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Moduli of analytic branches

Introduction

The purpose of this paper is to compute the underlying set of the moduli space for irreducible analytic branches embedded in affine-space over an algebraically closed field k of arbitrary characteristic.

Let:

$$C \hookrightarrow X = \mathrm{Spec}(k[[T_o,\ldots,T_m]])$$

be an analytic branch and $\{r_i\}_{i=o}^{\infty}$ the sequence of multiplicities of C and its successive quadratic transforms. We will denote by r the following:

$$r = h + 2 \sum_{i=o}^{\infty} r_i(r_i - 1)^2$$

where h is the number of quadratic transformations needed to desingularise C. Let C_r be the r-th quadratic transform of C let $X_C^r \longrightarrow X$ be the r-th quadratic transform of X in the direction of C and

$$\pi_r : X_C^r \longrightarrow X$$

the canonical map. If X_o^C is the reduced exceptional fibre of π_r we will denote by I the sheaf of ideals defining X_o and let $X_n^C \hookrightarrow X_C^r$ be the closed subscheme which is defined by I^{n+1}. Let $L = I/I^2$ be the conormal sheaf to X_o in X_C^r.

Definition: Two embedded branches $C \hookrightarrow X'$, $C' \hookrightarrow X$ are equisingular when X_1^C and $X_1^{C'}$ are isomorphic schemes. That is, one defines the equisingularity of $C \hookrightarrow X$ to be the scheme X_1^C.

Let X_1 be an equisingularity. Also let $M(X_1)$ be the set of analytic equivalence classes of branches with equisingularity X_1.

Let G_L^K denote the sheaf of $\mathcal{O}_{X_o}$-modules

$$G_L^K = L^2 \oplus L^3 \oplus L^4 \oplus \ldots \oplus L^K .$$

Similarly let G_L denote the sheaf

$$G_L = L^2 \oplus L^3 \oplus \ldots .$$

Let H^1 denote the k-vector space

$$H^1 = H^1(X_o, \mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{O}_{X_o}) \otimes_{\mathcal{O}_{X_o}} G_L^K) .$$

Main theorem: There exists a natural integer K and a certain quotient set M of H^1

$$\pi : H^1 \longrightarrow M$$

such that $M(X_1)$ is the subset of M defined by the vanishing of a 2-cycle obstruction class belonging to

$$H^2(X_o, \mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{O}_{X_o}) \otimes_{\mathcal{O}_{X_o}} G_L)$$

and associated to each point of M.

(More precision and details will be given below).

As a particular case; when $k = \mathbb{C}$, the complex field, and $\dim X = 2$ one gets a result of O. Zariski [5] stating that the moduli space for plane analytic branches over $\mathbb{C}$ is a quotient of a vector space.

In the present paper a sort of description of the fibres of π is given. I hope to come back to this problem in a future paper.

0. Notations

The ground field k will be algebraically closed and of arbitrary characteristic. We will denote the ring of formal power series in $m+1$ variables, with coefficients in k, by

$$A = k[[T_o, \ldots, T_m]] .$$

We will also denote by X the spectrum of A and by $\hat{X}$ the formal spectrum of A (in the sense of Grothendieck [2]).

Given a natural integer n and an analytic branch (always irreducible) $i : C \hookrightarrow X$, one defines the "n-th blowing up" of $\hat{X}$ in the direction of C as the sequence

$$\hat{X}(C_n) \longrightarrow \hat{X}(C_{n-1}) \longrightarrow \dots \longrightarrow \hat{X}(C_o) = \mathrm{Sp\,f}(A)$$

where $\hat{X}(C_i)$ is the formal blowing up of $\hat{X}(C_{i-1})$ along the closed point of C_{i-1} (i.e.: the formalisation of the local blowing up of $\hat{X}(C_{i-1})$) and C_i is the strict transform of C_{i-1} starting with $C_o = C$.

