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Blowing-up morphisms with Cohen-Macaulay associated graded rings

INTRODUCTION.

Let X be a Cohen-Macaulay variety, and let I be a coherent sheaf of ideals on X. Consider the blowing-up $\pi:\bar{X}\longrightarrow X$ of X with respect to I. Our purpose in this paper is to study the consequences of the Cohen-Macaulayness of $G_I\mathcal{O}_X = \mathcal{O}_X/I\oplus I/I^2\oplus I^2/I^3\oplus\dots$. The main result is the following:

<u>Theorem</u> 1.4. <u>If</u> Spec $G_I\mathcal{O}_X$ <u>is Cohen-Macaulay, then</u> $\bar{X}$ <u>is a Cohen-Macaulay variety and its dualizing sheaf</u> $\omega_{\bar{X}}$ <u>satisfies</u>

$$R^i\pi_*\omega_{\bar{X}} = 0 \quad \text{for} \quad i>0\ .$$

In this paper, variety means a equidimensional noetherian separated scheme locally of finite type over a field k. By a result of Nagata, any variety is an open subscheme of a proper variety Z. We denote by ω_X the restriction of the dualizing sheaf ω_Z to X. It is well known that ω_X is independent of the completion Z.

1. PROOF OF THEOREM 1.4.

<u>Notation</u> 1.1. We will write $C = \operatorname{Spec} G_I\mathcal{O}_X$ and $E = \operatorname{Proj} G_I\mathcal{O}_X$. The vertex V of the cone C is the subscheme corresponding to the irrelevant ideal of $G_I\mathcal{O}_X$. Note that V is also the center of the blowing-up, and $E\simeq\pi^{-1}(V)$.
Finally, we write $\mathcal{O}_{\bar{X}}(n) = I^n\cdot\mathcal{O}_{\bar{X}}$, $(n>0)$.

<u>Lemma</u> 1.2. <u>Let</u> $x\in V$ <u>be a closed point such that</u> $\mathcal{O}_{C,x}$ <u>is a Cohen-Macaulay ring of dimension</u> n. <u>Let</u> Y <u>be the fibre of the morphism</u> $\pi:\bar{X}\longrightarrow X$ <u>over</u> x. <u>Then</u>

$$H^i_Y(E,\mathcal{O}_E(-r)) = 0 \quad \text{for} \quad r>0 \quad \text{and} \quad i<n-1\ .$$

<u>Proof</u>. We will consider the natural morphisms

$$f:C\longrightarrow V \quad , \quad g:C-V\longrightarrow E\ .$$

Take $Y_o = f^{-1}(x)$, then there is a long exact sequence

$$\longrightarrow H^i_x(C,\mathcal{O}_C) \longrightarrow H^i_{Y_o}(C,\mathcal{O}_C) \xrightarrow{\lambda} H^i_{Y_o}(C-x,\mathcal{O}_C) \longrightarrow \quad . \qquad (*)$$

The groups of this sequence are calculated in the following way.

a) $H^i_x(C,\mathcal{O}_C) = 0$ for $i < n$, because $\mathcal{O}_{C,x}$ is Cohen-Macaulay.

b) As f is an affine morphism, we obtain

$$H^i_{Y_o}(C,\mathcal{O}_C) = H^i_x(V,f_*\mathcal{O}_C) = \bigoplus_{r\geq o} H_x(V,I^r/I^{r+1}) \quad .$$

c) As g is also an affine morphism, we have

$$H^i_{Y_o}(C-x,\mathcal{O}_C) = H^i_{Y_o}(C-V,\mathcal{O}_C) = H^i_Y(E,g_*\mathcal{O}_C) = \bigoplus_{r\in\mathbb{Z}} H^i_Y(E,\mathcal{O}_E(r)) \quad .$$

Therefore, the groups in the middle and on the right of (*) are graded $G_I\mathcal{O}_X$-modules. Moreover, it is easy to see that λ is a homogeneous morphism of degree zero. We conclude from (*) that $H^i_Y(E,\mathcal{O}_E(-r)) = 0$ for $r > 0$, $i < n-1$.

