $v \in l_R \cap \Lambda = \Lambda^0$, see Section 1.2). Therefore, $W^0 = K\Lambda^0$ is generated by reflections, and we have $W^0 = K$. Hence, $K.\Lambda = W = W^0 = K.\Lambda^0$ iff $\Lambda = \Lambda^0$. □

Corollary. The infinite r-groups W for which Lin W is generated by n reflections are exactly the groups Lin W.Λ, where Λ is a nonzero Lin W-invariant root lattice.

4.6. Description of the group of linear parts; proof.

We shall prove now the theorem from Section 2.5 and part of the theorem from Section 2.9.

Proof. a) ➡ b) is already proved in Section 4.3.

b) ➡ a) is trivial.

c) ➡ b) is also trivial: such a lattice is Tran W.

b) ➡ c). Let Γ be an invariant lattice of rank 2n. Then the semidirect product of K and $\Gamma^0$ is a crystallographic r-group because of the previous theorem.

b) ➡ d). Let us first prove that $\mathbb{Z}[\mathrm{Tr}K]$ coincides with the ring with unity generated over $\mathbb{Z}$ by all cyclic products.

We have $\mathbb{Z}[\mathrm{Tr}K] = \mathbb{Z}[\mathrm{Tr}\mathbb{Z}K]$. Indeed, clearly $\mathrm{Tr}K \subset \mathrm{Tr}\mathbb{Z}K$, hence $\mathbb{Z}[\mathrm{Tr}K] \subset \mathbb{Z}[\mathrm{Tr}\,\mathbb{Z}K]$. The reverse inclusion follows from the fact that Tr is an additive function. But $1-R_j$, $1 \leqslant j \leqslant s$, generate the ring $\mathbb{Z}K$ (here $R_j$, $1 \leqslant j \leqslant s$, is a generating system of reflections of K). Therefore, the monomials $(1-R_{j_1}) \dots (1-R_{j_r})$ generate $\mathbb{Z}$ as a $\mathbb{Z}$-module. We have:

$$\circledast \qquad (1-R_{j_1})(1-R_{j_r})(1-R_{j_{r-1}}) \dots (1-R_{j_2})e_{j_1} = c_{j_1\dots j_r}e_{j_1},$$

and it easily follows from this equality that

$$\mathrm{Tr}(1-R_{j_1})(1-R_{j_r})(1-R_{j_{r-1}}) \dots (1-R_{j_2}) = c_{j_1\dots j_r}$$

Therefore, $\mathbb{Z}[\mathrm{Tr}\mathbb{Z}K] = \mathbb{Z}[\dots, c_{j_1\dots j_r}, \dots]$ and we are done.
By now, let Γ be a K-invariant lattice of rank 2n. It follows from the equality ⊛ that

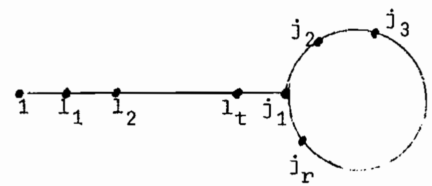


$$c_{j_1\dots j_r}\Gamma_{j_1} \subset \Gamma_{j_1}$$

for every $j_1, \dots j_r$. But $\mathrm{rk}\,\Gamma_j = 2$, see Section 4.1. Therefore, $c_{j_1\dots j_r}$ is an integral element of a certain purely imaginary quadratic extension of $\mathbb{Q}$; denote this extension by $L_{j_1}$.

Let $L = L_1$; we shall show that $c_{j_1\dots j_r} \in L$ for every $j_1, \dots, j_r$. K being irreducible, there exists

$$\alpha = c_{1l_1l_2\dots l_tj_1l_tl_{t-1}\dots l_1} \neq 0$$




for certain $l_1, \dots, l_t$. Let

$$\beta = c_{1l_1l_2\dots l_tj_1j_2\dots j_rj_1l_tl_{t-1}\dots l_1},$$

$$\gamma = c_{j_1j_2\dots j_r}.$$

Then $\alpha, \beta \in L$ and $\beta = \alpha\gamma$. But $\alpha \neq 0$, hence $\gamma = \beta/\alpha \in L$. Therefore the ring $\mathbb{Z}[\mathrm{Tr}K]$ lies in the maximal order of L.

d) ➡ e). This is proved in [7], Lemma 1.2.

d) ➡ a). Let $\mathbb{Z}[\mathrm{Tr}K] \subset D$, where D is the maximal order of a certain purely imaginary quadratic extension L of $\mathbb{Q}$. K being irreducible and D being integrally closed, one can again apply Lemma 1.2 from [7] and obtain that K is defined over D. Therefore there exists a K-invariant D-submodule Γ of V such that $\Gamma \otimes_D \mathbb{C} \to V$ is an isomorphism. But D is a Dedekind ring and Γ is a D-module of rank n without torsion. Therefore, Γ is isomorphic to a direct sum of n fractional ideals of the field L, see [8]. Let $J_1, \dots, J_n$

be these ideals. Then there exists a $\mathbb{C}$-basis $e_1, \ldots, e_n$ of V such that

$$\Gamma = J_1 v_1 + \ldots + J_n v_n.$$

But D is a lattice of rank 2 in $\mathbb{C}$ and for every fractional ideal J of L there exists $d \in D$, $d \neq 0$, with $d.J \subset D$. Hence J is also a lattice of rank 2 in $\mathbb{C}$.