The morphism:

$$\pi_n : \hat{X}(C_n) \longrightarrow \mathrm{Sp\ f}(A)$$

is algebraisable, that is: there exists an ideal I of A such that the formalisation of the blowing up $X(C_n)$ of X along I :

$$X(C_n) \longrightarrow \mathrm{Spec}(A)$$

is exactly:

$$\pi_n : \hat{X}(C_n) \longrightarrow \mathrm{Sp\ f}(A) .$$

1. Equivalence theorem. Upper bound for the conductor

One starts with the following: for every multiplicity sequence $\{r_n\}_{n=o}^{\infty}$, if h denotes the minimum integer such that $r_h = 1$, one defines r to be the integer:

$$r = h + 2 \sum_{i=o}^{\infty} r_i (r_i - 1)^2 \quad ;$$

then if

$$\hat{X}(C_r) \longrightarrow \hat{X}$$

is the r-th blowing up of $\hat{X}$ directed by the branch C with multiplicity sequence $\{r_n\}_{n=o}^{\infty}$ then $\hat{X}(C_r)$ determines the branch C up to analytic equivalence. More precisely:

Theorem 1.1: If C , C' are two branches with the same multiplicity sequence $\{r_n\}_{n=o}^{\infty}$ then C , C' are analytically equivalent if and only if the schemes $\hat{X}(C_r)$ and $\hat{X}(C'_r)$ are isomorphic.

To prove this theorem one needs some lemmas.

Lemma 1.2: C , C' are analytically equivalent if and only if they are isomorphic schemes; i.e.

$$C = \mathrm{Spec}(\mathcal{O}_C) \approx \mathrm{Spec}(\mathcal{O}_{C'}) = C' .$$

if one denotes by

$$\bar{C} = \mathrm{Spec}(k[[T]])$$

the desingularization of C and by $\mathfrak{C}$ the conductor, then $\mathfrak{C} = T^c k[[T]]$, where

$$c = \ell(\mathcal{O}_{\bar{C}}/C)$$

is the length of the conductor and one has

$$T^c k[[T]] \subset \mathcal{O}_C \quad .$$

Lemma 1.3: If $\ell \geq c, c'$ then $\mathcal{O}_C$ and $\mathcal{O}_{C'}$ are isomorphic if and only if the respective subalgebras of $k[[T]]/(T^\ell)$ they induce are isomorphic, i.e.: if and only if there exists an automorphism of $k[[T^\ell]]$ that maps one subalgebra onto the other.

We will now prove that the length c of the conductor of a branch C is bounded by the multiplicity sequence $\{r_n\}_{n=o}^{\infty}$. More precisely, there exists a positive integer K $(= 2 \sum_{i=o}^{\infty} r_i(r_i - 1)^2)$, depending only on the given multiplicity sequence, such that $c \leq K$.

Let $\mathcal{O}_{\bar{C}} = k[[T]]$ be the ring of the desingularization of C and, let ν_C be the valuation of the field of fractions of $\mathcal{O}_C$ induced by $\mathcal{O}_{\bar{C}}$. If m_C is the maximal ideal of $\mathcal{O}_C$ and t is an element of m_C with minimum value for ν_C then

$$\nu_C(t) = \text{multiplicity of } \mathcal{O}_C = r_o \quad .$$

Moreover, one has

$$\dim_k(m_C/m_C^2) \leq r_o = \dim_k(m_C/tm_C) \quad .$$

We will denote by d_C the embedding dimension of C :

$$d_C = \dim_k(m_C/m_C^2) \quad ;$$

it is clear that d_C is a formal analogue of elements which generate the k-algebra $\mathcal{O}_C$.

Lemma 1.4: For a plane branch C , i.e.: $d_C = 2$, then

$$m_C^n = t m_C^{n-1}$$

for all $n > r_o - 1$.

As a corollary one gets the following general result in the case $d_C \geq 1$. As the algebra

$$\mathcal{O}_C = k[[t,t_1,\dots,t_{d_C-1}]]$$

contains all the subalgebras $k[[t,t_i]]$ for $1 \leq i \leq d_C-1$, which are plane branches and with multiplicities $\leq r_o = \nu_C(t)$, one can apply lemma 1.4 to get