<u>Theorem</u> 1.3. <u>Let</u> x <u>be a closed point of</u> X <u>and let</u> Y <u>be the fibre of</u> $\pi:\bar{X} \longrightarrow X$ <u>over</u> x. <u>If</u> $\operatorname{Spec} G_I\mathcal{O}_X$ <u>is Cohen-Macaulay, then</u>

$$H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}) = 0 \quad \text{for} \quad i < n = \dim X \quad .$$

<u>Proof</u>. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(1) \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_E \longrightarrow 0 \quad .$$

Tensoring with $\mathcal{O}_{\bar{X}}(-r)$ and taking cohomology, we get

$$\longrightarrow H^i_Y(\mathcal{O}_{\bar{X}}(-(r-1))) \longrightarrow H^i_Y(\mathcal{O}_{\bar{X}}(-r)) \longrightarrow H^i_Y(\mathcal{O}_E(-r)) \longrightarrow \quad .$$

Furthermore, we have

a) $H^i_Y(\mathcal{O}_E(-r)) = 0$ for $r > 0$ and $i < n-1$ (lemma 1.2);

b) $H^i_Y(\mathcal{O}_{\bar{X}}(-r))^* = (R^{n-i}\pi_*\omega_{\bar{X}}(r))^\wedge_x = 0$ for $i < n$ and $r \gg 0$.

The first equality in b) is a duality formula (see Appendix).

We conclude by descending induction on r.

Theorem 1.4. Let X be a Cohen-Macaulay variety. Let $\pi : \bar{X} \longrightarrow X$ be the blowing-up with respect to a coherent sheaf of ideals I on X. Assume that $\operatorname{Spec} G_I\mathcal{O}_X$ is Cohen-Macaulay. Then $\bar{X}$ is a Cohen-Macaulay variety and its dualizing sheaf $\omega_{\bar{X}}$ satisfies

$$R^i\pi_*\omega_{\bar{X}} = 0 \quad \text{for} \quad i > 0 .$$

Proof. As $C-V$ is Cohen-Macaulay and $g : C-V \longrightarrow E$ is smooth, E is Cohen-Macaulay. This implies that $\bar{X}$ is Cohen-Macaulay. From 1.3 and a duality formula, we deduce that $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$.

Remark 1.5. Note that C is Cohen-Macaulay if and only if $\mathcal{O}_{C,x}$ is Cohen-Macaulay for each $x \in V$.

There is an analogous result for the Rees algebra of I, $R(I) = \mathcal{O}_X \oplus I \oplus I^2 \oplus \dots$.

Corollary 1.6. Let X and I be as in 1.4. If $R(I)$ is Cohen-Macaulay, then $\bar{X}$ is Cohen-Macaulay and $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$.

Proof. According to a result of Huneke [5], if $R(I)$ is Cohen-Macaulay, then $G_I\mathcal{O}_{\bar{X}}$ is Cohen-Macaulay. We conclude by 1.4.

Finally, we obtain a converse of 1.4:

Theorem 1.7. Let X be a Cohen-Macaulay variety. Let $\pi : \bar{X} \longrightarrow X$ be the blowing-up with a coherent sheaf of ideals I on X. Assume that $\bar{X}$ is Cohen-Macaulay and $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$. Then there exists an integer r_o such that for each $r \geq r_o$, $\operatorname{Spec} G_{I^r}\mathcal{O}_X$ is Cohen-Macaulay.

