# Introduction to Mixed Hodge Modules

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In this note we give an elementary introduction to the theory of Mixed Hodge Modules [S1-5]. Philosophically the Mixed Hodge Modules are the objects in char. 0 which correspond to the perverse mixed complexes in char. p (cf. [B2] [BBD]) by the dictionary of Deligne [D1]. For the definition of Mixed Hodge Modules we have to use essentially the theory of filterd D-Modules and vanishing cycle functors. But in this note we try to avoid the technical difficulties as much as possible; e.g. the knowledge of D-Modules is not supposed in §1-2.

## §1. How to use Mixed Hodge Modules.

1.1. Let $X$ be an algebraic variety over $\mathbb{C}$ assumed always separated and reduced. We associate to $X$ its cohomology groups $H^{\bullet}(X, \mathbb{Z})$ functorially, and Deligne's fundamental result [D1] says that these cohomologies carry the natural mixed Hodge structure functorially. This result can be generalized to homologies, local cohomologies [loc. cit] and Borel Moore homologies [B1] etc. But to do so more systematically and generalize these results, e.g. to define the pure Hodge structure on intersection cohomologies, we can argue as follow

1.2. We first make refinement of the cohomology theory due to Verdier etc. Instead of abelian groups $H^{\bullet}(X, \mathbb{Z})$ etc. we associate to $X$ the category $D_c^b(\mathbb{Q}_X)$ (the derived category of bounded $\mathbb{Q}$-complexes with constructible cohomologies [V1]). Here we change the coefficient $\mathbb{Z}$ by $\mathbb{Q}$ for the relation with the perverse sheaves. Then these categories are stable by the natural functors like $f_*$, $f_!$, $f^*$, $f^!$, $\mathbb{D}$, $\psi_g$, $\phi_{g,1}$, $\boxtimes$, $\otimes$, $\underline{Hom}$, where $f$ is a morphism of varieties and $g$ is a function. Here $\phi_{g,1}$ is the unipotent monodromy part of $\phi_g$ (cf. [D3] for the definition of vanishing cycle functors $\psi_g$, $\phi_g$). As to the direct images and the pull-backs we have the adjoint relations (cf. [V2]):

(1.2.1) $\mathrm{Hom}(f^*M,N) = \mathrm{Hom}(M,f_*N) \quad \mathrm{Hom}(f_!M,N) = \mathrm{Hom}(M,f^!N)$

and $\mathbb{D}^2 = \mathrm{id}$, $\mathbb{D}f_* = f_!\mathbb{D}$, $\mathbb{D}f^* = f^!\mathbb{D}$. Let $Z$ be a closed subvariety of $X$, and $i : Z \to X$ and $a_X : X \to pt$ the natural morphisms so that $\mathbb{Q}_X = a_X^*\mathbb{Q}$, $\mathbb{D}\mathbb{Q}_X = a_X^!\mathbb{Q}$ and $i_*i^! = \mathbb{R}\underline{\Gamma}_Z$. Then $H^\cdot(X,\mathbb{Q})$, $H_\cdot(X,\mathbb{Q})$, $H_\cdot^{BM}(X,\mathbb{Q})$ and $H_Z^\cdot(X,\mathbb{Q})$ are respectively the cohomologies of

$$(a_X)_*a_X^*\mathbb{Q},\ (a_X)_!a_X^!\mathbb{Q},\ (a_X)_*a_X^!\mathbb{Q} \ \text{ and } \ (a_X)_*i_*i^!a_X^*\mathbb{Q}.$$

Moreover the restriction morphism

$$f^\# : \mathbb{Q}_Y \to f_*\mathbb{Q}_X \quad \text{and} \quad f^\# : (a_Y)_*\mathbb{Q}_Y \to (a_X)_*\mathbb{Q}_X$$

is induced by the adjoint relation (1.2.1) putting $M = \mathbb{Q}_Y$, $N = \mathbb{Q}_X = f^*\mathbb{Q}_Y$, and the Gysin morphism

$$f_\# : f_!\mathbb{D}\mathbb{Q}_X \to \mathbb{D}\mathbb{Q}_Y \quad \text{and} \quad f_\# : (a_X)_!\mathbb{D}\mathbb{Q}_X \to (a_Y)_!\mathbb{D}\mathbb{Q}_Y$$

by the dual arguement. Note that $f_* = f_!$ if $f$ proper, and $\mathbb{D}\mathbb{Q}_X = \mathbb{Q}_X(d_X)[2d_X]$ if $X$ smooth, where $d_X = \dim X$. In particular we get the usual Gysin morphism if $X$, $Y$ are smooth and proper.

The main result of [S5, §4] (cf. also [S3-4]) is that the above theory of $\mathbb{Q}$-complexes underlies the theory of mixed Hodge Modules, i.e.

1.3. Theorem. For each $X$ we have $MHM(X)$ the abelian category of mixed Hodge Modules with the functor

$$\mathrm{rat} : D^bMHM(X) \to D_c^b(\mathbb{Q}_X)$$

which associates their underlying $\mathbb{Q}$-complexes to mixed Hodge Modules, such that $\mathrm{rat}(MHM(X)) \subset \mathrm{Perv}(\mathbb{Q}_X)$, i.e.

$\mathrm{rat} \circ H = {}^pH \circ \mathrm{rat}$ (cf. [BBD] for the definition of $\mathrm{Perv}(\mathbb{Q}_X)$ and ${}^p\underline{H}$). Moereover the functors $f_*$, $f_!$, $f^*$, $f^!$, $\mathbb{D}$, $\psi_g$, $\phi_{g,1}$, $\boxtimes$, $\otimes$, $\underline{Hom}$ are naturally lifted to the functors of $D^b\mathrm{MHM}(X)$, i.e. they are compatible with the corresponding functors on the underlying $\mathbb{Q}$-complexes via the functor rat.