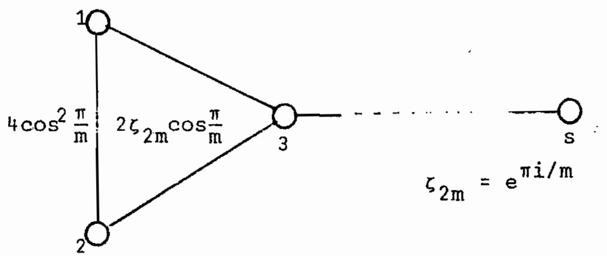
e) $\Rightarrow$ d). Let K be defined over a purely imaginary quadratic extension L of $\mathbb{Q}$. Then $\mathrm{Tr}P \in L$ for every $P \in K$, hence $\mathbb{Z}[\mathrm{Tr}K] \subset L$. But, TrP is an integral algebraic number. Therefore $\mathbb{Z}[\mathrm{Tr}K]$ lies in a maximal order of L.

d) $\Rightarrow$ f). Using Table 1, we can easily find those K for which $\mathbb{Z}[\mathrm{Tr}K]$ lies in the maximal order of a certain purely imaginary quadratic extension of $\mathbb{Q}$. It is convenient to use the following necessary condition on each generator of this ring $\mathbb{Z}[\mathrm{Tr}K]$:

$z \in \mathbb{C}$ <u>is an integral element of a certain purely imaginary quadratic extention of</u> $\mathbb{Q}$ <u>iff</u> $|z|^2 \in \mathbb{Z}$ <u>and</u> $2\,\mathrm{Re}\, z \in \mathbb{Z}$.

It appears <u>aposteriori</u> that this condition is also sufficient. After a certain amount of calculations we obtain the list of f).

<u>Example</u>. Let $K = K_2$, type $G(m,m,s)$, $s \geq 3$. The graph of K is




We have $\mathbb{Z}[\mathrm{Tr}K] = \mathbb{Z}[e^{\pi i/m}]$. The previous condition for the generator $2e^{\pi i/m}$ gives: $2\cos\frac{2\pi}{m} \in \mathbb{Z}$.



Therefore $m = 2,3,4,6$, and in these cases $\mathbb{Z}[\mathrm{Tr}K]$ lies in a maximal order of a certain purely imaginary extension of $\mathbb{Q}$. □

4.7. <u>Description of root lattices</u>.

We assume now that $k = \mathbb{C}$. We shall first describe (up to similarity) all invariant root lattices of rank 2n when $s = n$ (it follows from Section 4.5 that only these lattices are of interest to us).

<u>Theorem</u>. <u>Let</u> K <u>be an irreducible finite linear</u> r-<u>group and let</u> $\Lambda_j \subset \mathbb{C}e_j$, $1 \leq j \leq s$, <u>be a set of lattices of rank</u> 2 <u>and</u> $\Gamma = \Lambda_1 + \ldots + \Lambda_s$. <u>In order for</u> $\Gamma$ <u>to be a</u> K-<u>invariant root lattice with</u> $\Gamma_j = \Lambda_j$, $1 \leq j \leq s$, <u>it is necessary</u>, <u>and</u>, <u>if</u> $s = n$, <u>also sufficient</u>, <u>that</u>:

a) $\mathbb{Z}[\mathrm{Tr}K]\Lambda_j \subset \Lambda_j$, $1 \leq j \leq s$.

b) $(1-R_k)\Lambda_j \subset \Lambda_k$ and $(1-R_j)\Lambda_k \subset \Lambda_j$ for every $k \neq j$ such that $c_{kj} \neq 0$.

Moreover, b) is equivalent to

c) For every k and j, such that $j < k$ and $c_{kj} \neq 0$, one has

$$(1-R_k)\Lambda_j \subset \Lambda_k \subset c_{k_j}^{-1}(1-R_k)\Lambda_j.$$

<u>Proof</u>. Assertion a) follows from the formula $c_{j_1 \ldots j_r} \Gamma_{j_1} \subset \Gamma_{j_1}$ and the fact that $\mathbb{Z}[\mathrm{Tr}K]$ is generated over $\mathbb{Z}$ by cyclic products.

The "necessary part" of b) follows from the invariance of the lattice. Let us prove the "sufficient part". If $s = n$ then $\Gamma$ is in fact a direct sum of $\Lambda_j$, $1 \leq j \leq n$, hence $\Gamma$ is a lattice. This lattice is invariant under $1-R_k$ for every k: if $k \neq j$ then it follows from b); if $k = j$ then it follows from a). Hence, $\Gamma$ is K-invariant.

Now let us prove that b) ⇔ c).

b) ⇒ c). One obtains the proof by applying $1-R_k$ to both sides of $(1-R_j)\Lambda_k \subset \Lambda_j$.

c) ⇒ b). One obtains the proof by applying $1-R_j$ to both sides of $\Lambda_k \subset c_{kj}^{-1}(1-R_k)\Lambda_j$.