Proof. Take r_o such that the following conditions are fulfilled:

1. $\pi_*\mathcal{O}_{\bar{X}}(r) = I^r$ for $r \geq r_o$.
2. $R^i\pi_*\mathcal{O}_{\bar{X}}(r)$ for $i > 0$ and $r \geq r_o$.
3. $H^i_Y(\bar{X}, \mathcal{O}_{\bar{X}}(-r))^* = (R^{n-i}\pi_*\omega_{\bar{X}}(r))^{\wedge}_x = 0$ for $i < n = \dim X$ and $r \geq r_o$, where x and Y are as in 1.3. Moreover $H^i_Y(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ for $i < n$, by the duality formula.

Now, we will show that $G_J\mathcal{O}_X$ is Cohen-Macaulay for $J = I^r$, $r \geq r_o$. As in 1.1, we write $C = \operatorname{Spec} G_J\mathcal{O}_X$, $E = \operatorname{Proj} G_J\mathcal{O}_X$ and V for the vertex of C. Changing the notation, we write $\mathcal{O}_{\bar{X}}(1) = J \cdot \mathcal{O}_{\bar{X}}$. Then, as in the proof of 1.2, we have the exact sequence

$$\longrightarrow H^i_x(C,\mathcal{O}_C) \longrightarrow H^i_{Y_o}(C,\mathcal{O}_C) \longrightarrow H^i_{Y_o}(C-x,\mathcal{O}_C) \longrightarrow$$

where

$$H^i_{Y_o}(C,\mathcal{O}_C) = \bigoplus_{m\geq o} H^i_x(V,J^m/J^{m+1})$$

$$H^i_{Y_o}(C-x,\mathcal{O}_C) = \bigoplus_{m\in\mathbb{Z}} H^i_Y(E,{}_E(m)) \; .$$

Therefore, we have to prove the following facts:

a) $H^i_x(V,J^m/J^{m+1}) = H^i_Y(E,\mathcal{O}_E(m))$ for $m > 0$.

b) $H^i_x(V,\mathcal{O}_X/J) = H^i_Y(E,\mathcal{O}_E)$ for $i < n-1$, and $H^{n-1}_x(V,\mathcal{O}_X/J) \longrightarrow H^{n-1}_Y(E,\mathcal{O}_E)$ is injective.

c) $H^i_Y(E,\mathcal{O}_E(m)) = 0$ for $i < n-1$ and $m < 0$.

Proof of a). By 1 and 2 above, we have

$$R^i\pi_*\mathcal{O}_{\bar{X}}(m) = 0 \quad \text{for} \quad i > 0 \quad \text{and} \quad \pi_*\mathcal{O}_{\bar{X}}(m) = J^m \quad , \quad (m > 0)$$

Therefore, from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(m+1) \longrightarrow \mathcal{O}_{\bar{X}}(m) \longrightarrow \mathcal{O}_E(m) \longrightarrow 0$$

we deduce that

$$R^i\pi_*\mathcal{O}_E(m) = 0 \quad \text{for} \quad i > 0 \text{ , and } \quad \pi_*\mathcal{O}_E(m) = J^m/J^{m+1} \; .$$

We conclude, from an obvious spectral sequence, that

$$H^i_x(V,J^m/J^{m+1}) = H^i_Y(E,\mathcal{O}_E(m)) \quad .$$

Proof of b). Taking cohomology with supports in x and Y , respectively, in the exact sequences

$$0 \longrightarrow J \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/J \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(1) \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

we obtain the following diagram

$$\begin{array}{ccccccc} \longrightarrow H^i_x(X,J) & \longrightarrow & H^i_x(X,\mathcal{O}_X) & \longrightarrow & H^i_x(X,\mathcal{O}_X/J) & \longrightarrow & \\ \downarrow \wr & & \downarrow & & \downarrow & & \\ \longrightarrow H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}(1)) & \longrightarrow & H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}) & \longrightarrow & H^i_Y(E,\mathcal{O}_E) & \longrightarrow & . \end{array}$$

The vertical arrows on the left are isomorphisms, because $R^i\pi_*\mathcal{O}_{\bar{X}}(1) = 0$ for $i > 0$, and $\pi_*\mathcal{O}_{\bar{X}}(1) = J$. Moreover, the groups in the middle vanish for $i < n-1$. The assertion follows easily from the diagram.