As to the relation with Deligne's mixed Hodge structure we have

1.4. Theorem. $\mathrm{MHM}(pt)$ is the category of polarizable $\mathbb{Q}$-mixed Hodge structures.

In particular we have uniquely $\mathbb{Q}^H \in \mathrm{MHM}(pt)$ such that $\mathrm{rat}(\mathbb{Q}^H) = \mathbb{Q}$ and $\mathbb{Q}^H$ is of type $(0,0)$. Put $\mathbb{Q}_X^H = a_X^*\mathbb{Q}^H$. Then the same argument as in 1.2 applies replacing $D_c^b(\mathbb{Q}_X)$, $\mathbb{Q}_X$, etc. by $D^b\mathrm{MHM}(X)$, $\mathbb{Q}_X^H$, etc. In particular we get the mixed Hodge structure on the cohomology groups, etc. with the restriction and Gysin morphisms in the category of mixed Hodge structures (or Modules). Here we have proved a little bit stronger result, because $(a_X)_*a_X^*\mathbb{Q}^H$, etc. are complexes of mixed Hodge structures (compare to [B1]). We have also the multiplicative structure on $(a_X)_*\mathbb{Q}_X^H$ by the morphism in $D^b\mathrm{MHM}(pt)$:

$$(a_X)_*\mathbb{Q}_X^H \otimes (a_X)_*\mathbb{Q}_X^H = (a_{X\times X})_*\mathbb{Q}_{X\times X}^H \xrightarrow{\Delta^*} (a_X)_*\mathbb{Q}_X^H,$$

because $\mathbb{Q}_{X\times X}^H = \mathbb{Q}_X^H \boxtimes \mathbb{Q}_X^H$, where $\Delta : X \to X\times X$ is the diagonal embedding.

As suggested by the terminology 'mixed' (cf. [BBD]), we have the following

1.5. Proposition. Each $M \in \mathrm{MHM}(X)$ has a finite increasing filtration $W$ in $\mathrm{MHM}(X)$, called the weight filtration of $M$, such that the functors $M \to W_iM$, $\mathrm{Gr}_i^WM$ are exact.

1.6. Definition. $M \in D^b\mathrm{MHM}(X)$ is mixed of weight $\leq n$ (resp. $\geq n$) if $\mathrm{Gr}_i^WH^jM = 0$ for $i > j+n$ (resp. $i < j+n$), and pure of weight $n$ if $\mathrm{Gr}_i^W H^j M = 0$ for $i \neq j+n$.

The followings are the analogy of [BBD].

1.7. Proposition. If $M$ is of weight $\leq n$ (resp. $\geq n$), so are $f_! M$, $f^* M$ (resp. $f_* M$, $f^! M$).

1.8. Corollary. $f_* M$ is pure of weight $n$ if $M$ is pure of weight $n$ and $f$ is proper.

1.9. Proposition. For any $M \in \mathrm{MHM}(X)$, $\mathrm{Gr}_i^W M$ is a semisimple object of $\mathrm{MHM}(X)$.

1.10. Corollary. $\mathrm{Ext}^i(M,N) = 0$ for $M$ mixed of weight $\leq m$ and $N$ of weight $\geq n$, if $m < n+i$.

1.11. Corollary. We have a noncanonical isomorphism

$$M = \oplus H^j M[-j] \quad \text{in} \quad D^b\mathrm{MHM}(X),$$

if $M$ is pure of weight $n$.

1.12. Theorem. If $M$ is pure and $f$ is proper, we have a noncanonical isomorphism in $D^b\mathrm{MHM}(Y)$:

$$f_* M = \oplus H^j f_* M[-j].$$

1.13. To get the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber (after taking rat), we have to explain about the intersection complexes. Assume $X$ is irreducible (or, more generally, equidimensional). Let $j : U \to X$ be a non-singular affine open dense subset. Then the intersection complex is defined by

$$\underline{IC}_X \mathbb{Q} = \mathrm{Im}(j_! \mathbb{Q}_U[d_X] \to j_* \mathbb{Q}_U[d_X]) \in \mathrm{Perv}(\mathbb{Q}_X),$$

which is independent of the choice of $U$, cf. [BBD]. We define $\underline{IC}_X \mathbb{Q}^H \in \mathrm{MHM}(X)$ replacing $\mathbb{Q}_U$ by $\mathbb{Q}_U^H$ so that

$\mathrm{rat}(\underline{IC}_X\mathbb{Q}^H) = \underline{IC}_X\mathbb{Q}$. Then $\underline{IC}_X\mathbb{Q}^H$ has no subobject and no quotient object supported in $X \setminus U$. In particular it is simple and <u>pure of weight</u> $d_X$, beacuse so is $\mathbb{Q}_U^H[d_X]$. Substituting $\underline{IC}_X\mathbb{Q}^H$ to $M$, we get the decomposition theorem of BBDG after taking rat, and the pure Hodge structure on the intersection cohomology $IH^{j+d_X}(X,\mathbb{Q}) = H^j(X,\underline{IC}_X\mathbb{Q})$. Note that these results are generalized to the case of intersection complexes (or cohomologies) with coefficient in polarizable variations of Hodge structures, cf. 2.3.

As to the relation between $\mathbb{Q}_X^H$ and $\underline{IC}_X\mathbb{Q}^H$, we have the following

1.14. <u>Proposition</u>. $H^j\mathbb{Q}_X^H = 0$ $(j>d_X)$, $\mathrm{Gr}_i^W H^{d_X}\mathbb{Q}_X^H = 0$ $(i>d_X)$ and $\mathrm{Gr}_{d_X}^W H^{d_X}\mathbb{Q}_X^H = \underline{IC}_X\mathbb{Q}^H$.