Corollary. Let $\Gamma$ be a nonzero K-invariant lattice. If $|c_{kj}| = 1$ then $\Gamma_j$ and $\Gamma_k$ determine each other uniquely by the formulas

$$\Gamma_k = (1-R_k)\Gamma_j \text{ and } \Gamma_j = (1-R_j)\Gamma_k.$$

Proof. We have by c):

$$(1-R_k)\Lambda_j \subset \Lambda_k \subset c_{kj}^{-1}(1-R_k)\Lambda_j$$

(we assume that $j < k$). The index of the left lattice in the right lattice is $|c_{kj}|^2 = 1$. Therefore the inclusions are in fact equalities. Applying $1-R_j$, we get

$$c_{jk}\Lambda_j \subset (1-R_j)\Lambda_k \subset \Lambda_j.$$

Again, the inclusions are in fact equalities. □

If $s = n + 1$, we also need to know $\Gamma^*$ for $\Gamma$ a K'-invariant lattice, and to select those $\Gamma \subset \Lambda \subset \Gamma^*$ for which

a) $\Lambda$ is $R_{n+1}$-invariant,

b) $\Lambda_j = \Gamma_j$, $1 \leqslant j \leqslant s$.

Therefore, we also need to know $(\Gamma^*)^0$. We have the following description of this lattice:

Theorem. $$\Gamma_j^* = \bigcap_{\substack{k \\ \text{such that} \\ c_{jk} \neq 0}} c_{jk}^{-1}(1-R_j)\Gamma_k, \qquad 1 \leqslant j \leqslant s.$$



In particular, $\Gamma_j^* = \Gamma_j$ if there is a number $k$ such that $|c_{jk}| = 1$.

Proof. We have

$$\lambda e_j^* \in \Gamma_j^* \quad \text{iff} \quad (1-R_k)\lambda e_j \in \Gamma_k, \quad 1 \leqslant k \leqslant s.$$

Assume that $c_{jk} \neq 0$, i.e. $\langle e_j | e_k \rangle \neq 0$. If $\lambda(1-R_k)e_j \in \Gamma_k$ then, applying $1-R_j$ to the both sides of this relation, we obtain $\lambda c_{jk}e_j \in (1-R_j)\Gamma_k$, i.e. $\lambda e_j \in c_{jk}^{-1}(1-R_j)\Gamma_k$.

Vice versa, if $\lambda e_j \in c_{jk}^{-1}(1-R_j)\Gamma_k$ for some $\lambda$ then applying $1-R_k$, we obtain: $(1-R_k)\lambda e_j \in c_{jk}^{-1}c_{jk}\Gamma_k = \Gamma_k$, i.e. $\lambda e_j \in \Gamma^*$.

In order to prove the second assertion, let us apply $1-R_j$ to $\Gamma_k \supset (1-R_k)\Gamma_j$. We obtain $\Gamma_j \supset (1-R_j)\Gamma_k \supset c_{jk}\Gamma_j$ and, if $|c_{jk}| = 1$ then $c_{jk}\Gamma_j = \Gamma_j$. Thus the inclusion is in fact an equality; hence $c_{jk}^{-1}(1-R_j)\Gamma_k = \Gamma_j$. □

Let us assume now that $s = n$.

An algorithm for constructing K-invariant root lattices of full rank: case 1.

We shall only consider here groups K from the theorem in Section 1.6 with the property: every two nodes of the graph of K (see Table 1) can be connected by a path of edges such that the absolute value of the weight of each edge equals 1.

These groups are:

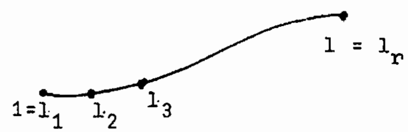
$K_1$; $K_2$, type $G(6,1,s)$, $s \geqslant 2$, type $G(m,m,s)$, $m = 2,4,6$, $s \geqslant 3$ and $m = 3$, $s \geqslant 2$; $K_3$, $m = 2,3,4,6$; $K_4$; $K_8$; $K_{24}$, $K_{25}$, $K_{29}$, $K_{32}$, $K_{33}$; $K_{34}$, $K_{35}$, $K_{36}$; $K_{37}$.

Algorithm:

Let $\Delta$ be an arbitrary lattice in $\mathbb{C}$ of rank 2, which is invariant under $\mathbb{Z}[\mathrm{Tr}K]$.

Let: $\Lambda_1 = \Delta e_1$.

For $l \geq 2$ let us consider an arbitrary path of edges from 1 to $l$, for which the absolute value of the weight of each edge equals 1.




Let

$$\Lambda_l = (1-R_{l_r}) \dots (1-R_{l_2})\Lambda_1 =$$

$$= \Delta\Big(\prod_{j=2}^{r} (1-\mu_{l_j}) \langle e_{l_{j-1}} | e_{l_j} \rangle\Big)\, e_{l_r}.$$

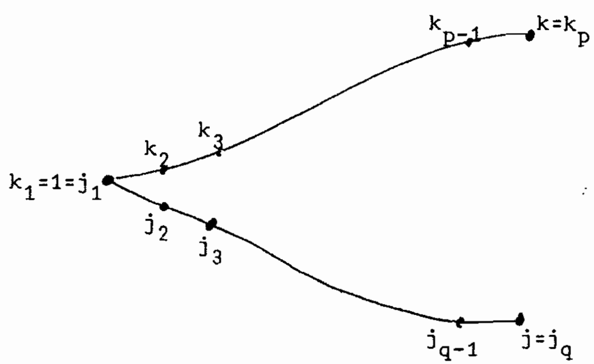
<u>We claim that</u>:

a) $\Gamma = \Lambda_1 + \dots + \Lambda_n$ <u>is a</u> K-<u>invariant lattice of full rank</u>.

b) $\Gamma$ <u>does not depend on the construction</u> (<u>i.e. on the choice of the paths</u>).

c) <u>each invariant lattice is obtained in this way</u>.