Lemma 1.5: $m_C^n = t\, m_C^{n-1}$ for all $n > (r_o - 1)(d_C - 1) = q$ and so one has

$$m_C^n = t^{n-q}\, m_C^q \quad .$$

If $\mathcal{O}_{C_1}$ is the first quadratic transform of $\mathcal{O}_C$, then

$$\mathcal{O}_{C_1} = \bigcup_{i=o}^{\infty} \frac{m^i}{t^i} \subset \sum \qquad \text{(field of fractions of } \mathcal{O}_C\text{)}$$

and also $d_C \leq r_o$. The lemma applies and gives

Corollary 1.6: $m_C^{n-1}\mathcal{O}_C = m_C^{n-1}$ for all $n > (r_o - 1)^2$ and so

$$\ell(\mathcal{O}_{C_1}/\mathcal{O}_C) \leq r_o(r_o - 1)^2 \quad .$$

Proof: The second part results from the fact that

$$m_C^q\, \mathcal{O}_{C_1} = m_C^q$$

(recall that $q = (r_o - 1)(d_C - 1) > (r_o - 1)^2$) . So one has

$$\ell(\mathcal{O}_{C_1}/\mathcal{O}_C) \leq \ell(\mathcal{O}_{C_1}/m_C^q\mathcal{O}_{C_1}) = \ell(\mathcal{O}_{C_1}/t^q\mathcal{O}_{C_1}) = q \cdot r_o$$

because $r_o = \ell(\mathcal{O}_{C_1}/t\,\mathcal{O}_{C_1})$.

Corollary 1.7: $\ell(\mathcal{O}_{\bar{C}}/\mathcal{O}_C) \leq \sum_{i=o}^{\infty} r_i(r_i - 1)^2$.

If one uses the inequality

$$\ell(\mathcal{O}_C/\mathfrak{C}) \leq \ell(\mathcal{O}_{\bar{C}}/\mathcal{O}_C)$$

one concludes that

$$c = \ell(\mathcal{O}_{\bar{C}}/\mathfrak{C}) \leq \ell(\mathcal{O}_{\bar{C}}/\mathcal{O}_C) + \ell(\mathcal{O}_C/\mathfrak{C}) \leq 2 \sum_{i=o}^{\infty} r_i(r_i - 1)^2 \,.$$

Corollary 1.8: The length c of the conductor of a branch C with multiplicity sequence $\{r_n\}_{n=o}^{\infty}$ is bounded as follows:

$$c \leq 2 \sum_{i=o}^{\infty} r_i(r_i - 1)^2 \ .$$

Let h be the number of quadratic transformations necessary to desingularise C, that is h is the least integer such that $r_h = 1$.

Let

$$\pi_h : \hat{X}(C_h) \longrightarrow \mathrm{Sp}\ f(A)$$

be the desingularisation map of C. Suppose $C' \hookrightarrow X$ is another branch and $\bar{C}' \hookrightarrow \hat{X}(C_h)$ its strict transform for π_h (C' is supposed to have the same multiplicity sequence as C). Then one has

Lemma 1.9: If C_h and $\bar{C}'$ have a contact of order $m \geq 1$, then $\mathcal{O}_C$ and $\mathcal{O}_{C'}$ induce isomorphic subalgebras in $k[[T]]/(T^m)$.

Proof: As C, C' have the same multiplicity sequence, if C_h and $\bar{C}'$ intersect, then $\bar{C}' = C'_h$ is simple and C, C' are direct identical h-blowing ups of $\hat{X}$. Let T be a function on $\hat{X}(C_h)$ that is a parameter for C_h and C'_h. If $A = k[[T_o, \ldots, T_m]]$, then by the contact condition $T_o, \ldots, T_m$ will have the same expansion up to order m along both branches C, C'.

This means that the following diagram of natural maps is commutative:

$$\begin{array}{ccccc} & & \mathcal{O}_C & & \\ & \nearrow & & \searrow & \\ A & & & & k[[T]]/(T^m) \\ & \searrow & & \nearrow & \\ & & \mathcal{O}_{C'} & & \end{array} \ .$$

Lemma 1.10: For the formal schemes $\hat{X}(C_{h+m})$ and $\hat{X}(C'_{h+m})$ to be isomorphic it is necessary and sufficient that there exists an automorphism τ of X such that the desingularisation $\tau(C)_h$ of $\tau(C)$ has a contact of order m with C'_h.