In particular we get the (quotient) morphism

$$(1.14.1) \qquad \mathbb{Q}_X^H[d_X] \to H^{d_X}\mathbb{Q}_X^H \to \underline{IC}_X\mathbb{Q}^H \quad \text{in } D^b\mathrm{MHM}(X)$$

inducing the identity on $U$. (This morphism is unique.)

1.15. Let $i: Z \to X$ be a closed irreducible subvariety. We have a natural morphism in $D^b\mathrm{MHM}(X)$:

$$\mathbb{Q}_X^H \xrightarrow{i^\#} i_*\mathbb{Q}_Z^H \to i_*\underline{IC}_Z\mathbb{Q}^H[-d_Z].$$

Composing this with its dual, we get the cycle class of $Z$:

$$\begin{aligned} cl_Z^H &\in \mathrm{Hom}(\mathbb{Q}_X^H, \mathbb{D}\mathbb{Q}_X^H(-d_Z)[-2d_Z]) \\ &= \mathrm{Hom}(\mathbb{Q}^H, (a_X)_*a_X^!\mathbb{Q}^H(-d_Z)[-2d_Z]), \end{aligned}$$

because $\mathbb{D}\,\underline{IC}_Z\mathbb{Q}^H = \underline{IC}_Z\mathbb{Q}^H(d_Z)$. Note that $(a_X)_*a_X^!\mathbb{Q}^H$ corresponds to the Borel-Moore homology(cf. 1.2), and if $X$ is smooth $cl_Z^H$ belongs to the $\mathbb{Q}$-Deligne cohomology, because

$$\mathrm{Hom}(\mathbb{Q}^H, (a_X)_*\mathbb{Q}_X^H(p)[2p]) \subset H_{\mathcal{D}}^{2p}(X,\mathbb{Q}(p)), \text{ cf. [B1]},$$

where $p = \mathrm{codim}\, Z$. (The above inclusion becomes the equality, if $X$ is smooth and proper.) Here $(n)$ is the Tate twist for $n \in \mathbb{Z}$, and defined, for example, by $\boxtimes \mathbb{Q}^H(n)$, where $\mathbb{Q}^H(n)$ is the mixed Hodge structure of type $(-n,-n)$, cf. [D1]. We can show that the above construction induces the cycle map

$$CH_d(X) \otimes \mathbb{Q} \to \mathrm{Hom}(\mathbb{Q}^H,\ (a_X)_* a_X^! \mathbb{Q}^H(-d)[-2d]),$$

and if $X$ is smooth and proper, it induces Griffiths' Abel-Jacobi map tensored by $\mathbb{Q}$.

1.16. Remark. Let $X$ be a smooth and proper variety over $\mathbb{C}$. Then we have an exact sequence (choosing $i = \sqrt{-1}$ ):

$$0 \to J^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p)) \to H^{p,p}(X,\mathbb{Q}) \to 0$$

where $$J^p(X)_{\mathbb{Q}} = H^{2p-1}(X,\mathbb{Q})\backslash H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C})$$

$$H^{p,p}(X,\mathbb{Q}) = F^pH^{2p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{Q}).$$

Let $f : X \to S = \mathbb{P}^1$ be a Lefschetz pencil, and put $S' = S \setminus f(\mathrm{Sing}\, f)$ so that $f' : X' \to S'$ the restriction of $f$ over $S'$ is smooth. Then we have

$$0 \to J^p(X_t)_{\mathbb{Q}} \to H^{2p}(X_t,\mathbb{Q}(p)) \to H^{p,p}(X_t,\mathbb{Q}) \to 0$$

where $X_t = f^{-1}(t)$ for $t \in S'$. For $\xi \in H^{p,p}(X,\mathbb{Q})$ we choose a lift $\xi^{\mathcal{D}}$ of $\xi$ in $H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))$. Restricting $\xi^{\mathcal{D}}$ to $X_t$, we get $\xi_t^{\mathcal{D}} \in H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))$, and it belongs to $J^p(X_t)_{\mathbb{Q}}$ if $\xi_t$ is zero in $H^{2p}(X_t,\mathbb{Q}(p))$. In this case $\xi_t^{\mathcal{D}}$ determine the normal function with value in $J^p(X_t)_{\mathbb{Q}}$ and we can show that they give an element of $\mathrm{Ext}^1(\mathbb{Q}^H_{S'},\ (R^{2p-1}f'_*\mathbb{Q}_{X'})^H)$, where $\mathrm{Ext}^1$ is taken in $\mathrm{MHM}(S')$; in particular, the corresponding variation of mixed Hodge structure is admissible in the sense of Steenbrink-Zucker [SZ]. Here note that $\xi_t^{\mathcal{D}}$ are not uniquely determined by $\xi$, but depend on the choice of lift $\xi^{\mathcal{D}}$ (e.g. if $2p = \dim X$ and $H^{2p-1}(X) \neq 0$). Therefore to prove the Hodge conjecture, we have to choose a good lift; otherwise $\xi_t$ would not belong to the image of the Abel-Jacobi map. Note that for the proof of the Hodge conjecture we can restrict to a non-empty open subset (i.e., it is enough to construct a cycle in $X'$), because the Hodge conjecture is equivalent to the following:

(1.16.1) For a smooth proper variety $X$ and $\xi \in H^{pp}(X,\mathbb{Q})$, there exists a nonempty Zariski open subset $U$ such that the restriction of $\xi$ to $U$ is zero.

In particular we may assume $X$ is projective using [D1].