<u>Proof</u>. The assertions b) and c) follow from the corollary above. Let us prove a). We check the conditions a) and b) of the theorem proved at the beginning of this section. The condition a) is clearly fulfilled, so we need only check b). We have






$\Lambda_k = (1-R_{k_p}) \dots (1-R_{k_2})\Lambda_1$ and $\Lambda_j = (1-R_{j_q}) \dots (1-R_{j_2})\Lambda_1$.

Let $P = (1-R_{k_1}) \dots (1-R_{k_{p-1}})$. Then $P\Lambda_k = c_{k_1k_2\dots k_{p-1}k_pk_{p-1}\dots k_2}\Lambda_1 \subset \Lambda_1$.

(thanks to the construction of $\Lambda_1$).

But $\quad c_{k_1k_2\dots k_{p-1}k_pk_{p-1}\dots k_2} = c_{k_1k_2}c_{k_2k_3}\dots c_{k_{p-1}k_p}$.

Hence, $|c_{k_1k_2\dots k_{p-1}k_pk_{p-1}\dots k_2}| = 1$, so that

$$P\Lambda_k = \Lambda_1.$$

Let us consider $(1-R_k)\Lambda_j$. We have

$$P(1-R_k)\Lambda_j = (1-R_{k_1}) \dots (1-R_{k_{p-1}})(1-R_{k_p})(1-R_{j_q}) \dots (1-R_{j_2})\Lambda_1$$

$$= c_{k_1\dots k_{p-1}k_pj_q\dots j_2}\Lambda_1 \subset \Lambda_1. \text{ Therefore,}$$

$$P(1-R_k)\Lambda_j \subset \Lambda_1 = P\Lambda_k.$$

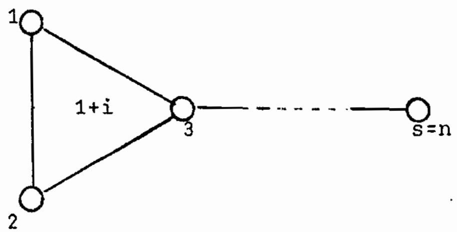
But the restriction of $P$ to $\mathbb{C}e_k$ has a trivial kernel, because $P\Lambda_k = \Lambda_1$. Therefore, $(1-R_k)\Lambda_j \subset \Lambda_k$. Using the same arguments we obtain also $(1-R_j)\Lambda_k \subset \Lambda_j$. □

Therefore we only need to find all $\Delta \subset \mathbb{C}$ such that $\mathbb{Z}[\mathrm{Tr}K]$ is in the ring of integers of the field it generates. It is well known how to do it (see [9]). We obtain, after checking, that in fact if $\mathbb{Z}[\mathrm{Tr}K] \neq \mathbb{Z}$ then $\mathbb{Z}[\mathrm{Tr}K]$ (in the cases under consideration) is the maximal order in its fraction field. Hence, up to similarity, we have

$$\Delta = \mathbb{Z}[\mathrm{Tr}K], \text{ if } \mathbb{Z}[\mathrm{Tr}K] \neq \mathbb{Z}.$$

Example.

$K = K_2$, type $G(4,4,s)$, $s \geq 3$. The graph is




The path we choose connecting 1 and $l \geq 3$ will be $1,3,4,\ldots,l$; and the path, connecting 1 and 2 will be 1,3,2.

Here $\mathbb{Z}[\mathrm{Tr}K] = \mathbb{Z}[i]$ and $\langle e_1|e_3\rangle = \langle e_2|e_3\rangle = \langle e_3|e_4\rangle = \cdots = \langle e_{n-1}|e_n\rangle = -\frac{1}{2}$.

Hence,

$$\Delta = [1,i],$$
$$\Lambda_1 = [1,i]e_1,$$
$$\Lambda_l = [1,i](1-(-1))^{l-2}(-\tfrac{1}{2})^{l-2}e_l = [1,i]e_l \text{ if } l \geq 3,$$
$$\Lambda_2 = [1,i](1-(-1))^2(-\tfrac{1}{2})^2 e_2 = [1,i]e_2.$$

Therefore

$$\Lambda = [1,i]e_1 + \ldots + [1,i]e_n.$$

Checking all cases as done in this example, one obtains exactly those lattices that are in the column Tran W with Lin W = K of Table 2.

An algorithm for constructing K-invariant root lattices of full rank: case 2.

We shall consider now the remaining irreducible finite r-groups K, i.e. the groups



$K_2$, type $G(m,1,s)$, $s \geq 2$, $m = 2,3,4$, and $G(6,6,2)$;

$K_5$; $K_{26}$; $K_{28}$.

We see that the graph of K in these cases is a chain. Taking a suitable numbering, we can assume that $c_{12}, c_{23}, \ldots, c_{n-1,n}$ are the only nonzero $c_{jk}$ (the numbering in Table 1 has this property).

Algorithm.

Let $\Delta_1$ be a $\mathbb{Z}[\mathrm{Tr}K]$-invariant lattice in $\mathbb{C}$.