Proof: Given an isomorphism

$$\Phi : \hat{X}(C_{h+m}) \longrightarrow X(C_{h+m}) \quad ,$$

by taking global sections one gets an automorphism τ of X. Conversely, given a τ with the properties of the lemma, τ induces an isomorphism

$$\Phi : \hat{X}(C_{h+m}) \longrightarrow \hat{X}_{\tau(C)_{h+m}} = \hat{X}_{C'_{h+m}} \ .$$

Proof (of theorem 1.1): Taking $m = 2 \sum_{i=o}^{\infty} r_i(r_i - 1)^2$ and applying lemmas 1.10, 1.9, then corollary 1.8 and lemmas 1.3, 1.2 (in this order) one concludes that if the formal schemes $\hat{X}(C_r)$ and $\hat{X}(C'_r)$ are isomorphic, then the branches C, C' in X are analytically equivalent. The converse is immediate.

2. Characterization of the "blowing ups" which are directed by a branch

Let $\{r_n\}_{n=0}^{\infty}$ be a multiplicity sequence of a branch C embedded in X. Let

$$r = h + 2 \sum_{i=0}^{\infty} r_i(r_i - 1)^2$$

where h is minimal with the condition that $r_h = 1$. For every embedded branch C' in X, with multiplicity sequence $\{r_n\}_{n=0}^{\infty}$, one denotes by

$$\pi'_r : \hat{X}(C'_r) \longrightarrow \hat{X}$$

the r-th formal blowing up in the direction of C'; X'_0 will denote the exceptional reduced fibre of π'_r and L' the conormal sheaf to X'_0 in $\hat{X}(C'_r)$.

We now fix an embedded branch C in X with multiplicity sequence $\{r_n\}_{n=0}^{\infty}$. Let (X_0, L) be the exceptional fiber and the conormal sheaf in the r-th blowing up

$$\hat{X}(C_r) \longrightarrow \hat{X}$$

in the direction of C.

Let $\hat{X}'$ be a formal scheme along a closed subscheme isomorphic to X_0. Suppose that the conormal sheaf to X_0 in $\hat{X}'$ is isomorphic to L. The main result characterizing the r-th blowing up is the following.

Theorem: $\hat{X}'$ is isomorphic to the composition of r formal blowing ups starting with $\hat{X}'$ and with centers at closed points if and only if the sheaf $\mathcal{O}_{\hat{X}'}$ is locally isomorphic to $\mathcal{O}_{X(C_r)}$ along X_0.

Proof: Let

$$\pi'_r : \hat{X}(C'_r) \longrightarrow \hat{X}$$

be an r-blowing up of $\hat{X}$ in the direction of $C' \hookrightarrow X$. Suppose also that X'_0 and X_0 are isomorphic. One can prove easily by induction on r that if $x \in X'_0$ is a closed point, then the local ring at x is:

$$\mathcal{O}_{\hat{X}(C'_r),x} = \left[\begin{array}{c} k[Y_0,\dots,Y_m][[Y_0 \cdot Y_1 \cdot \dots \cdot Y_{t-1}]] \\ \text{localization at the origin} \end{array}\right],$$

where t is the number of irreducible connected components of X'_0 which pass through x. This proves the only if part.

Conversely, let X_o be embedded in $\hat{X}'$ with the conditions of the theorem. Suppose

$$X_o = X_o^1 \cup \ldots \cup X_o^r$$

is the decomposition of X_o into irreducible components, and

$$\hat{X}(C_r) \xleftarrow{\ i\ } X_o \xrightarrow{\ i'\ } \hat{X}'$$

are the given embeddings. Then there are positive integers $n_1,\ldots,n_r$ such that $\mathcal{O}(-(n_1X_o' + \ldots + n_rX_o^r))$ is an ample line sheaf for

$$\pi_r : X(C_r) \longrightarrow \hat{X} \ .$$

So its inverse image $i^*\mathcal{O}(-(n_1X_o^1 + \ldots + n_rX_o^r))$ is ample on X_o , This implies that the other inverse images $i'^*\mathcal{O}(-(n_1X_o^1 + \ldots + n_rX_o^r))$ are also ample on X_o (because by the hypothesis $i^*\mathcal{O}(-X_o) = L \approx L' = i'^*\mathcal{O}(-X_o))$. One can then apply M. Artin's theorem on contractions ([4], Corollary (6.10)) to conclude that there exists a modification (see [4] for definition):