1.17. Remark. For $i : Z \to X$ a closed immersion of varieties we define:

$$_{\mathcal{H}}H_Z^j(X,\mathbb{Q}(n)) = \mathrm{Hom}(\mathbb{Q}_Z^H,\ i^! a_X^* \mathbb{Q}^H(n)[j])$$

$$_{\mathcal{H}}H_j(X,\mathbb{Q}(n)) = \mathrm{Hom}(\mathbb{Q}_X^H,\ a_X^! \mathbb{Q}^H(-n)[-j]).$$

Then they form a Poincare duality theory with support in the sense of [BO]. In fact, (1.3.1)(Cap product with supports)

$$_{\mathcal{H}}H_k(X,\mathbb{Q}(m)) \otimes {}_{\mathcal{H}}H_Z^j(X,\mathbb{Q}(n)) \to {}_{\mathcal{H}}H_{k-j}(Z,\mathbb{Q}(m-n))$$

is given by the composition of $u$ and $i^! v$ for $u \in {}_{\mathcal{H}}H_Z^j(X,\mathbb{Q}(n))$ and $v \in {}_{\mathcal{H}}H_k(X,\mathbb{Q}(m))$, and (1.3.4)(Fundamental class) is constructed in 1.15, so that (1.3.5)(Poincare duality) becomes trivial, because $\eta_X$ is the natural isomorphism $\mathbb{Q}_X^H \xrightarrow{\sim} a_X^! \mathbb{Q}^H(-d)[-2d]$ if $X$ is smooth of dimension $d$. Moreover the well-definedness of the cycle map in 1.15 implies (1.5)(Principal triviality).

1.18. The following application is suggested by Durfee. Assume $X$ is analytically irreducible (or equidimensional) at $x \in X$, and put $j : U := X \setminus \{x\} \to X$ and $i : \{x\} \to X$. By restricting $X$ to an analytic neighborhood of $\{x\}$, we may assume $U \simeq I \times L$ (topologivally) by the cone theorem, where $I$ is an open interval and $L$ is the neighborhood boundary $\partial X$. Therefore $IH(L)$ the intersection cohomology of $L$ is given by the cohomology of

$$i^* j_* \underline{IC}_U \mathbb{Q}^H = C(j_! \underline{IC}_U \mathbb{Q}^H \to j_* \underline{IC}_U \mathbb{Q}^H).$$

up to the shift of complex by $n = d_X$. In particular we get the mixed Hodge structure on $IH(L)$ and the duality of mixed Hodge structure

$$IH^j(L) \otimes IH^{2n-1-j}(L) \to \mathbb{Q}(-n),$$

because $\mathbb{D}\, j_! = j_* \mathbb{D}$ and $\mathbb{D}(\underline{IC}_U \mathbb{Q}^H) = \underline{IC}_U \mathbb{Q}^H(n)$. We have also the estimate of weight:

$IH^j(L)$ is of weight $\leq j$ for $j < n$ and $> j$ for $j \geqq n$,

because the assertion for $j < n$ follows from the isomorphism

$$\tau_{<0} i^* j_* \underline{IC}_U \mathbb{Q}^H = i^* \underline{IC}_X \mathbb{Q}^H$$

and we use the duality for $j \geqq n$.

1.19. Let $g$ be a non zero divizor of $\Gamma(X, \mathcal{O}_X)$, and put $i : Y = g^{-1}(0)_{red} \to X$, $n = \dim X$ and $M = \underline{IC}_X \mathbb{Q}^H$ (or more generally, $M$ is pure of weight $n$ and has no subobject supported in $Y$). Then we have an exact sequence in $MHM(Y)$:

$$(1.19.1) \quad 0 \to i^* M[-1] \to \psi_{g,1} M \to \phi_{g,1} M \to 0$$

The weight filtration $W$ of $\psi_g M$ and $\phi_{g,1} M$ are the monodromy filtration shifted by $n-1$ and $n$. Let $P_N$ denote the primitive part of $Gr_j^W$. Then we have

$$(1.19.2) \quad \begin{aligned} P_N Gr_j^W \psi_{g,1} M &= P_N Gr_j^W \phi_{g,1} M \quad \text{for } j \geqq n \\ P_N Gr_{n-1+j}^W \psi_{g,1} M(j) &= Gr_{n-1-j}^W i^* M[-1] \quad \text{for } j \geqq 0. \end{aligned}$$

For $x \in Y$ put $i_x : \{x\} \to Y$. If $X$ is smooth and $M = \mathbb{Q}_X^H[n]$, we have

$$i_x^* i^* M[-1] = \mathbb{Q}^H[n-1]$$

and $i_x^* \phi_g M$ gives the reduced cohomologies of the Milnor fiber around $x$, where $\phi_g M = \psi_{g,\neq 1} \oplus \phi_{g,1} M$. Note that $Gr_\cdot^W \psi_g M$ is calculated by $\phi_g M$ using (1.19.2) and 1.14 (i.e.

$P_N \mathrm{Gr}^W_{n-1}\psi_{g,1}M = \mathrm{Gr}^W_{n-1}\mathbb{Q}^H_Y[n-1] = \underline{IC}_Y\mathbb{Q}^H$). If $X$ and $Y$ have an isolated singular point at $x$ and $M = \underline{IC}_X\mathbb{Q}^H$, we have a natural inclusion of spectral sequences in $\mathrm{MHM}(pt)$:

$$(1.19.3)\quad \begin{array}{ccc} E_1^{-k,j+k} = H^j i_x^* \mathrm{Gr}^W_k i^* M[-1] & \Longrightarrow & H^j i_x^* i^* M[-1] \\ \downarrow & & \downarrow \\ E_1^{-k,j+k} = H^j i_x^* \mathrm{Gr}^W_k \psi_{g,1} M & \Longrightarrow & H^j i_x^* \psi_{g,1} M \end{array}$$