Proof of c). Taking cohomology with supports in Y in the exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(1+m) \longrightarrow \mathcal{O}_{\bar{X}}(m) \longrightarrow \mathcal{O}_E(m) \longrightarrow 0$$

we conclude by condition 3.

2. EXAMPLES

a) The obvious example is when I is generated (locally) by a regular sequence. In this case, it is well known that $\operatorname{Spec} G_I\mathcal{O}_X$ is Cohen-Macaulay. Therefore the blowing-up $\pi:\bar{X} \longrightarrow X$ of I satisfies $R^i\pi_*\omega_{\bar{X}} = 0$, for $i > 0$.
A particular case is the blowing-up of a smooth variety along a smooth subvariety.

b) Let X be a hypersurface of a smooth variety Z. Let $\pi:\bar{X} \longrightarrow X$ be the blowing-up of X along a smooth subvariety Y. If p is the ideal defining Y, it is easy to see that $G_p\mathcal{O}_X$ is Cohen-Macaulay. Therefore $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$.
Moreover, if m is the multiplicity of X at the generic point of Y and d is the codimension of Y in X, then $\pi_*\omega_{\bar{X}} = p^{m-d} \otimes \omega_X$. With these results, one can calculate the variation of the arithmetic genus in the morphism π, (see [11]).

c) Let X be a Cohen-Macaulay variety and let $\pi:\bar{X} \longrightarrow X$ be the blowing-up at a closed point x. Let m be the maximal ideal corresponding to x. We will write m = multiplicity of X at x, $d = \dim \mathcal{O}_{X,x}$ and δ = length m/m^2. According to a result of Abhyankar [1], $d \leq \delta \leq m+d-1$. In general $G_m\mathcal{O}_X$ is not Cohen-Macaulay, though in the case of high embedding dimension we have the following result

Theorem (Sally, [8], [10]). If $\delta = m+d-1$, then $G_m \mathcal{O}_X$ is Cohen-Macaulay. If $\mathcal{O}_{X,x}$ is Gorenstein and $\delta = m+d-2$ or $m+d-3$, then $G_m \mathcal{O}_X$ is Cohen-Macaulay.

Therefore, we obtain the following

Corollary 2.1. If $\delta = m+d-1$, then $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$. If $\mathcal{O}_{X,x}$ is Gorenstein and $\delta = m+d-2$, $m+d-3$, then $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$.

Remark 2.2. It is easy to see that if $\mathcal{O}_{X,x}$ is Gorenstein and $m \leqq 5$, then $\delta = d, d+1, m+d-3$, or $m+d-2$.

J. Sally also obtained the Hilbert function in the cases of 2.1. This allows us to calculate the variation of the arithmetic genus and the groups of cohomology in the blowing-up π. If one knows the Hilbert function $H(n) = \dim. \chi(X, I^n/I^{n+1})$ (or the Samuel function $S(n) = \dim. \chi(X, \mathcal{O}_X/I^n)$) corresponding to an ideal I on X, then one can calculate the variation of the arithmetic genus in the blowing-up of I. The following theorem shows that the variation is the last coefficient of Samuel's polynomial.

Theorem 2.3 (see also Ramanujam [7]). Let X be a r-dimensional variety, proper over k, and let I be a coherent sheaf of ideals on X. Let $\pi: \bar{X} \longrightarrow X$ be the blowing-up of X with respect to I. If $E = cl^{\cdot}(\mathcal{O}_{\bar{X}}) - cl^{\cdot}(\mathcal{O}_{\bar{X}}(1)) \in K^{\cdot}(\bar{X})$, then one has

$$S(n) = \chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) - \sum_{i=1}^{r} (-1)^i \chi(E^i)\binom{n}{i} \quad , \quad n \gg 0 \ .$$

Note 2.4. $K^{\cdot}(\bar{X})$ denotes, as usual, the Grothendieck group of coherent locally free sheaves on $\bar{X}$.