$$\pi' : \hat{X}' \longrightarrow \bar{X}$$

where $\hat{\hat{X}}$ is the formal spectrum of a complete local ring. By Grothendieck's algebraization theorem [3], there is a scheme X^r containing X_o as a closed subscheme and such that $\hat{X}'$ is the formalization of X' along X_o ; corollary (6.11) of [4] applied to the subscheme $X_o^r = \mathbb{P}_m(k)$ allows us to contract X_o^r to a point. That is, there exists a contraction

$$f : X' \longrightarrow X''$$

such that the formalization of f along $f(X_o) = f(X_o') \cup \ldots \cup f(X_o^{r-1})$ is a formal blowing up with center at the closed point $f(X_o^r)$ of X'' . One concludes by induction on the number r of irreducible components of X_o .

3. The scheme of r-blowing ups and a theorem of boundedness

Let

$$\hat{X}(C_r) \xrightarrow{\ \pi_r\ } \hat{X} \longleftarrow \hat{X}(C_r')$$

be two r-blowing ups, and I , I' be the respectives sheaves of ideals defined by the reduced exceptional fibres. Denote by X_K , X_K' , the subschemes defined respectively by I^{K+1} , I'^{K+1} .

We want to prove the

Theorem 3.1 (Boundedness): For every r there exists a K such that the formal schemes $\hat{X}(C_r)$ and $\hat{X}(C'_r)$ are isomorphic if and only if the schemes X_K and X'_K are isomorphic.

Proof: The condition is obviously necessary for every K. To see the sufficiency suppose the following (*) is true:

(*) "for every positive integer r there exists a λ such that the r-th blowing up

$$\hat{X}(C_r) \longrightarrow \hat{X} = \mathrm{Sp}\ f(A) \quad ,$$

with $A = k[[T_o,\ldots,T_m]]$, is the blowing up of an ideal α of A such that

$$m^{\lambda} \subset \alpha \subset A \quad ,$$

where m is the maximal ideal of A and $\pi_*(\alpha \mathcal{O}_{X(C_r)}) = \alpha$ "..

It is clear that there exists a positive integer K_1 such that

$$m\mathcal{O}_X \subset I^{K_1}$$

for every r-blowing up $\hat{Z}$ of $\hat{X}$ (it is enough to take $K_1 = r!$ and to prove it by induction on r). Let $K = (\lambda+1)K_1$. Suppose

$$\Phi : X_K \approx X'_K$$

is an isomorphism. This induces another isomorphism between the subschemes defined by the sheaves $m^{\lambda+1}\mathcal{O}_X$ and $m^{\lambda+1}\mathcal{O}_{X'}$. By taking global sections one gets an automorphism $\bar{\tau}$ of $\mathrm{Spec}(A/m^{\lambda+1})$. Let τ be an automorphism of $\mathrm{Spec}(A)$ that induces $\bar{\tau}$. Let α be an ideal of A such that

$$\pi_r : \hat{X}(C_r) \longrightarrow \hat{X}$$

is the blowing up along α, and which satisfies the condition (*) above, then $\tau_*(\alpha)$ is another ideal in $\hat{X}$ such that

$$\tau_*(\alpha)\mathcal{O}_{X(C'_r)}$$

is locally principal (to see this use Nakayama and note that $\tau_*(\alpha)\mathcal{O}_{X'_K} = \Phi_*(\alpha\mathcal{O}_{X_K}))$.
So the morphism

$$\hat{X}(C_r) \longrightarrow \mathrm{Sp}\ f(A)$$

factors through the r-blowing up

$$\hat{X}(C^r_r) \longrightarrow \mathrm{Sp}\ f(A) \xrightarrow{\ \tau\ } \mathrm{Sp}\ f(A) \quad ,$$

that is, there exists a morphism

$$f:\hat{X}(C_r)\longrightarrow\hat{X}(C'_r)$$

such that

$$\pi_r = \tau\circ\pi'_r\circ f \quad .$$

One concludes that f is an isomorphism, because $X(C_r)$ and $X(C'_r)$ are r-blowing ups.