By assumption $E_1^{-k,j+k} = 0$ except for $j = 0$ and $j < 0$, $k = n-1$, and we have for $j \neq 0$:

$$(1.19.4)\quad H^j i_x^* \mathrm{Gr}^W_{n-1} i^* M[-1] = H^j i_x^* \mathrm{Gr}^W_{n-1}\psi_{g,1}M = H^j i_x^* \underline{IC}_Y\mathbb{Q}^H.$$

In particular $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$ is zero except for $(p,q) = (1-n,n-2)$, and the image $I_r \subset E_r^{1-n+r,n-1-r}$ of $d_r$ is independent of the two spectral sequences. Then we have

$$(1.19.5)\quad H^j i_x^* i^* M[-1] = H^j i_x^* \psi_{g,1}M \hookrightarrow H^j i_x^* \underline{IC}_Y\mathbb{Q}^H \quad \text{for } j \neq 0$$

where the inclusion becomes an isomorphism for $j < -1$, and $\mathrm{Gr}^W_{n-1-r}$ of the cokernel of the inclusion for $j = -1$ is $I_r$ for $r > 0$. We have also

$$(1.19.6)\quad \mathrm{Gr}^W_{n-1+k}\psi_{g,1}M = \begin{cases} i_{x*}\mathrm{Gr}^W_{n-1+k} H^0 i_x^* \psi_{g,1}M & \text{for } k>0 \\ i_{x*}\mathrm{Gr}^W_{n-1} H^0 i_x^* \psi_{g,1}M \oplus \underline{IC}_Y\mathbb{Q}^H & \text{for } k=0 \\ i_{x*}\mathrm{Gr}^W_{n-1-k} H^0 i_x^* \psi_{g,1}M(-k) & \text{for } k<0, \end{cases}$$

(same for $i^*M[-1]$ if $k=0$), where the induced filtration $W$ on $H^0 i_x^* \psi_{g,1}M$ (or $H^0 i_x^* i^* M[-1]$) is the weight filtration (this is not true for $H^j$ with $j \neq 0$). As a corollary, we have a direct sum decomposition $\mathrm{Gr}^W_\bullet H^0 i_x^* \psi_{g,1}M = L_\bullet \oplus L^!_\bullet$ as a graded $\mathbb{Q}[N]$-modules such that $L_\bullet$ (resp. $L^!_\bullet$) is symmetric with center $n-1$ (resp. $n$) and

$$(1.19.7)\quad \begin{array}{lll} P_N L_{n-1+k} = \mathrm{Gr}^W_{n-1-k} H^0 i_x^* i^* M[-1](-k) & & \text{for } k \geq 0 \\ P_N L'_{n-1+k} = I_k(-k) & & \text{for } k > 0, \end{array}$$

using (1.19.2), cf. [S1,§4]. As for $\psi_{g,\neq 1}$, we have

(1.19.8) $\mathrm{Gr}_k^W \psi_{g,\neq 1} M = i_{x*} \mathrm{Gr}_k^W H^0 i_x^* \psi_{g,\neq 1} M$

because $\mathrm{supp}\ \psi_{g,\neq 1} M \subset \{x\}$. Note that if $X$ is furthermore smooth and $n > 1$, $H^j i_x^* i^* M[-1] = 0$ for $j > 1-n$ and we have $L. = 0$ and $I_k = \mathrm{Gr}_{n-1-k}^W\ H^{-1} i_x^* \underline{IC}_Y \mathbb{Q}^H$ as is well-known. To get the information about $\psi_{g,\neq 1}$, we can also replace $X$ by $\{g = t^m\} \subset X \times \mathbb{C}$, where $m$ is a positive integer such that the m-th power of the monodromy becomes unipotent. Then (1.19.7) is compatible with the join theorem of mixed Hodge structure on the Milnor cohomologies.

Now we assume $X$ is smooth, and put $Z = (\mathrm{Sing}\ g)_{\mathrm{red}}$, $Z' = Z \setminus \{x\}$ (same for $X'$, $Y'$) and $M = \mathbb{Q}_X^H[n]$. We assume that $g$ is locally topologically trivial along $Z'$ and $Z'$ is smooth. For $z \in Z'$, we assume:

(1.19.9) $H^{-d} i_z^* \psi_{g,1} M = 0$ , where $d = d_Z$,

(1.19.10) the monodromy of $H^{-d} i_z^* \psi_g\ M$ is semi-simple.

Then by (1.19.9) $\mathbb{Q}_{Y'}^H[n-1] = \underline{IC}_{Y'} \mathbb{Q}^H$, i.e. $\mathrm{supp}\ \mathrm{Gr}_k^W i^* M \subset \{x\}$, and for the second spectral sequence in (1.19.3), $E_1^{-k,j+k} = 0$ except for $j \neq 0$ and $j < 0$, $k = n-1$. Therefore as for $\psi_{g,1} M$ the same argument as above holds. As for $\psi_{g,\neq 1} M$ we have the spectral sequence as in (1.19.3) with the vanishing of $E_1$-term as above by (1.19.10), but the calculation of $\mathrm{Gr}_{n-1}^W \psi_{g,\neq 1} M$ is not so easy. If $d = 1$, $H^{-1} i_x^* j_{!*} j^{-1} \psi_{g,\neq 1} M$ is the invariant part by the monodromy along each irreducible component of $Z$ , where $j : Y' \to Y$. But in general we have to caluculate the cohomology of the local fundamental group of $(Z,x)$.

1.20. For $M \in D^b\mathrm{MHM}(X)$ and $f : X \to Y$, $g : Y \to Z$, we have the (perverse) Leray spectral sequence in $\mathrm{MHM}(Z)$:

(1.20.1) $E_2^{pq} = H^p g_* H^q f_* M \Longrightarrow H^{p+q} (gf)_* M$ ,

which degenerates at $E_2$ if $f$ is proper and $M$ is pure.

