# AMPLE DIVISORS ON GORENSTEIN VARIETIES

by

Andrew John SOMMESE

Let $L$ be an ample line bundle on a normal irreducible $n$ dimensional complex projective variety $X$. Assume that $L$ is spanned by global sections at all points of $X$. Assume further that $X$ is Gorenstein and that the set $\mathrm{Irr}(X)$ of non rational singularities on $X$ is finite. In this paper we give some structure theorems for such pairs $(X,L)$.

Theorem [(0.4), (0.4.1)] :

Let $X$ and $L$ be as above. Then $K_X \otimes L^n$ is spanned by global sections unless $(X,L) = \left[\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)\right]$.

If $K_X \otimes L^n$ is spanned at all points and the image of the map associated to $\Gamma(K_X \otimes L^n)$ has dimension $< n$ then either :

a) $X$ is biholomorphic to a quadric $\mathcal{Q}$ in $\mathbb{P}^{n+1}$ and $L$ is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ to $\mathcal{Q}$,

or

b) $X$ is a $\mathbb{P}^{n-1}$ bundle over a curve $C$, and the restriction of $L$ to a fibre is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Theorem [(0.5.2), (1.1), (1.2)] :

The following conditions are equivalent for the pair $(X,L)$ :

a) $K_X \otimes L^{n-1}$ is numerically effective,

b) the image of the map associated to $\Gamma(K_X \otimes L^n)$ is $n$ dimensional,

c) there is a $t > 0$ such that $(K_X \otimes L^{n-1})^t$ is spanned by global sections at all points.

d) $h^0(K_X \otimes L^{n-1}) \neq 0$,

e) $g(L) \neq h^{1,0}(X)$.

Finally in (1.4) the structure of the map associated to $\Gamma\left[(K_X \otimes L^{n-1})^t\right]$ is partially described.

These results are proved by induction on the stronger results [So6] for Gorenstein surfaces. For smooth varieties ([So1], [So2], [So3], [So4], [B+P]) and Gorenstein varieties with milder singularities [(Fa+So)2], [Li+So] stronger results are true. See also [Sa].

In section 2 we prove an unstable version of the preceding results.

Theorem (2.0) :

Let $L$ be an ample and spanned line bundle on a normal irreducible Gorenstein projective variety $X$ of dimension $n \geq 2$ such that $\mathrm{Irr}(X)$ is finite. Assume that the map $\phi : X \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma(L)$ is generically one to one and an immersion off $\mathrm{Sing}(X)$. Then the following are equivalent :

a) $g(L) \neq h^{1,0}(X)$,

b) $\Gamma(K_X \otimes L^{n-1})$ spans $K_X \otimes L^{n-1}$ at all points.

For smooth surfaces this result can be found in [So1] and [VdV]. The case of $n \geq 3$ immediately reduces to the case of $n = 2$ (as in [So1, § 3], [So7], and [I]). The proof for $n = 2$ simplifies in the case of a smooth surface to a slight variant of the proof in [So1]. This variant in the smooth case shortens the proof in [So1]; the original proof in [So1] has the virtue of classifying all pairs $(X,L)$ such that $L$ is very ample on $X$ and there is a line $P \subseteq X$ such that a generic $C \in |L|$ with $P \subseteq C$ is of the form $P+R$ where $R$ is disconnected.

In § 3 we give a few simple applications of the above to the classification of embeddings of normal Gorenstein surfaces $X$ in $\mathbb{P}_{\mathbb{C}}$ with the genus of the desingularization of a general hyperplane section $\leq 3$. Results such as this have a long history (see [So1], [I], [Liv], and the very nice recent thesis [S-G]).

I would like to thank the Max Planck Institut für Mathematik, the University of Notre Dame, the University of Nancy I, and the National Science Foundation [DMS 82-00629 and DMS 84-20315] for their support.

§ 0.- Background Material :

We will follow the notation of [So2 , (Fa+So)2] . One exception is that we will denote the dualizing sheaf of a variety $X$ , by $K_X$ .

If $X$ is a normal variety then $Irr(X)$ , the irrational locus of $X$ , is the set of all non rational singularities of $X$ . The following is an immediate consequence of the usual Kodaira vanishing theorem.

(0.1) Kodaira Vanishing Theorem [(Fa+So)1 , So6] :

Let $L$ be an ample line bundle on an irreducible normal projective variety $X$ . Then $H^p(X,K_X \otimes L) = 0$ for $p > \max(\dim Irr(X),0)$ .

We need a form of Bertini's Lemma.