To finish one has only to prove (*). Let X be the spectrum of a local k-algebra of finite type. One has:

Theorem 3.2 (Representability): There exists a noetherian scheme of finite type over k and a blowing up

$$\bar{\pi}_n:\bar{X}\longrightarrow X\times\Delta_n$$

such that for every closed point $x_n\varepsilon\Delta_n$ the blowing up of X induced by π_n on the closed subscheme $X=X\times x_n$ of $X\times\Delta_n$ is an n-blowing up of X and every n-blowing up is obtained in this way. Moreover, if $p_{x\times\Delta_n}$ is the sheaf of ideals of the subscheme $x\times\Delta_n$ of $X\times\Delta_n$, then the blowing up $\bar{\pi}_n$ is defined by a sheaf of ideals α such that

$$p^K_{x\times\Delta_n}\subset\alpha\subset p_{x\times\Delta_n} \quad .$$

Proof: We will only give the construction of Δ_n. The properties of Δ_n follow from the the general properties of a blowing up and the definition of Δ_n.

Firstly it is easy to see that if

$$\bar{\pi}:\bar{X}'\longrightarrow X'$$

is a blowing up of schemes over Y, and Z is a flat scheme over Y, and

$$X'_Z = X'\times_Y Z\longrightarrow Z$$

the map obtained by base change, then the blowing up that $\bar{\pi}$ induces on X'_Z is precisely

$$\bar{X}' \times_Y Z \xrightarrow{\bar{\pi}\times 1} X_Z$$

Construction of Δ_n : let

$$\pi_1 : \bar{X}_1 \longrightarrow X$$

be the blowing up of X at its closed point and Δ_2 the exceptional fibre. Let

$$i_2 : \Delta_2 \hookrightarrow \bar{X}_1 \times \Delta_2$$

be the diagonal, and

$$\pi_2 : \bar{X}_2 \longrightarrow \bar{X}_1 \times \Delta_2$$

be the blowing up of $\bar{X}_1 \times \Delta_2$ along $i_2(\Delta_2)$. Inductively one defines

$$\pi_n : \bar{X}_n \longrightarrow \bar{X}_{n-1} \times \Delta_n$$

by blowing up the closed subscheme $i_n : \Delta_n \hookrightarrow \bar{X}_{n-1} \times \Delta_n$, and Δ_{n+1} as the fibre over $i_n(\Delta_n)$ of π_n and where

$$i_{n+1} : \Delta_{n+1} \hookrightarrow \bar{X}_n \times \Delta_{n+1}$$

is the diagonal. It is clear that $\bar{X}_n$ is obtained from

$$R = X \times \Delta_1 \times \ldots \times \Delta_n$$

(with $\Delta_1 = x$) by a sequence of blowing ups. One also has projections

$$f_i : \Delta_n \longrightarrow \Delta_i$$

$(i \leq n)$. Let

$$f : X \times \Delta_n \longrightarrow X \times \Delta_1 \times \ldots \times \Delta_n$$

be the closed immersion defined by

$$f = (\pi_X, f_1 \circ \pi_{\Delta_n}, \ldots, f_n \circ \pi_{\Delta_n})$$

where π_X , π_{Δ_n} are the projections on the factors (with $f_n = \mathrm{Id}_{\Delta_n}$) . Consider

$$\begin{array}{ccc} \bar{X} & \hookrightarrow & \bar{X}_n \\ \downarrow{\scriptstyle \bar{\pi}_n} & & \downarrow{\scriptstyle \pi} \\ X \times \Delta_n & \hookrightarrow & R \end{array}$$

where π is the blowing up constructed above and $\bar{\pi}_n$ the one induced on $X \times \Delta_n$ by π.

(*) Follows from this, because if

$$\pi : \tilde{X} \longrightarrow X$$

is a blowing up, there exists a closed point $x_n \in \Delta_n$ such that

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \bar{X} \\ \downarrow \pi & & \downarrow \bar{\pi} \\ X = X \times x_n & \hookrightarrow & X \times \Delta_n \end{array}$$

where π is the blowing up induced by $\bar{\pi}$. But $\bar{\pi}$ is defined by the blowing up of the sheaf of ideals $\bar{\alpha}$ such that

$$P^K_{x \times \Delta_n} \subset \alpha \subset P_{x \times \Delta_n} .$$

Restricting everything to $X = X \times x_n$ one has that π is defined by the blowing up of a sheaf of ideals α such that

$$m^K_x \subset \alpha \subset m_x .$$

Moreover, one can easily see that $\beta = \pi_*(\alpha \mathcal{O}_{\tilde{X}})$ defines the same blowing up as α, and it verifies

$$\pi_*(\beta \mathcal{O}_{\tilde{X}}) = \pi_*(\alpha \mathcal{O}_{\tilde{X}}) = \beta ,$$

with which one concludes the proof of condition (*).