Proof of 2.3. One has $R^i\pi_*\mathcal{O}_{\bar{X}}(n) = 0$ for $i > 0$ and $n \gg 0$, and $\pi_*\mathcal{O}_{\bar{X}}(n) = I^n$ for $n \gg 0$. Therefore $\chi(\bar{X}, I^n \cdot \mathcal{O}_{\bar{X}}) = \chi(X, I^n)$ for $n \gg 0$. Writing $1 = cl^{\cdot}(\mathcal{O}_{\bar{X}})$, one has

$$S(n) = \chi(\mathcal{O}_X) - \chi(I^n) = \chi(\mathcal{O}_X) - \chi(I^n \cdot \mathcal{O}_{\bar{X}}) = \chi(\mathcal{O}_X) - \chi((1-E)^n) =$$

$$= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_{\bar{X}}) - \sum_{i=1}^{r} (-1)^i \chi(E^i)\binom{n}{i} \ .$$

The last equality results from $E^i = 0$ for $i > r$, in the Grothendieck group $K_{\cdot}(\bar{X})$ of coherent sheaves (S.G.A. 6, exp. X).

Remark 2.5. Note that $H(n) = S(n+1) - S(n)$. In particular, we have that $H(n)$ is a polynomial for $n \gg 0$, as is well known. Moreover, if we write $P(n)$ for that polynomial, it is easy to see that the last coefficient of Samuel's polynomial is

$$\chi(X,\mathcal{O}_X) - \chi(\bar{X},\mathcal{O}_{\bar{X}}) = \sum_{n=0}^{\infty} (H(n) - P(n)) .$$

Theorem 2.6. Let X be a Cohen-Macaulay variety, proper over k . Let $\pi:\bar{X} \longrightarrow X$ be the blowing-up at a closed point x . Suppose one of the following conditions is satisfied:

a) $\delta = m+d-1$ and $d > 1$,

b) $\mathcal{O}_{X,x}$ is Gorenstein, $\delta = m+d-2$ and $d > 2$,

c) $\mathcal{O}_{X,x}$ is Gorenstein, $\delta = m+d-3$ and $d > 3$.

Then $H^i(X,\mathcal{O}_X) = H^i(\bar{X},\mathcal{O}_{\bar{X}})$.

Proof. $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$, by 2.1. Therefore, we are done if $\pi_*\omega_{\bar{X}} = \omega_X$. In any case, there is a natural morphism $f:\pi_*\omega_{\bar{X}} \hookrightarrow \omega_X$. Taking Euler characteristics in the exact sequence

$$0 \longrightarrow \pi_*\omega_{\bar{X}} \longrightarrow \omega_X \longrightarrow \text{Coker } f \longrightarrow 0$$

we obtain

$$\chi(X,\omega_X) - \chi(\bar{X},\omega_{\bar{X}}) = \text{length Coker } f .$$

However J. Sally shows ([9], [10]) that the Hilbert function is a polynomial for $n \geq 0$. This means, by 2.5, that the variation of the Euler characteristic is zero. Therefore Coker $f = 0$ and we have finished.

d) Let I be an ideal of a Gorenstein ring $\mathcal{O}_X$. We set $v(I)$ equal to the least number of generators of I . We assume that I is an almost complete intersection which is a generic complete intersection. This means that $v(I) = ht(I) + 1$, $v(I_p) = h(I)$ for all $p \in \text{Min } \mathcal{O}_X/I$. If $\mathcal{O}_X/I$ is Cohen-Macaulay, then $G_I\mathcal{O}_X$ is Gorenstein (Valla, [12]). Therefore, we have

Theorem 2.7. Let X be a Gorenstein variety and let $\pi:\bar{X} \longrightarrow X$ be the blowing-up with respect to a coherent sheaf of ideals I . We assume that Spec $\mathcal{O}_X/I$ is Cohen-Macaulay and that I is locally an almost complete intersection which is a generic complete intersection. Then, $R^i\pi_*\omega_{\bar{X}} = 0$ for $i > 0$.

e) Let X be a Cohen-Macaulay variety over k and let $\pi:\bar{X} \longrightarrow X$ be the blowing-up with respect to a coherent sheaf of ideals I .