1.21. For an application to the representation theory, see [T].

§2. Naturality.

2.1. Assume $X$ smooth, and let $MHM(X)_s$ be the full subcategory of $MHM(X)$ consisting of smooth mixed Hodge Modules, where $M \in MHM(X)$ is called smooth iff $rat(M)$ is a local system. Let $VMHS(X)_{ad}$ be the category of admissible variation of mixed Hodge structures, where a variation of mixed Hodge structure is called admissible if it is graded polarizable and for any morphism $f : S \to X$ with $\dim S = 1$, its pull-back by $f$ satisfies the conditions of [SZ], cf. [K2]. Then we have the equivalence of categories:

2.2. Theorem. $MHM(X)_s \simeq VMHS(X)_{ad}$ .

This implies that a polarizable variation of Hodge structure of weight $n$ is a smooth mixed Hodge Module and pure of weight $n + \dim X$. In particular, the polarizable Hodge Modules are the pure Hodge Molules by the stabilty by intermediate direct images $j_{!*} = Im(j_! \to j_*)$, and for $X$ irreducible we have

2.3. Theorem. $MH_X(X,n)^p \simeq VHS(X,n-\dim X)^p_{gen}$ .

Here the left hand side is the category of polarizable Hodge Modules of weight $n$ with strict support $X$ (i.e. having no subobject and no quotient object supported in a proper subvariety of $X$) and the right hand side is the category of polarizable variation of Hodge structures of weight $n - \dim X$ defined on some nonempty smooth open subset of $X$, whose local monodromies are quasi-unipotent.

As a corollary of 2.2, we get a canonical mixed Hodge structure on $H(X,L)$ if $L$ underlies an admissible variation of mixed Hodge structure. (This result can be generalized to the analytic case if $X$ has a Kähler compatification, using [KK].)

2.4. Let $g$ be a function on $X$ . Put $Y = g^{-1}(0)_{red}$, $U = X \setminus Y$. Let $MHM(U,Y)_{gl}$ be the category whose objects are $\{M', M'', u, v\}$ where $M' \in MHM(U)$, $M'' \in MHM(Y)$, $u \in Hom(\psi_{g,1}M', M'')$, $v \in Hom(M'', \psi_{g,1}M'(1))$ such that $vu = N$

(the logarithm of the unipotent part of the monodromy, tensored by $(2\pi i)^{-1}$). Then we have an equivalence of categories (compare to [V3]):

2.5. Theorem. $\mathrm{MHM}(X) \simeq \mathrm{MHM}(U,Y)_{gl}$.

Here we associate $\{M|_U, \phi_{g,1}M, \mathrm{can}, \mathrm{Var}\}$ to $M \in \mathrm{MHM}(X)$. Because the definition of mixed Hodge Module is Zariski local, every object of MHM(X) can be constructed by induction on the dimension of support using 2.2 and 2.5.

## §3. Definition.

3.1. To explain more precisely about the statements in §2, we have to speak about the definition of mixed Hodge Modules. For simplicity we assume X is smooth. The general case can be reduced to this case using local embeddings into smooth varieties. Let $MF_h(\underline{D}_X)$ be the category of filtered $\underline{D}_X$-Modules $(M,F)$ such that $M$ is regular holonomic[Bo] and $Gr^F M$ is coherent over $Gr^F \underline{D}$. (We can also use analytic $\underline{D}_X$-Modules, because the final result is the same by GAGA and the extendability of mixed Hodge Modules.) By [K1] we have a faithful and exact functor $DR : MF_h(\underline{D}_X) \to \mathrm{Perv}(\mathbb{C}_X)$, and we define $MF_h(\underline{D}_X,\mathbb{Q})$ to be the fiber product of $MF_h(\underline{D}_X)$ and $\mathrm{Perv}(\mathbb{Q}_X)$ over $\mathrm{Perv}(\mathbb{C}_X)$, i.e. the objects are $(M,F,K) \in MF_h(\underline{D}_X) \times \mathrm{Perv}(\mathbb{Q}_X)$ with an isomorphism $\alpha : DR(M) \xrightarrow{\sim} \mathbb{C} \otimes K$, and the morphisms are the pairs of morphisms compatible with $\alpha$. A filtration $W$ of $(M,F,K)$ is a pair of filtrations $W$ on $M$ and $K$ compatible with $\alpha$. Let $MF_hW(\underline{D}_X,\mathbb{Q})$ be the category of the objects of $MF_h(\underline{D}_X,\mathbb{Q})$ with a finite increasing filtration $W$. Then MHM(X) the category of mixed Hodge Modules is a full subcategory of $MF_hW(\underline{D}_X,\mathbb{Q})$ and $W$ gives the weight filtration in 1.5. For $(M,F,K;W) \in \mathrm{MHM}(X)_s$ we can show that $(M,F,L;W)$ is an admissible variation of mixed Hodge structure, where $L$ is the local system on $X$ such that $K = L[-d_X]$ and the functor in 2.2 is induced in this way. Here we use the convension $F_pM = F^{-p}M$ and the Griffiths transversality follows from $F_1\underline{D}_X F_p M \subset F_{p+1}M$.

3.2. To define MHM(X), we have to define first MH(X,n) the category of <u>Hodge Modules of weight</u> n , cf. [S1-2]. This is a full subcategory of $MF_h(\underline{D}_X,\mathbb{Q})$ and satisfies:

(3.2.1) MH(pt,n) is the category of $\mathbb{Q}$-Hodge structures of weight n,

(3.2.2) If supp $\underline{M} = \{x\}$ for $\underline{M} \in MH(X,n)$, there exists $\underline{M}' \in MH(pt,n)$ such that $i_{x*}\underline{M}' = \underline{M}$, where $i_x : \{x\} \to X$.