We have

$$\Delta_1 \subset c_{12}^{-1}\Delta_1.$$

Let us take an arbitrary $\mathbb{Z}[\mathrm{Tr}K]$-invariant lattice $\Delta_2$ between these two lattices (such a lattice exists, e.g. $\Delta_1$ has this property):

$$\Delta_1 \subset \Delta_2 \subset c_{12}^{-1}\Delta_1.$$

And so on:

$$\Delta_1 \subset \Delta_2 \subset c_{12}^{-1}\Delta_1,$$
$$\Delta_2 \subset \Delta_3 \subset c_{23}^{-1}\Delta_2,$$
$$\ldots.$$
$$\Delta_{n-1} \subset \Delta_n \subset c_{n-1,n}^{-1}\Delta_{n-1}.$$

Let

$$\Lambda_l = \Delta_l(1-R_l)(1-R_{l-1}) \ldots (1-R_2)e_l =$$
$$= \Delta_l\Big(\prod_{j=2}^{l} (1-\mu_j)\langle e_{j-1}|e_j\rangle\Big)e_j.$$

We claim that $\Gamma = \Lambda_1 + \ldots + \Lambda_n$ is an invariant lattice and that any invariant lattice is of this form.

<u>Proof</u>. Let us check the conditions a) and c) of the theorem proved in the beginning of this section. We need in fact only check c), because a) is obvious. We have

$$\Lambda_l = \Delta_l(1-R_l)(1-R_{l-1}) \ldots (1-R_2)e_1$$

$$\Lambda_{l+1} = \Delta_{l+1}(1-R_{l+1})(1-R_l) \ldots (1-R_2)e_1 .$$

Therefore,

$$(1-R_{l+1})\Lambda_l = \Delta_l(1-R_{l+1})..(1-R_2)e_1 \subset \Delta_{l+1}(1-R_{l+1})..(1-R_2)e_1 =$$

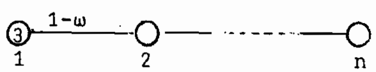
$$= \Lambda_{l+1} \subset c^{-1}_{l,l+1}\Delta_l(1-R_{l+1})\ldots(1-R_2)e_1 = c^{-1}_{l,l+1}(1-R_{l+1})\Lambda_l . \qquad \square$$

It appears that, as in the case 1,

$\mathbb{Z}[\mathrm{Tr}K]$ is the maximal order and $\Delta_1$ is similar to $\mathbb{Z}[\mathrm{Tr}K]$, if $\mathbb{Z}[\mathrm{Tr}K] \neq \mathbb{Z}$ .

<u>Example</u>.

$K = K_2$, type $G(3,1,s)$, $s \geqslant 2$. The graph is




We have $\mathbb{Z}[\mathrm{Tr}K] = \mathbb{Z}[\omega]$ and $\langle e_1 | e_2 \rangle = 1/\sqrt{2}$, $\langle e_2 | e_3 \rangle = \ldots =$ $= \langle e_{n-1} | e_n \rangle = -\frac{1}{2}$, $c_{12} = 1-\omega$, $c_{23} = \ldots = c_{n-1,n} = 1$.
Therefore, $(1-R_l) \ldots (1-R_2)e_1 = (1-(-1))^{l-1}(-\frac{1}{2})^{l-2}\frac{1}{\sqrt{2}}e_l =$ $= (-1)^l\sqrt{2}e_l$, $l = 2,\ldots,n$. We have

$$\Delta_1 = [1,\omega] ,$$

$$\Delta_2 = \ldots = \Delta_n ,$$

$$[1,\omega] \subset \Delta_2 \subset (1-\omega)^{-1}[1,\omega] .$$



But $|1-\omega|^2 = 3$. Hence,

$$\Delta_2 = [1,\omega] \text{ or } (1-\omega)^{-1}[1,\omega] = \frac{1}{\sqrt{3}}[1,\omega] .$$

Thus, we have only two possibilities: either

$$\Lambda = [1,\omega]e_1 + [1,\omega]\sqrt{2}e_2 + \ldots + [1,\omega]\sqrt{2}e_n ,$$

or

$$\Lambda = [1,\omega]e_1 + [1,\omega]i\sqrt{\tfrac{2}{3}}\, e_2 + \ldots + [1,\omega]i\sqrt{\tfrac{2}{3}}e_n .$$

One can check that, <u>using this algorithm, one obtains, the lattices that are in the column</u> Tran W <u>with</u> Lin W = K <u>of Table</u> 2 <u>as well as other lattices in the cases</u> $K = K_2$, <u>types</u> $G(2,1,2)$, $G(6,6,2)$; $K = K_5$ <u>and</u> $K = K_{28}$. As a matter of fact, these last lattices are not included in Table 2 because they do not give new r-groups (i.e. the semidirect product of K and the corresponding lattice is <u>equivalent</u> to one of the groups from Table 2). We omit the check of these last assertions.

4.8. <u>Invariant lattices in the case</u> $s = n+1$.

<u>Now we shall briefly consider the case</u> $s = n+1$.

These are the groups:

$G(4,2,n)$, $G(6,2,n)$, $G(6,3,n)$, $K_{12}$ and $K_{31}$.

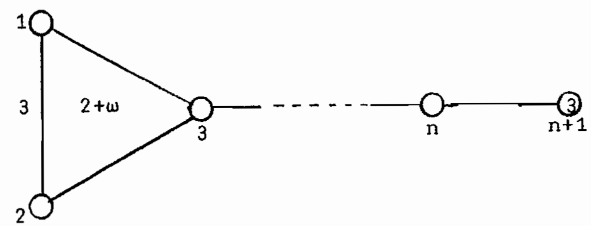
We shall explain the approach by several examples.

<u>Examples</u>.