(0.2) Lemma [So6] :

Let $L$ be an ample line bundle on a normal irreducible Gorenstein projective variety $X$ of dimension $\geq 3$ . Assume that $L$ is spanned by global sections at all points of $X$ . Given any $x \in X$ , a generic $A \in |L|$ passing through $x$ is normal, irreducible and Gorenstein. Further if $\dim Irr(X) \leq 0$ then $\dim Irr(A) \leq 0$ .

We need a slight generalization of a result of Kobayashi and Ochiai [K+O] .

(0.3) Theorem :

Let $X$ be an irreducible normal projective Gorenstein variety with $\dim Irr(X) \leq 0$ . Let $L$ be an ample line bundle spanned by global sections at all points of $X$ . Assume that $K_X = L^{-t}$ . Then $t \leq \dim X + 1$ . If $t = \dim X + 1$ then $(X,L) = [\mathbb{P}^{t-1}, \mathcal{O}_{\mathbb{P}^{t-1}}(1)]$ .

If $t = \dim X$ then $X$ is biholomorphic to a quadric $\mathcal{Q}$ in $\mathbb{P}^{t+1}$ and $L$ is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^{t+1}}(1)$ to $X$ .

Proof :

The proof follows by the inductive argument of [K+O] using (0.1) and (0.2) instead of the usual Kodaira vanishing and Bertini theorems.

□

By a geometrically ruled surface we mean a pair $(S,L)$ consisting of :

a) a normal irreducible Gorenstein projective surface $S$ ,

b) an ample line bundle $L$ on $S$ such that $(K_S \otimes L^2)^n$ is spanned for $n >> 0$ and the map associated to $\Gamma\left[(K_S \otimes L^2)^n\right]$ has a one dimensional image.

Note that, by [So6] this class consists precisely of pairs $(S,L)$ where :

a) $S$ is a $\mathbb{P}^1$ bundle over a curve $C$ ,

b) $L$ is an ample line bundle over $S$ such that $L_F \approx \mathcal{O}_F(1)$ for a fibre $F$ of the bundle,

c) $L$ is not the degree $1,1$ line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ .

Except for the exclusion of c), this is the usual class of geometrically ruled surfaces. Note that $\Gamma\left[(K_S \otimes L^2)^n\right]$ gives the ruling on $S$ .

(0.4) Theorem :

Let $L$ be an ample line bundle on an $n$ dimensional normal, irreducible projective variety. Assume that $L$ is spanned by global sections at all points of $X$ and that $\mathrm{Irr}(X)$ is finite. Then $K_X \otimes L^n$ is spanned by global sections at all points of $X$ unless $(X,L) \approx \left[\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)\right]$ .

Proof :

This result is easily seen to be true for $n = 1$ and it is true for $n = 2$ by theorem (3.4) of [So6] . We can thus assume without loss of generality that $n \geq 3$ . Assume by induction the theorem is true for dimensions $< n$ . For an arbitrary $x \in X$ , choose a generic $A \in |L|$ that passes through $X$ .

By lemma (0.2) it follows that $A$ is irreducible normal and Gorenstein and that $\mathrm{Irr}(A)$ is finite. Consider :

$$o \to K_X \otimes L^{n-1} \to K_X \otimes L^n \to K_A \otimes L^{n-1} \to o \ .$$

By the Kodaira vanishing theorem (0.1) and the induction hypothesis, it follows that $K_X \otimes L^n$ is spanned by global sections at $x$ . Since $x$ was arbitrary this proves the theorem.

□

(0.4.1) Corollary :

Let X and L be as in (0.4) and assume that $K_X \otimes L^n$ is spanned by global sections at all points of X . Let $\phi : X \to \mathbb{P}_{\mathbb{C}}$ be the map associated to $\Gamma(K_X \otimes L^n)$ . If $\dim \phi(X) < n$ then either :

a) X is biholomorphic to a quadric $\mathcal{Q}$ in $\mathbb{P}^{n+1}$ and L is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ to $\mathcal{Q}$,

or

b) X is a $\mathbb{P}^{n+1}$ bundle over a curve C and the restriction of L to a fibre is isomorphic to $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ .

Proof :

If $\dim \phi(X) = 0$ then $K_X = L^{-n}$ and this follows from (0.3).

If $\dim \phi(X) > 0$ let $s \circ r = \phi$ be the Remmert-Stein factorization of $\phi$, i. e. $r : X \to Y$ is a holomorphic surjection with connected fibres onto a normal variety and $s : Y \to \mathbb{P}_{\mathbb{C}}$ has finite fibres. Let F be a generic fibre of r . Then F is irreducible, Gorenstein and normal. Further $k_F \approx L^{-n}$ . By (0.3) and $\dim \phi(X) > 0$ it follows that $(F, L_F) \approx (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}})$ . The rest of the proof is by standard arguments.