Observation: The scheme Δ_n parametrizes the analytic branches in $\hat{X}$ up to order n modulo the relation:

$C \sim C'$ if and only if the both direct the same n-blowing up.

4. Classification theorem

Let $\{r_n\}_{n=0}^{\infty}$ be the multiplicity sequence of an embedded branch in X. We will suppose all branches have multiplicity sequence equal to $\{r_n\}_{n=0}^{\infty}$. Let

$$\pi_r : \hat{X}(C_r) \longrightarrow \mathrm{Sp}\ f(A) ,$$

with $A = k[[T_0, \ldots, T_m]]$, be the r-blowing up of X in the direction of a branch C

embedded in X . We will denote by X_o the exceptional reduced fibre of π_r , by I the sheaf of ideals defining X_o in $\hat{X}(C_r)$ and by $L=I/I^2$ the conormal sheaf. Let X_n be the closed subscheme of $\hat{X}(C_r)$ defined by I^{n+1} .

To a given branch C embedded in X we associate the scheme X_1 . Note that the pair (X_o,L) is part of the information X_1 carries which we have defined as the equisingularity of C as embedded in X . In this paragraph we want to classify embedded branches with the same, up to isomorphism, associated scheme X_1 . So we will fix the scheme X_1 . The pair (X_o,L) is also fixed and we will also fix, as a reference to classify, the formal scheme $\hat{X}(C_r)$.

By theorem (1.1), one knows that to classify, up to analytic equivalence, the branches embedded in X with multiplicity sequence $\{r_n\}_{n=o}^{\infty}$ and given associated pair (X_o,L) amounts to classifying isomorphism classes of formal schemes obtained by r-blowing ups of X $(r=h+2\sum_{i=o}^{\infty} r_i(r_i-1)^2)$ such that their associated pair is (X_o,L) . But by the theorem of equivalence (section 2) to classify these classes of formal schemes is the same as to classify the isomorphism classes of formal schemes $\hat{X}'$ which contain X_o as closed subscheme, which are complete along X_o and such that the conormal sheaf to X_o in X' is isomorphic to L and $\mathcal{O}_{\hat{X}'}$ is locally isomorphic to $\mathcal{O}_{\hat{X}^r}$ on X_o . Besides, by the boundedness theorem (3.1), a formal scheme such as X' determined, up to isomorphism, by the closed subscheme X'_K of $\hat{X}'$, where X'_K is the subscheme defined by the sheaf of ideals I'^{K+1} , I' being the defining sheaf of X_o in $\hat{X}'$ (and K is a positive integer whose existence is guaranteed by theorem 3.1).

So, it is enough to classify the schemes X'_K so obtained.

Consider now, for each pair of positive integers $n \geq m$, the sheaf $\mathbf{Aut}_{X_m}(X_n)$ of groups over X_o consisting of local automorphisms of the scheme X_n which restrict the identity on the subscheme X_m .

Denote by

$$\tau_n^m:\mathbf{Aut}_{X_1}(X_n) \longrightarrow \mathbf{Aut}_{X_1}(X_n)$$

(for $n \geq m \geq 1$) the natural restriction maps.

Lemma 4.1. For each $r>1$, the sheaf of groups $\mathbf{Aut}_{X_{n-1}}(X_n)$ is canonically isomorphic to the sheaf of groups

$$\mathbf{Der}_k(\mathcal{O}_{X_1},L^n) \ .$$

In particular

$$\mathbf{Aut}_{X_{n-1}}(X_n) \hookrightarrow \mathbf{Aut}_{X_1}(X_n)$$

is a subsheaf contained in the center of this last sheaf of groups.