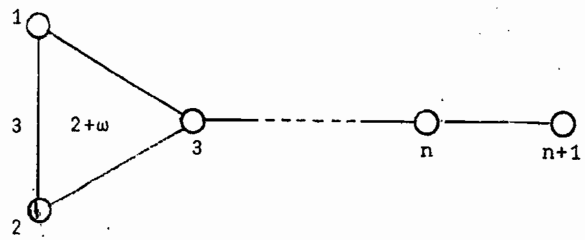
a) $K = K_2$, type $G(6,2,n)$ or $G(6,3,n)$.

The graphs of these groups are

G(6,2,n)



G(6,3,n)




We see that $K' = K_2$, type $G(6,6,n)$, cf. Table 1.
Using the previous theory, one checks that, up to similarity, there exists only one $K'$-invariant <u>root</u> lattice of full rank, i.e.

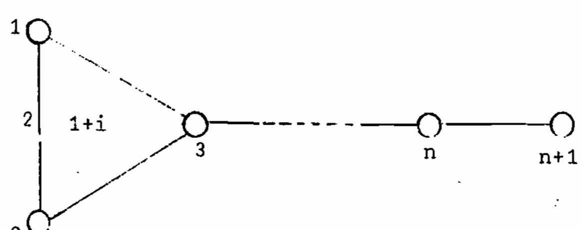
$$\Gamma = [1,\omega]e_1 + \ldots + [1,\omega]e_n.$$

But for $K'$ we have

$$\det S = 4\sin^2 \frac{\pi}{m}\Big|_{m=6} = 1,$$

see the example b) in Section 4.4. It follows from the corollary in Section 4.4 that, up to similarity, $\Gamma$ is a unique K-invariant lattice of full rank. But K <u>must</u> have an invariant lattice because of the theorem in Section 2.5. Therefore this lattice has to be $\Gamma$ (of cource, one can check that $\Gamma$ is a K-invariant lattice also by means of a straightforward computation).

b) $K = K_2$, type $G(4,2,n)$, $n \geq 3$. The graph is




We see that $K' = K_2$, type $G(4,4,n)$. One can check that there exists only one (up to similarity) $K'$-invariant root lattice of full rank:

$$\Lambda = [1,i]e_1 + \ldots + [1,i]e_n.$$



So, to describe all K-invariant lattices of full rank we need to find all $\Gamma$ such that:

a) $\Gamma^0 = \Lambda$ (with respect to $K'$);

b) $\Lambda \subset \Gamma \subset \Lambda^* = S^{-1}\Lambda$ (with respect to $K'$);

c) $(1-R_{n+1})\Gamma \subset \Gamma$.

We see that on every node there is an edge, whose weight is 1. Hence $(\Lambda^*)^0 = \Lambda$, see the theorem above. Therefore, b) $\Rightarrow$ a).

The matrix of S with respect to the basis $e_1,\ldots,e_n$ has the form

$$\begin{pmatrix} 2 & 1-i & -1 & & & \\ 1+i & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

We have $\det S = 4\sin^2\frac{\pi}{4} = 2$, see Section 4.4. Therefore the coefficients of $S^{-1}$ lie in $\mathbb{Z}[i,\frac{1}{2}]$ and (cf. Section 4.4), the order of $\Lambda^*/\Lambda$ is equal to 4. It follows from these facts that $\Lambda^*/\Lambda \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It is not difficult to see that $e_j = Sf_j$, $j = 1,2$, where

$$f_1 = \frac{n}{2}e_1 + \frac{-1-(n-1)i}{2}e_2 + \frac{(1-i)(n-2)}{2}e_3 + \frac{(1-i)(n-3)}{2}e_4 + \ldots + \frac{(1-i)1}{2}e_n,$$

$$f_2 = \frac{-(n-1)-i}{2}e_1 + \frac{ni}{2}e_2 + \frac{(i-1)(n-2)}{2}e_3 + \frac{(i-1)(n-3)}{2}e_4 + \ldots + \frac{(i-1)1}{2}e_n.$$

It is readily checked that $f_1$, $f_2$, $f_3$ are representatives of different nonzero elements of $\Lambda^*/\Lambda$. Therefore every K-invariant lattice of full rank has to be similar to one of the lattices:

$\circledast$ $\Lambda$; $\Lambda \cup (\Lambda+f_j)$, $j = 1,2,3$; $\Lambda \cup \bigcup_{j=1}^{3} (\Lambda+f_j)$

Using the equalities

$$e_{n+1} = \frac{-1-i}{\sqrt{2}}e_1 + \frac{-1+i}{\sqrt{2}}e_2 - \sqrt{2}e_3 + \sqrt{2}e_4 - \ldots - \sqrt{2}e_n,$$

$$(1-R_{n+1})v = \begin{cases} 0 \text{ if } v = e_1,\ldots,e_{n-1},f_3, \\ \sqrt{2}e_{n+1} \text{ if } v = e_n, \\ \frac{1-i}{\sqrt{2}}e_{n+1} \text{ if } v = f_1, \\ \frac{i-1}{\sqrt{2}}e_{n+1} \text{ if } v = f_2, \end{cases}$$

one can straightforwardly verify that <u>all the lattices</u> $\circledast$ <u>are</u> K-<u>invariant</u>.

The same considerations can be carried out for other groups K from the list above, and the description of <u>all</u> (up to similarity) K-invariant lattices of full rank will be obtained. We leave this to the reader (the most complicated case is $K = K_2$, type $G(4,2,2)$).



5. <u>The structure of r-groups in the case</u> $s = n + 1$.