□

(0.5) Definition :

$\mathcal{R}$ is the set of all pairs (X,L) such that :

a) X is an irreducible normal Gorenstein projective variety,

b) L is an ample line bundle on X spanned by global sections at all points of X ,

c) given some irreducible normal Gorenstein n fold $A \in |L|$ with $n \geq 2$ , $(A, L_A) \in \mathcal{R}$,

d) $\mathcal{R}$ contains the geometrically ruled surfaces, $[\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)]$ , and the 2 dimensional irreducible Gorenstein quadrics $[\mathcal{Q}^2, \mathcal{O}_{\mathcal{Q}^2}(1)]$ .

(0.5.1) Theorem :

Let $(X,L) \in \mathcal{R}$ with $\dim X = n \geq 2$ . Then either :

a) $(X,L) = \left[\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)\right]$ ,

or

b) $X$ is a $\mathbb{P}^{n-1}$ bundle over a curve $C$ and $L$ restricted to any fibre is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ,

or

c) $X$ is biholomorphic to a quadric $\mathcal{Q}$ in $\mathbb{P}^{n+1}$ and $L$ is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ to $\mathcal{Q}$ .

Proof :

Since by the definition of $\mathcal{R}$ this is true when $n = 2$ we can assume by induction that $n \geq 3$ and that the result is true for $n-1$ folds.

Choose the $A \in |L|$ which by (0.5c) is irreducible, normal, and Gorenstein. If $(A,L_A) \approx \left[\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1)\right]$ then by the adjunction formula $K_X = L^{-(n+1)}$ and the theorem is true by theorem (0.3). If $A$ is a quadric and $K_A \otimes L_A^{n-1} = \mathcal{O}_A$ , then by the adjunction formula $K_X = L^{-n}$ and the theorem is true by (0.3) again. If $A$ is a $\mathbb{P}^{n-2}$ bundle $f : A \to C$ over a curve $C$ with $L_F \approx \mathcal{O}_F(1)$ for a fiber $F$ of $f$ then $K_A \otimes L_A^{n-1}$ is trivial on $F$ . Let $\phi : X \to \mathbb{P}_{\mathbb{C}}$ be map associated to $\Gamma(K_X \otimes L^n)$ by theorem (0.4). Then since $\dim \phi(A) < \dim A$ it follows that $\dim \phi(X) < n$ and therefore the theorem is done by corollary (0.4.1). □

(0.5.2) Theorem :

Let $L$ be an ample and spanned line bundle on a normal irreducible Gorenstein variety of dimension $n \geq 2$ . The following are equivalent :

a) $g(L) = h^1(\mathcal{O}_X)$ ,

b) $h^\circ(K_X \otimes L^{n-1}) = 0$ and $\mathrm{Irr}(X)$ finite,

c) $(X,L) \in \mathcal{R}$ .

Proof :

First we do the case of $n = 2$ . Note that a) is equivalent to b). To see this consider :

$$*) \qquad o \to K_X \to K_X \otimes L \to K_C \to o$$

for a general $C \in |L|$ . Note that $h^1(K_X \otimes L) = 0$ by (0.1). If $h^\circ(K_X \otimes L) = 0$ , then $g(L) = h^\circ(K_C) = h^1(K_X)$ and since $X$ is a normal surface $h^1(K_X) = h^1(\mathcal{O}_X)$ . Conversely if $g(L) = h^1(\mathcal{O}_X)$ then we conclude from *) that $h^\circ(K_X) = h^\circ(K_X \otimes L)$ . Since $h^\circ(K_X) \neq 0$ and $h^\circ(L) \neq 0$ imply that $h^\circ(K_X \otimes L) \geq h^\circ(L) + h^\circ(K_X) - 1$ we conclude that $h^\circ(K_X) = 0$ . Hence $h^\circ(K_X \otimes L) = 0$ .

Theorem (1.3) and (1.4) of [So6] imply that a) and c) are equivalent. Thus the theorem is true if $n = 2$.

In general by using (0.5.1) c) is easily checked to imply a) and b).

Thus we can by induction assume that if $n \geq 3$ the theorem is true for dimensions $< n$. Let $A$ be a general element of $|L|$. $A$ is normal and $Irr(A)$ is finite by (0.2). Consider :

$$o \to K_X \to K_X \otimes L \to K_A \to o \ .$$

Tensoring with $L^{n-2}$ and using (0.1) we see that b) implies that $(A, L_A)$ satisfies b). By induction and property c) of the definition of $\mathcal{R}$ we conclude that b) $\Rightarrow$ c).