Theorem 2.8. Suppose one of the following conditions is satisfied:

a) $k=\mathbb{C}$ and $\bar{X}$ is smooth,

b) $\dim X=2$ and $\bar{X}$ is normal.

Then Spec $G_{I^n}\mathcal{O}_X$ is Cohen-Macaulay for $n \gg 0$.

Proof: The vanishing theorem $R^i\pi_*\omega_{\bar{X}}=0$ for $i>0$, holds in the two cases (Grauert-Riemenschneider [2] and Lipman [6]).

3. APPENDIX: A DUALITY FORMULA

Our purpose is to prove the following

Theorem 3.1. Let $\pi: Z \longrightarrow X$ be a proper morphism and let Y be the fibre over a closed point x. If Z is an n-dimensional Cohen-Macaulay variety and L is a coherent locally free sheaf on Z, one has

$$H_Y^i(Z,L)^* = (R^{n-i}\pi_*(\omega_Z \otimes \check{L}))_x^{\wedge} \quad ,$$

where the dual is taken over the base field k.

Proof: Let p be the sheaf of ideals corresponding to Y. Then by the theorem of formal functions and global duality, one has:

$$(R^{n-i}\pi_*(\omega_Z \otimes \check{L}))_x^{\wedge} = \varprojlim_r H^{n-i}(\omega_Z \otimes \check{L} \otimes \mathcal{O}_Z/p^r) = \varprojlim_r \mathrm{Ext}^i_{\mathcal{O}_Z}(\omega_Z \otimes \check{L} \otimes \mathcal{O}_Z/p^r, \omega_Z)^* =$$

$$= [\varinjlim_r \mathrm{Ext}^i_{\mathcal{O}_Z}(\omega_Z \otimes \mathcal{O}_Z/p^r, L \otimes \omega_Z)]^* \overset{(1)}{=} \varinjlim_r \mathrm{Ext}^i_{\mathcal{O}_Z}(\mathcal{O}_Z/p^r, L)]^* = H_Y^i(Z,L)^* \ .$$

Equality (1) is a consequence of the spectral sequences with the same abutment:

$$E_2^{p,q} = [\varinjlim \mathrm{Ext}^p_{\mathcal{O}_Z}(\mathbf{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Z/p^r, \omega_Z), L \otimes \omega_Z)]^* = \varprojlim H^{n-p}(\mathbf{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Z/p^r, \omega_Z) \otimes \check{L})$$

$$\bar{E}_2^{p,q} = [\varinjlim \mathrm{Ext}^p_{\mathcal{O}_Z}(\mathcal{O}_Z/p^r, \mathbf{Ext}^q_{\mathcal{O}_Z}(\omega_Z, \omega_Z) \otimes L)]^* \ .$$

A computation gives $E_2^{p,q}=0$ for $q \neq 0$, and the second spectral sequence is also degenerate by the following fact (see [4], Satz 6.1)

$$\mathbf{Ext}^q_{\mathcal{O}_Z}(\omega_Z,\omega_Z) = 0 \quad \text{for} \quad q>0 \quad , \quad \mathbf{Hom}_{\mathcal{O}_Z}(\omega_Z,\omega_Z) = \mathcal{O}_Z \qquad \text{q.e.d.}$$

Theorem 3.1 is probably known. There are analogous results in Hartshorne ([3]) and Lipman [6].

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