Proof: The map

$$\Phi : \mathbf{Aut}_{X_{n-1}}(X_n) \longrightarrow \mathrm{Der}_k(\mathcal{O}_{X_1}, \mathcal{L}^n)$$

defined by

$$\Phi(\tau) = \tau - \mathrm{Id} = D_\tau$$

is a morphism of sheaves of groups. The image of Φ is contained in

$$j_*\mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{L}^n)$$

where

$$j : X_1 \hookrightarrow X_n$$

is the canonical injection. Conversely, given a derivation $D \in \mathrm{Der}_k(\mathcal{O}_{X_1}, \mathcal{L}^n)$ one defines $\Phi^{-1}(D)$ to be

$$\Phi^{-1}(D) = \mathrm{Id} + j_*D = \tau_D \quad .$$

It is easily seen that τ_D is an automorphism of X_n giving the identity on X_1. The rest follows from this.

Corollary 4.2. The sheaves of groups $\mathbf{Aut}_{X_1}(X_n)$ for $n > 1$ have resolutions by sheaves of coherent $\mathcal{O}_{X_o}$-modules of the form

$$\mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{L}^h)$$

for $2 \leqq h \leqq n$. More precisely, the sequences

$$(n) \qquad 0 \longrightarrow \mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{L}^n) \xrightarrow{i_n} \mathbf{Aut}_{X_1}(X_n) \longrightarrow \mathbf{Aut}_{X_1}(X_{n-1}) \longrightarrow 0$$

are exact, where i_n is defined as in the above lemma, identifying

$$\mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{L}^n) = \mathbf{Aut}_{X_{n-1}}(X_n) \quad .$$

To simplify the notation, let us write

$$D^n = \mathbf{Der}_k(\mathcal{O}_{X_1}, \mathcal{O}_{X_o}) \otimes \mathcal{L}^n$$

$$A^n = \mathrm{Aut}_{X_1}(X_n)$$

$$G^n = \mathcal{L}^2 \otimes \mathcal{L}^3 \otimes \ldots \otimes \mathcal{L}^n \quad .$$

Corollary 4.3: For every n there exists a quotient M_1 of the abelian group

$$H^1(X_o, D^o \otimes_{X_o} G^n)$$

and a map

$$f_n : M_n \longrightarrow H^2(X_o, D^o \otimes G^n)$$

which identifies

$$H^1(X_o, A^n)$$

with the subset of M_n defined the elements $c \in M_n$ such that $f_n(c) = 0$.

Proof: By induction on n . For $n = 2$ one has

$$M_1 = H^1(X_o, D^2)$$

and $f_1 = 0$ by applying lemma 5.1. If $n > 2$ the cohomology sequence associated with (n) gives an exact sequence

$$B = H^1(D^n)/\delta H^o(A^{n-1}) \hookrightarrow H^1(A^n) \longrightarrow H^1(A^{n-1}) \xrightarrow{\overline{\delta^n}} H^2(D^n)$$

(see [1]). The first term on the left is an abelian group and acts freely on the left on $H^1(A^n)$. The orbits of this action are the fibres of

$$\mathrm{Im}(\bar{\tau}^n) = (\overline{\delta^n})^{-1}(o) \quad .$$

So

$$H^1(A^n) = \mathrm{Im}(\bar{\tau}^n) \times B \subset H^1(A^{n-1}) \times B \subset M_{n-1} \times B \quad .$$

We define

$$f_{n-1} : M_n \longrightarrow H^2(D^n) \oplus H^2(D^o \oplus G^{n-1}) = H^2(D^o \oplus G^n)$$

to be

$$f_{n-1} = \bar{f}_{n-1} \circ \pi \quad ,$$

where

$$\pi : M_{n-1} \times B \longrightarrow M_{n-1}$$

is the natural projection and

$$\bar{f}_{n-1} = f_{n-1} + g$$

where $g : M_{n-1} \longrightarrow B$ is equal to $\bar{\delta}^n$ on $H^1(A^{n-1})$ and is zero on the complement.

As the sheaves $\mathcal{O}_{X'_K}$ are locally isomorphic to $\mathcal{O}_{X_K}$, they are classified by the set

$$H^1(X_1, \mathbf{Aut}_{X_1}(X_K))$$

(see [1]). The quotient of this H^1 by the action of the group $Aut(X_1)$ classified the schemes X'_K which contain X_1 as a closed subscheme, and whose structure sheaf $\mathcal{O}_{X'_K}$ is locally isomorphic to $\mathcal{O}_{X_K}$.

From Corollary 4.3 and the considerations at the beginning of this paragraph the main theorem of this introduction follows easily.

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