Choose a set of general elements $\{A_1, \ldots, A_{n-1}\} \subseteq |L|$ which meet transversaly in a smooth curve $C$.

Let $\mathcal{A}_0 = X$, $\mathcal{A}_1 = A_1$, $\mathcal{A}_2 = \mathcal{A}_1 \cap A_2, \ldots, \mathcal{A}_{n-1} = \mathcal{A}_{n-2} \cap A_{n-1}$. We have :

*) $$o \to [\mathcal{A}_i]^{-1} \to \mathcal{O}_{\mathcal{A}_{i-1}} \to \mathcal{O}_{\mathcal{A}_i} \to o \ .$$

Since the genericity of the $A_i$ and (0.2) implies that the $\mathcal{A}_i$ are all normal we have $H^1(\mathcal{A}_i, [\mathcal{A}_{i+1}]^{-1}) = 0$ for $i = 0, \ldots, n-2$ by Munford's vanishing theorem. Thus by **)

$$h^1(\mathcal{O}_X) \leq h^1(\mathcal{O}_{\mathcal{A}_1}) \leq \ldots \leq h^1(\mathcal{O}_C) \ .$$

Since $h^1(\mathcal{O}_X) = g(L) = h^1(\mathcal{O}_C)$ we conclude that for all $i \leq n-2 (\mathcal{A}_i, L_{\mathcal{A}_i})$ satisfies a).

To finish we need show that a) $\Rightarrow$ c). Assume a). By property c) of the definition of $\mathcal{R}$ and induction a) $\Rightarrow$ c).

□

(0.6) Let $L$ be an ample line bundle on a normal Gorenstein surface $X$. Let $C \in |L|$. Let $\pi : \overline{X} \to X$ be a resolution of singularities and let $\overline{C} = \pi^{-1}(C) \in |\pi^*L|$. Consider the diagram :

#)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_X & \longrightarrow & K_X \otimes L & \xrightarrow{\text{Res}} & K_C & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \pi_*} & & \uparrow{\scriptstyle \pi_*} & & \uparrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & K_{\overline{X}} & \longrightarrow & K_{\overline{X}} \otimes \pi^*L & \xrightarrow{\text{Res}} & K_{\overline{C}} & \longrightarrow & 0 \\
 & & & & & \stackrel{\beta}{\nearrow} & & & \\
 & & & & T^*_{\overline{X}} & & & &
\end{array}$$

Here the 2 horizontal sequences are standard ; $K_C$ and $K_{\overline{C}}$ are the dualizing sheaves of $C$ and $\overline{C}$ and the horizontal sequences can be taken as their definitions (see [A+K]) .

The maps $\pi_*$ are standard [A+K] and they imply the existence of $\alpha$. The map $\beta$ is more or less standard. It is defined as follows. Fix a section $s$ of $\pi^*L$ that vanishes precisely on $\overline{C}$. On an open set $U$ of $\overline{X}$, write $s_U = f \otimes e$ where $e$ is a nowhere vanishing section of $\pi^*L$ over $U$. Given $\eta$, a local section of $T^*_{\overline{X}}$ over $U$, define $\beta(\eta) = \text{Res}\,[(\eta \wedge df)e]$. To see that it only depends on $s$ and not $U$ write $s_U = (fg) \otimes e'$ where $g$ is a nowhere vanishing function on $U$ and $e'$ is a nowhere vanishing section of $(\pi^*L)_U$. Then

$$\begin{aligned}
&\text{Res}\,[\eta \wedge d(fg) \otimes e'] = \\
&\text{Res}(\eta \wedge df \otimes e) + \text{Res}\,[f(\eta \wedge dg) \otimes e'] \\
&= \text{Res}(\eta \wedge df \otimes e)
\end{aligned}$$

since $f$ vanishes on $\overline{C} \cap U$.