(3.2.3) If $\underline{M} \in MH(X,n)$, $\underline{M}$ is regular and quasi-unipotent along g, $Gr_i^W\psi_g\underline{M}$, $Gr_i^W\phi_{g,1}\underline{M} \in MH(U,i)$ for any i, $\phi_{g,1}$ = Im can $\oplus$ Ker Var, for any g defined on an open subset U, where W is the monodromy filtration shifted by n-1 and n.

Here for a closed immersion $i : X \to Y$ of codimension k such that $X = \{f_1 = \cdots = f_k = 0\}$, the direct image $i_*(M,F)$ of a filtered $\underline{D}_X$-Module is defined by

$$(i_*M = M[\partial_1,\ldots,\partial_k],F) \text{ with } F_p i_* M = \textstyle\sum_{q+|\nu|\leq p-k} F_q M \otimes \partial^\nu$$

where $\partial_i$ are vector fields such that $[\partial_i, f_j] = \delta_{ij}$, cf. [Bo]. We say that $(M,F,K) \in MF_h(\underline{D}_X,\mathbb{Q})$ is <u>regular and quasi-unipotent along</u> g , if the monodromy of $\psi_g K[-1]$ is quasi-unipotent and $(\tilde{M},F) = i_{g*}(M,F)$ satisfies

$$\text{(3.2.4)}\quad \begin{aligned} t : F_p V^\alpha \tilde{M} &\xrightarrow{\sim} F_p V^{\alpha+1}\tilde{M} && \text{for } \alpha > -1 \\ \partial_t : F_p Gr_V^\alpha \tilde{M} &\xrightarrow{\sim} F_{p+1} Gr_V^{\alpha-1}\tilde{M} && \text{for } \alpha < 0, \end{aligned}$$

where $i_g : X \to X \times \mathbb{C}$ is the immersion by graph of g, t is the coordinate of $\mathbb{C}$ and V is the filtration of Malgrange-Kashiwara[K3] indexed by $\mathbb{Q}$ such that $t\partial_t - \alpha$ is nilpotent on $Gr_V^\alpha \tilde{M}$. In this case we define

$$\text{(3.2.5)}\quad \begin{aligned} \psi_g(M,F,K) &= (\oplus_{-1<\alpha\leq 0} Gr_V^\alpha(\tilde{M},F),\ \psi_g K[-1]) \\ \phi_{g,1}(M,F,K) &= (Gr_V^{-1}(\tilde{M},F[-1]),\ \phi_{g,1}K[-1]), \end{aligned}$$

and can : $\psi_{g,1} \to \phi_{g,1}$ and Var : $\phi_{g,1} \to \psi_{g,1}(-1)$ are induced respectively by $-\partial_t$ and $t$, where $F[m]_i = F_{i-m}$. Here we use left $\underline{D}$-Modules. For the correspondence with the right Modules we use $(\Omega_X^{d_X}, F) \otimes$ with $Gr_i^F \Omega_X^{d_X} = 0$ for $i \neq -d_X$. Actually MH(X,n) is defined to be the largest full subcategory of $MF_h(\underline{D}_X, \mathbb{Q})$ satisfying (3,2,1-3). This is well-defined by induction on dim supp M. (In the analytic case we have to care about the difference of global and local irreducibility.) Let Z be a closed irreducible subvariety of X. We say that (M,F,K) has strict support Z, if M (or K) has no subobject and no quotient object supported in a proper subvariety of Z and supp M = Z. Let $MH_Z(X,n)$ denote the full subcategory of the objects with strict support Z. Then we have the strict support decomposition

(3.2.6) $$MH(X,n) = \oplus_Z MH_Z(X,n).$$

A polarization of $(M,F,K) \in MH_Z(X,n)$ is a pairing $S : K \otimes K \to a_X^! \mathbb{Q}(-n)$ satisfying

(3.2.7) If $Z = \{x\}$, there is a polarization S' of Hodge structure $\underline{M}'$ [D1] such that $S = i_{x*}S'$, where $i_x$ and $\underline{M}'$ are as in (3.2.2).

(3.2.8) S is compatible with the Hodge filtration F, i.e. the corresponding isomorphism $K \xrightarrow{\sim} (\mathbb{D}K)(-n)$ is extended to an isomorphism $(M,F,K) \xrightarrow{\sim} \mathbb{D}(M,F,K)(-n)$.

(3.2.9) For any g as in (3.2.3) such that $g^{-1}(0) \not\subset Z$, the induced pairing

$$^p\psi_g S \circ (id \otimes N^i) : Gr^W_{n-1+i}\psi_g K[-1] \otimes Gr^W_{n-1+i}\psi_g K[-1] \to a_U^! \mathbb{Q}(1-n-i)$$

is a polarization on the primitive part $P_N Gr^W_{n-1+i} \psi_g(M,F,K)$,

cf. [S2] for the definition of $\mathbb{D}(M,F,K)$ and $^p\psi_g S$. Here the condition (3.2.9) is again by induction on dim supp M. We say that $(M,F,K) \in MH(X,n)$ is polarizable, if it has a polarization, and we denote by $MH(X,n)^p$ the full sub category of polarizable Hodge Modules. Here a polarization of $\underline{M} = \oplus \underline{M}_Z$ with $\underline{M}_Z \in MH_Z(X,n)$ is a direct sum of polarizations on $\underline{M}_Z$.