When $s = n + 1$ it is no longer true in general that an infinite irreducible complex crystallograph r-group W is the semidirect product of Lin W and Tran W. We shall explain here how one can find the corresponding extensions of Tran W by Lin W in this case.

We assume that $k = \mathbb{C}$.

5.1. <u>The Cocycle c</u>.

We recall that the structure of the extension is given by a 1-cocycle c of $\widehat{K}$, where K = Lin W, with values in V; see Section 2.4.

Let $\Gamma$ = Tran W and $a \in E$. Taking a as an origin, we can identify A(E) and GL(V)V; then W consists of the elements $(P,c(P)+\Gamma)$, $P \in K$.

First of all, <u>one can assume</u>, <u>upon replacing c by a suitable cocycle homologuous to c</u>, <u>that</u>

$$c(r_1) = \ldots = c(r_n) = 0.$$

We assume that the order of $R_j$ is equal to $m(H_{R_j})$, $1 \leqslant j \leqslant s$, see Section 3.2.

<u>Proof</u>. Let $\gamma_1,\ldots,\gamma_n \in W$ be the reflections with Lin $\gamma_j = R_j$, $1 \leqslant j \leqslant n$. We assume, as usual, that the group K' generated by $R_j$, $1 \leqslant j \leqslant n$, is irreducible. We have $\bigcap_{j=1}^{n} H_{R_j} = 0$, hence $\bigcap_{j=1}^{n} H_{\gamma_j} = b \in E$. Then we have

$$\kappa_b(\gamma_j) = (R_j,0),\ j = 1,\ldots,n,$$

and we are done. $\square$

We shall assume now that

$$c(r_1) = \dots = c(r_n) = 0.$$

Therefore c is defined by only one vector $c(r_{n+1})$. Moreover, one can take

$$c(r_{n+1}) = \lambda e_{n+1}, \ \lambda \in \mathbb{C}.$$

because there exists a reflection $(R_{n+1},v)$ in W.

Therefore, given an invariant lattice $\Gamma$ of full rank, one has to solve the following problems

Find those $\lambda \in \mathbb{C}$ for which

a) the cocycle c of $\widetilde{K}$ with values in V, given by condition

$$c(r_j) = 0, \ 1 \leq j \leq n; \ c(r_{n+1}) = \lambda e_{n+1}, \ \lambda \in \mathbb{C},$$

satisfies

$$c(F) \in \Gamma$$

for any relation F of K (i.e. for every element F of Ker $\phi$, see Section 2.4).

b) Condition a) holds and the corresponding group W is an r-group.

<u>Theorem</u> 1) <u>If</u> $\Gamma = \Gamma^0$ <u>then</u> a) $\Rightarrow$ b).

2) If $c(F) \in \Gamma^0$ for <u>every relation</u> F then b) $\Rightarrow \Gamma = \Gamma^0$.

<u>Proof</u>. 1) Let $\Gamma = \Gamma^0$. We know that

$$\Gamma^0 = \Gamma_1 + \dots + \Gamma_{n+1}.$$

But $\Gamma' = \Gamma_1 + \dots + \Gamma_n$ is a root lattice for K' and the condition $c(r_1) = \dots = c(r_n) = 0$ shows that the semidirect product of K' and $\Gamma'$ lies in W. This semidirect product is an r-group, see Section 4.5.



We also have that W contains the reflections $(R_{n+1},\lambda e_{n+1}+t)$, $t \in \Gamma_{n+1}$.

Let $\gamma \in W$ be an arbitrary element, $\gamma = (P,v)$. Then P is a product of reflections $R_j$, $1 \leq j \leq n + 1$, in some order. Therefore, multiplying $\gamma$ by $(R_j,0)$, $1 \leq j \leq n$, and by $(R_{n+1},\lambda e_{n+1})$ in a suitable order, one can obtain an element of the form $(1,t)$. But the elements $(1,t')$, $t' \in \Gamma'$, are also products of reflections. Therefore, multiplying $\gamma$ by reflections, one can obtain $(1,r)$, $r \in \Gamma_{n+1}$. But

$$(R_{n+1},\lambda e_{n+1})(R_{n+1},\lambda e_{n+1}+r)(1,r) = (1,0).$$

Therefore $\gamma$ is a product of reflections.

2) We have $c(F) \in \Gamma^0$. Therefore c, in fact, defines a 1-cocycle of K with values in $V/\Gamma^0$. Let W' be the group defined by c, with Lin W' = K and Tran W' = $\Gamma^0$. It is an r-group because of 1). Also we have a group W defined by c, with Lin W = K and Tran W = $\Gamma$.

Let $\gamma \in W$ be a reflection. By Section 3.2, we know, there exists $\delta \in W'$ with $\delta \gamma \delta^{-1} = (R^l_j,t)$ for certain l and j. This is also a reflection, hence $t \perp H_{R_j}$. But $t = c(r_j^l) + v$, $v \in \Gamma$. We have

$$c(r_j^l) = (1+r_j+r_j^2 + \dots + r_j^{l-1})\, c(r_j).$$

By definition of c, we have $c(r_j) \perp H_{R_j}$. Hence, also $c(r_j^l) \perp H_{R_j}$. Therefore, $v \perp H_{R_j}$, or, in other words, $v \in \Gamma^0$. This means that $\delta\gamma\delta^{-1} \in W'$. It follows now from $\delta \in W'$ that $\gamma \in W'$; hence W = W' and $\Gamma = \Gamma^0$. □