We also have a map

##) $$H^0(T^*_{\overline{X}}) \xrightarrow{\gamma} H^1(K_{\overline{X}}) \to 0$$

obtained in the following manner. First note that $H^\circ(T^*_{\bar{X}}) \xrightarrow{\wedge c_1(\pi^*L)} H^1(K_{\bar{X}})$ is an isomorphism. To see this note $c_1(\pi^*L)$ is represented by $\omega$, a positive semidefinite (1,1)-form that is positive definite on a dense open set of $\bar{X}$. Therefore if $\eta$ is a holomorphic on form on $X$

$$\sqrt{-1}\,\omega \wedge \eta \wedge \bar{\eta}$$

is a positive semidefinite (2,2)-form that is positive definite on a dense open set of $\bar{X}$. In particular

$$\int_X \sqrt{-1}\,\omega \wedge \eta \wedge \bar{\eta} > 0$$

and thus $\wedge\, c_1(\pi^*L)$ is an injective map from $H^\circ(T^*_{\bar{X}})$ to $H^1(K_{\bar{X}})$. Since by Serre duality and Hodge theory

$$h^\circ(T^*_{\bar{X}}) \approx h^2(T^*_{\bar{X}}) \approx h^1(K_{\bar{X}})$$

we conclude that $\wedge\, c_1(\pi^*L)$ is an isomorphism. Compose this with $\pi^* H^1(K_{\bar{X}}) \to H^1(K_X)$ to get $\gamma$.

A simple variant of a result of Matsumura [Ma, Prop. (2.2), pg. 38] shows that

$$\begin{array}{ccccc} \Gamma(K_{\bar{X}} \otimes \pi^*L) & \to & \Gamma(K_{\bar{C}}) & \to & H^1(K_{\bar{X}}) \\ & & \uparrow{\scriptstyle \mathcal{B}} & \nearrow{\scriptstyle \wedge\, c_1(\pi^*L)} & \\ & & \Gamma(T^*_{\bar{X}}) & & \end{array}$$

commutes. Combining this with #) and # #) we get the following useful lemma (the Gorenstein condition is not needed).

(0.6.1) Lemma :

Let $L$ be an ample line bundle on a normal irreducible Gorenstein surface $X$. Let $C \in |L|$. Then the following diagram commutes where $\alpha$ and $\gamma$ are as defined above and $\pi : \bar{X} \to X$ is any desingularization of $X$.

$$\begin{array}{ccccc} & & & & 0 \\ & & & \nearrow & \\ \Gamma(K_X \otimes L) \to & \Gamma(K_C) & \to & H^1(K_X) & \\ & \uparrow \alpha & \gamma \nearrow & & \\ & \Gamma(T^*_X) & & & \end{array}$$

(0.6.2) Corollary :

Let $L$ be an ample and spanned line bundle on a normal irreducible Gorenstein surface $X$ . Assume that $L.L \geq 3$ . Assume that $g(L) \neq h^{1,0}(X)$ and let $C \in |L|$ . If $x$ is a smooth (and hence reduced) point of $C$ then

$$\Gamma(K_K \otimes L) \quad \text{spans} \quad (K_X \otimes L)_x$$

unless the irreducible component $P$ of $C$ that contains $x$

a) is a rational curve,

b) belongs to $X_{reg}$ .

Proof :

If $\Gamma(K_C)$ spans $K_{C,x}$ then by lemma (0.6.1) it follows that $\Gamma(K_X \otimes L)$ not spanning $K_{C,x}$ can happen only if $\alpha(\eta)$ is non zero at $x$ for some $\eta \in \Gamma(T^*_{\overline{X}})$ . In particular composing with the desingularization $\overline{P}$ of $P$ we get a non zero holomorphic one form in $\overline{P}$ . So a) is true.

Next, note by theorem (3.1) of [So6] it follows that $K_X \otimes L$ is spanned in a neighborhood of $Sing(X)$ . Thus if $P$ meet $Sing(X)$ then $\Gamma(K_C)$ spans $K_{C,p}$ on a dense open set of $P$ . Consider :

#) $$0 \to K_C \otimes m_x \to K_C \to \mathbb{C}_x \to 0$$

where $m_x = [x]^{-1}$ is the ideal sheaf of $x$ on $C$ .

Using

$$0 \to K_X \to K_X \otimes L \to K_C \to 0$$

and (0.1) we conclude that $h^1(K_C) = 1$ . Thus by #) it follows that if $K_C$ doesn't span $K_{C,x} \approx \mathbb{C}_x$ then $h^1(K_C \otimes m_x) = 2$ . By Serre duality $h^\circ(C,[x]) = 2$ . This implies that $h^\circ(C,[y]) = 2$ for $y$ near $x$ and thus $H^1(C,K_C \otimes m_y) = 2$ . This implies that $\Gamma(K_C)$ does not span $K_C$ on dense set of $P$ . Therefore if b) were false we conclude by contradiction that $\Gamma(K_C)$ span $K_{C,x}$ . Since by a) $\mathcal{B}$ must be $0$ we get a contradiction from (0.6.1). □

§ 1.- The Stable Adjunction Mapping :

(1.0) Throughout this section $L$ is an ample line bundle on a normal irreducible Gorenstein $n$ dimensional projective variety $X$ with $\mathrm{Irr}(X)$ finite. Further it is assumed that $L$ is spanned by global sections at all points of $X$.