The main result of [S2] is that $MH(X,n)^p$ is stable by projective direct image. Here for the projection $p : X\times Y \to Y$ and $(M,F) \in MF(\underline{D}_{X\times Y})$, we define the direct image $p_*(M,F)$ to be the usual direct image of the filtered complex $DR_{X\times Y/Y}(M,F)[n]$ where $n = \dim X$ and

$$F_p DR_{X\times Y/Y} M = [F_p M \to \Omega_X^1 \otimes F_{p+1} M \to \cdots \to \Omega_X^n \otimes F_{p+n} M]$$

(Note that the assumption $p$ smooth is not enough to get an object of the derived category of filtered $\underline{D}_Y$-Modules.) Combining with the case of closed immersion, we get the definition of the general case (cf. [S2, §2] for a more intrinsic definition). Then for $(M,F,K) \in MH(X,n)^p$ and $f : X \to Y$ projective, we can prove that $f_*(M,F)$ is strict and $\underline{H}^j f_*(M,F,K) = (\underline{H}^j f_*(M,F), {}^p\underline{H}^j f_* K)$ belongs to $MH(Y,n+j)^p$. We also verify that $\mathbb{Q}_X^H[d_X] = (\underline{O}_X, F, \mathbb{Q}_X[d_X]) \in MH(X,d_X)^p$ where $Gr_p^F \underline{O}_X = 0$ for $p\neq 0$.

3.3. The mixed Hodge Modules are roughly speaking obtained by extensions of polarizable Hodge Modules. Here the extension is not arbitrary and to control this, we use again the vanishing cycle functors.

Let $MHW(X)$ be the full subcategory of $MF_h W(\underline{D}_X, \mathbb{Q})$ such that $Gr_i^W$ belongs to $MH(X,i)^p$ (i.e. the extension is arbitrary). Let $g$ be a function on $X$. Put $(\tilde{M},F,W) = i_{g*}(M,F,W)$ for $\underline{M} = (M,F,K,W) \in MHW(X)$. We say that <u>the vanishing cycle functors along</u> $g$ <u>are well-defined</u> for $\underline{M}$ if

(3.3.1) the relative monodromy filtration $W$ (cf. [D2]) of $(\psi_g K, L)$, $(\phi_{g,1} K, L)$ exists,

(3.3.2) $F, V, W$ on $\tilde{M}$ are compatible [S2,§1].

Here $L_i \psi_g K = \psi_g W_{i+1} K$ and $L_i \phi_{g,1} K = \phi_{g,1} W_i K$. If (3.3.1-2) are satisfied, we define

$$\psi_g\underline{M} = (\psi_g(M,F,K),W), \quad \phi_{g,1}\underline{M} = (\phi_{g,1}(M,F,K),W),$$

cf. (3.2.5). Let $j : U \to X$ be an open immersion such that the complement is a divisor. We say that <u>the direct images</u> $j_!$ <u>and</u> $j_*$ <u>are well-defined for</u> $\underline{M} \in MHW(U)$, if there exist $\underline{M}_!$ and $\underline{M}_*$ such that $rat(\underline{M}_!) = j_! rat(\underline{M})$ and the vanishing cycle functors along $g$ are well-defined for $\underline{M}_!$ and $\underline{M}_*$ for any local (not necessarily reduced) equation of the divisor. Here $rat(\underline{M}) = K$ if $\underline{M} = (M,F,K,W)$. We can show that $\underline{M}_!$ and $\underline{M}_*$ are at most unique if we fix $g$, but they might depend on the choice of the ideal generated by $g$. To avoid this ambiguity, we take the above definition.

The category of <u>mixed Hodge Modules</u> $MHM(X)$ is defined to be the largest full subcategory of $MHW(X)$ stable by the functors $\psi_g$, $\phi_{g,1}$, $j_!$, $j_*$, $\mathbb{Q}_Y^H[d_X]$ for any locally defined function $g$, partial compactification of an open subset $j : U \to U'$ such that the complement is a divisor, and smooth $Y$. Here we assume that the vanishing cycle functors along $g$ (resp. the direct images $j_!$ and $j_*$) are well-defined, when we say that it is stable by such functors, cf. [S3, S5].

3.4. Remark. The condition (3.13)ii) in [SZ] is not stable by base change. This condition is reasonable only in the unipotent monodromy case. In general we have to take a unipotent base change, or use the V-filtration and assume the compatibility of $F$, $W$, $V$ on Deligne's extension, because the V-filtration is essentially induced by the m-adic filtration on the pull-back by a unipotent base change.

3.5. Remark. Let $Z$ be a projective variety with an ample line bundle $L$ such that $Z$ is embedded in $X = \mathbb{P}^N$ by $L^m$. Then for $(M,F,K,W) \in MHM_Z(X)$ we have the <u>Kodaira vanishing</u>

$$H^i(Z, Gr_p^F DR_X(M,F) \otimes_{\mathcal{O}_Z} L^{\pm 1}) = 0 \quad \text{for} \quad i \gtrless 0.$$

This implies a vanishing of Ohsawa-Kollár (where $(M,F) = \underline{H}^j f_*(\underline{O}_Y,F)$ and $Gr_p^F DR_X(M,F) = R^j f_*\omega_Y$ for $f : Y \to X$ with

Y smooth), and that of Guillén-Navarro-Puerta

$$H^j(Z, \mathrm{Gr}_F^p \underline{\underline{\Omega}}_Z^{\cdot} \otimes L) = 0 \quad \text{for} \quad j > \dim Z$$

$$\underline{H}^j \mathrm{Gr}_F^p \underline{\underline{\Omega}}_Z^{\cdot} = 0 \quad \text{for} \quad j < p \quad \text{or} \quad j > \dim Z.$$

We can also generalize Kollár's torsion freeness to the proper Kähler case using [KK]. (This can be also generalized to the assertion for the first nonzero Hodge filtration of pure Hodge Modules.)

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