The following simple observation is very useful in practice because it gives strong restrictions on the choice of $\lambda$:

Theorem. Let a) be satisfied and let $P \in K'$ be such that

$$R_{n+1} P R_{n+1}^{-1} \in K'.$$

Then

$$\lambda(1-R_{n+1} P R_{n+1}^{-1})e_{n+1} \in \Gamma.$$

Proof. It follows from the definition of c that $(P,0)$ and $(R_{n+1},\lambda e_{n+1}) \in W$. We have now:

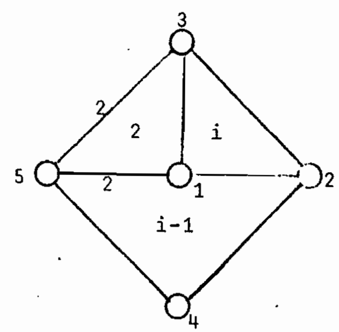
$$(R_{n+1},\lambda e_{n+1})(P,0)(R_{n+1},\lambda e_{n+1})^{-1} =$$

$$= (R_{n+1} P R_{n+1}^{-1}, -R_{n+1} P R_{n+1}^{-1}(\lambda e_{n+1}) + \lambda e_{n+1})$$

and we are done. □

5.2. Example.

$K = K_{31}$. The graph is




The vectors $e_1,e_2,\ldots e_5$ are given in Table 2. Note that $e_5 = ie_1 + e_2 + e_3$.

There exists only one (up to similarity) K-invariant lattice $\Gamma$ of full rank

$$\Gamma = [1,i]e_1 + \ldots + [1,i]e_4.$$

The presentation (i.e. the sets of generators and relations) of K is known, see [4].



The relations are:

$$R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^3 = (R_3R_1)^3 = (R_1R_2)^3 = 1,$$
$$(R_2R_1R_3R_1)^4 = 1,$$
$$(R_4R_5)^3 = 1.$$
$$R_5^2 = (R_5R_2)^2 = (R_5R_1R_3R_1)^2 = (R_5R_3)^4 =$$
$$= R_1(R_5R_3R_2R_3)R_1(R_5R_3R_2R_3)^{-1} = 1$$
$$R_4^2 = (R_4R_1)^2 = (R_4R_3)^2 = (R_4R_2)^3 = 1$$

It follows from $R_1(R_5R_3R_2R_3)R_1(R_5R_3R_2R_3)^{-1} = R_5^2 = 1$ that

$$R_5R_1R_5 \in K'.$$

Therefore

$$\Gamma \ni \lambda(1-R_5R_1R_5)e_5 = \lambda((1+i)e_1 + 2e_2 + 2e_3)$$

and hence $\lambda = \frac{a+bi}{2}$, $a,b \in \mathbb{Z}$, and $a \equiv b \pmod 2$. But $[1,i]e_5 \in \Gamma$. Therefore, we may assume that

$$\lambda = \frac{1+i}{2}.$$

It only remains to check whether $c(F) \in \Gamma$ or not for this particular $\lambda$. This has to be done by straightforward computations:

$$c(r_5^2) = c(r_5) + r_5c(r_5) = \frac{1+i}{2}(1+r_5)e = 0 \in \Gamma.$$
$$c((r_4r_5)^3) = (1+r_4r_5+(r_4r_5)^2)(c(r_4) + r_4c(r_5)) =$$
$$= \frac{1+i}{2}(1+r_4r_5+(r_4r_5)^2)(r_4e_5) = 0 \in \Gamma,$$

and so on (one only needs to consider the relations which involve $R_5$). This check shows that $\lambda = \frac{1+i}{2}$ gives a cocycle of K, indeed, and, hence, defines an r-group W with Lin W = K, Tran W = $\Gamma$.

It can be proved in a straightforward manner that this cocycle c is not a coboundary, i.e. W is not a semidirect product.

The same considerations can be given for each K with $s = n + 1$, and, as a result, one obtains Table 2. It appears a posteriori that in all cases either $\Gamma$ is a root lattice or $c(F) \in \Gamma^0$ for every relation F of K (therefore, for an r-group W, Tran W is always a root lattice).

By performing similar straightforward computations in the remaining cases, a proof of the theorems in Section 2.8. is obtained.

As for the proof of the theorem from Section 2.9, the first part of it is given in Section 4.6; the second part (about minimality of $\mathbb{Z}$[TrK]) follows from [9]. The third part follows from Table 1.



References:

[1] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV, V, VI, Hermann, Paris 1968.

[2] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, Canad. J. Math. 6 (1954), 274 - 304.

[3] A.M. Cohen, *Finite complex reflection groups*, Ann. Scient. Ec. Norm. Sup.4, t.9. (1976), 379 - 436.

[4] H.S.M. Coxeter, *Regular Complex Polytopes*, Cambridge University Press, London, 1974.

[5] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. I, II, III, Hermann, Paris, 1960, 1972.

[6] T.A. Springer, *Invariant theory*, Lecture Notes in Mathematics 585, Springer, Berlin, etc. 1977.

[7] E.B. Vinberg, *Rings of definition of dense subgroups of semisimple linear groups*, Izv. Akad. Nauk SSSR. Ser. Math. Tom 35 (1971) p. 45 - 55.

[8] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Intersc. Publ., New York, 1962.

[9] Z. Borevich and I. Shafarevich, *Number theory*, Moscow, 1963.