(1.1) Theorem :

Let $(X,L)$ be as above. Then there is a $t > 0$ such that $(K_X \otimes L^{n-1})^t$ is spanned by global sections unless either :

a) $(X,L) \approx [\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)]$,

or

b) $(X,L) \approx [\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)]$ where $\mathbb{Q}^n$ is an $n$ dimensional quadric in $\mathbb{P}^{n+1}$ and $\mathcal{O}_{\mathbb{Q}^n}(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ to $\mathbb{Q}^n$,

or

c) $X$ is a $\mathbb{P}^{n-1}$ bundle over a curve $C$ and the restriction of $L$ to a fibre $F$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Proof :

This result is true if $n = 2$ by [So6] and is trivial for $n = 1$. We need a few lemmas.

(1.1.1) Lemma :

Let $L$ be an ample line bundle on an irreducible Gorenstein surface $S$. Assume that $L$ is spanned by global sections at all points of $S$. If $K_S \otimes L$ is numerically effective then $K_S^t \otimes L^{t+1}$ is spanned for all $t \geq 1$.

Proof :

The case of $t = 1$ is theorem (3.4) of [So6]. Assume that the lemma is true for $t \leq t_0$. Note that $K_S^{t_0+1} \otimes L^{t_0+2} = K_S \otimes (K_S^{t_0} \otimes L^{t_0+1}) \otimes L$ is spanned at all points by theorem (3.4) of [So6] and that

$K_S^{t_0} \otimes L^{t_0+1}$ is spanned at all points of $S$ by the induction hypothesis.

□

Let $\mathcal{S}$ denote the class of all pairs $(X,L)$ such that :

a) $X$ is a normal irreducible Gorenstein projective variety with $\mathrm{Irr}(X)$ finite,

b) $L$ is an ample line bundle on $X$ that is spanned by global sections at all points of $X$,

c) $(X,L)$ does not belong to the class $\mathcal{R}$ of (0.5).

Consider the following assertion.

Assertion $S(n,t)$ :
For $(X,L) \in \mathcal{A}$, $\Gamma(K_X^t \otimes L^{(n-1)t+1})$ spans $K_X^t \otimes L^{(n-1)t+1}$ at all points.

(1.1.2) Lemma :

If $t \geq 0$ and $n \geq 3$ then $S(n-1,t+1)$ and $S(n,t)$ imply $S(n,t+1)$.

Proof :

Let $(X,L) \in \mathcal{A}$ with $\dim X = n$. Let $x \in X$. By (0.2) and (0.5c) we can choose $A \in |L|$ such that $(A,L_A)$ belongs to $\mathcal{A}$. Consider the exact sequence :

$$0 \to K_X \otimes (K_X^t \otimes L^{(n-1)t+1}) \otimes L^{n-2} \to K_X^{t+1} \otimes L^{(n-1)(t+1)+1} \to K_A^{t+1} \otimes L_A^{(n-2)(t+1)+1} \to 0 .$$

By $S(n,t)$ the bundle $(K_X^t \otimes L^{(n-1)t+1}) \otimes L^{n-2}$ is ample and by (0.1), $H^1(K_X^{t+1} \otimes L^{(n-1)t+1} \otimes L^{n-2}) = 0$. Thus by the fact that $x$ is an arbitrary point of $X$ and assertion $S(n-1,t+1)$ is follows from the above exact sequence that $S(n,t+1)$ is true □

Note that lemma (1.1.1) asserts that $S(2,t)$ is true for $t \geq 0$.

(1.1.3) Lemma :

Let $(X,L) \in \mathcal{A}$ with $\dim X = n \geq 2$. Then $S(n,1)$ is true.

Proof :

This is simply a restatement of (0.4).

□

By induction $S(n,t)$ is true for all $n \geq 2$, $t \geq 0$.

Let $(X,L)$ be as in (1.0).

Thus $K_X^t \otimes L^{(n-1)t+1}$ is spanned by global sections at all points for arbitrarily large $t$. From this it follows that $K_X \otimes L^{(n-1)}$ is numerically effective. Choose $A \in |L|$ containing an arbitrary point of $X$. It can be assumed that $A$ is irreducible normal Gorenstein with $\mathrm{Irr}(A)$ finite by (0.2).

We have the exact sequence :

$$0 \to K_X \otimes (K_X \otimes L^{n-1})^t \otimes L^{n-2} \to (K_X \otimes L^{n-1})^{t+1} \to (K_A \otimes L^{n-2})^{t+1} \to 0 .$$

Since $(K_X \otimes L^{n-1})^t$ is numerically effective, $(K_X \otimes L^{(n-1)t} \otimes L^{n-2}$ is ample if $n \geq 3$ . By (0.1) and induction the theorem follows from the above exact sequence and the fact that $x$ is arbitrary.

□

(1.2) Corollary :

Let (X,L) be as in (1.0). The following assertions are equivalent.

a) $K_X \otimes L^{n-1}$ is numerically effective,

b) there is a $t > 0$ such that $K_X^t \otimes L^{(n-1)t}$ is spanned by global sections.

Proof :

Immediate corollary of (1.1).

□

(1.3) Definition :

Let (X,L) be as in (1.0). Assume that $K_X \otimes L^{n-1}$ is numerically effective. Choose $t > 0$ such that :

(1.3.1) $(K_X \otimes L^{n-1})^t$ is spanned at all points by global sections, (1.3.2) the map $\phi : X \to \phi(X) \subseteq \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma\left[(K_X \otimes L^{n-1})^t\right]$ has connected fibres and a normal image.

Then $\phi : X \to \phi(X)$ is called the stable adjunction mapping associated to (X,L) .

By (1.2) and standard reasoning $\phi$ exists. By the usual argument $\phi$ does not depend on $t$ .

(1.4) Theorem :

Let (X,L) be as in (1.0). Assume that $K_X \otimes L^{n-1}$ is numerically effective. Let $\phi : X \to \phi(X)$ be the stable adjunction mapping associated to (X,L) . Then $\dim \phi(X) = n$ or 0 or 1 or 2 . If $\dim \phi(X) = 2$ then $\phi$ is a $\mathbb{P}^{n-2}$ bundle with $L_F \approx \mathcal{O}_F(1)$ for any fibre. If $\dim \phi(X) = 1$ then the general fibre of $\phi$ is a quadric. If $\dim \phi(X) = 0$ then $K_X \otimes L^{n-1} = \mathcal{O}_X$ .

Proof :

Standard reasoning using (0.1), (0.2), and (0.3) instead of the usual Kodaira vanishing theorem, Bertini's theorem, and the theorem of Kobayashi-Ochiai ; e. g. see [So4] and [(Fa+So)1] .

□

## § 2.- The Unstable Adjunction Mapping :

(2.0) Theorem :

Let $L$ be an ample and spanned line bundle on a normal irreducible Gorenstein projective variety $X$ with $\mathrm{Irr}(X)$ finite and $n = \dim X \geq 2$. Assume that the map $\phi : X \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma(L)$ is an immersion off a set of codimension 2. Assume further that $\phi$ is generically one to one. Then the following are equivalent :

a) $h^{\circ}\left[K_X \otimes L^{\dim X-1}\right] \neq 0$ ,

b) $\Gamma\left[K_X \otimes L^{\dim X-1}\right]$ spans $K_X \otimes L^{\dim X-1}$ at all points of $X$ ,

c) $g(L) \neq h^{1,0}(X)$ ,

d) $(X,L) \notin \mathcal{R}$ .

Proof :

The equivalence of a), c), and d) follows from (0.5.2). Clearly a) is implied by d). Thus we must show for $n \geq 2$ that :

$*)_n$ $\quad (X,L) \notin \mathcal{R} \Rightarrow$ b) where $\dim X = n$ .

Claim :

If $*)_2$ is true then $*)_n$ is true for all $n$ .

Proof of Claim :

Assume $*)_k$ is true for $k < n$ where $n \geq 3$ . Let $x \in X$ and choose a general $A \in |L|$ with $x \in A$ . By (0.2) and the definition of $\mathcal{R}$ , we can use $*)_{n-1}$ to conclude that $K_A \otimes L_A^{(n-1)-1}$ is spanned by global sections.

Consider :

$$0 \to K_X \otimes L^{n-2} \to K_X \otimes L^{n-1} \to K_A \otimes L_A^{n-2} \to 0 .$$

If $n > 2$ then $h^1(K_X \otimes L^{n-2}) = 0$ by (0.1). Thus $K_X \otimes L^{n-1}$ is spanned by global sections at $x$ if $K_A \otimes L_A^{n-2}$ is spanned by global sections on $A$ . This proves the claim.

□

By the claim we are reduced to the case $\dim X = 2$ , By ([So6], (3.1)) we must only show that $\Gamma(K_X \otimes L)$ spans $(K_X \otimes L)_x$ for smooth points $x \in X$ . $\pi : \overline{X} \to X$ be a desingularization of $X$ where $\pi$ is a biholomorphism in a neighborhood of $\pi^{-1}(x)$ . For simplicity of notation identify $x$ and $\pi(x)$ . If $K_{\overline{X}}$ has a global section $\sigma$ that is non zero at $x$ then $\pi^*\sigma$ is a global section of $K_X$ non-zero at $x$ and we conclude that $\Gamma(K_X \otimes L)$ spans $K_X \otimes L$ at $x$ .

Therefore we can assume that $\Gamma(T^*_{\overline{X}})$ does not span $T^*_{\overline{X},x}$ . There are two cases :

a) all sections of $T^*_{\overline{X}}$ are zero at $x$ ,

b) the sections of $T^*_{\overline{X}}$ span a one dimensional subspace $E$ of $T^*_{\overline{X},x}$ .

Assume a). Choose a general $C \in |L|$ that passes through $x$ . $C$ is smooth and $\pi^{-1}(C) \approx C$ . By assumption $g(L) > 0$ and thus $K_C$ is spanned by global sections. Thus by (0.6.1) and assumption a) $\Gamma(K_X \otimes L)$ spans $(K_X \otimes L)_x$ .

Finally assume b). Let $v$ be a non-zero tangent vector at $x$ annihilated by $E$ . Choose a general $C \in |L|$ with $C$ tangent to $v$ at $x$ . It is easy to see that $C$ is smooth at $x$ and that $C$ is generally reduced. By (0.6.2) we conclude that the theorem can be false only if $C = \mathcal{P} + \mathcal{R}$ where $x \in \mathcal{P}$ and $\mathcal{P}$ is a rational curve belonging to $X_{reg}$ . By the fact that $\mathcal{P} \subseteq X_{reg}$ and the reasoning of ([So1], lemma (0.10.2)) it follows that either $\mathcal{P} = C$ or $L \cdot \mathcal{P} = 1$ and $\mathcal{P}$ is smooth and transverse to $\mathcal{R}$ where $\mathcal{R}$ is smooth and $\mathcal{R} \subseteq X_{reg}$ . The former assumption and (0.1) imply the contradiction that $h^1(\mathcal{O}_X) = g(L) = 0$ .

Thus $C = \mathcal{P} + \mathcal{R}$ , $\{\mathcal{P}, \mathcal{R}\} \subseteq X_{reg}$ , $\mathcal{R}$ is smooth and transverse to $\mathcal{P}$ and $L \cdot \mathcal{P} = 1$ . If $\mathcal{R} \cdot \mathcal{P} > 1$ then it is classical that $\Gamma(K_C)$ spans $K_{C,\mathcal{P}}$ on $C_{reg} \cap \mathcal{P}$ . The argument of (0,6.2b) would then show that $\Gamma(K_X \otimes L)$ spans $K_{C,x}$ . Thus $\mathcal{P} \cdot \mathcal{R} \leq 1$ . By ampleness of $L\mathcal{R} \cdot \mathcal{P} \geq 1$ . Thus $\mathcal{R} \cdot \mathcal{P} = 1$ . Further $1 = L \cdot \mathcal{P} = \mathcal{R} \cdot \mathcal{P} + \mathcal{P} \cdot \mathcal{P}$ . From this we conclude $\mathcal{P} \cdot \mathcal{P} = 0$ . Note $\mathcal{R} \cdot \mathcal{R} = 0$ also. Otherwise $(L-\mathcal{P}) \cdot (L-\mathcal{P}) > 0$ and the argument of ([So1], lemma (0.10.1) and (1.1.1)) would imply that $\Gamma(K_X \otimes L)$ spans $(K_X \otimes L)_{\mathcal{P}}$ . Thus $L \cdot \mathcal{R} = 1$ which implies that $\mathcal{R}$ and $C = \mathcal{R} + \mathcal{P}$ are genus $0$ . Thus $h^1(\mathcal{O}_X) = g(L) = 0$ . This contradiction establishes the theorem.

□

## § 3.- Applications :

(3.0) Throughout this section $L$ is a very ample line bundle on a normal irreducible Gorenstein projective surface $X$. For simplicity :

$g$ denotes $g(L)$ ,

$d$ denotes $L \cdot L$

$d'$ denotes $(K_X+L)\cdot(K_X+L)$ .

(3.1) Theorem :

If $d' \geq 5$ then $K_X \otimes L$ is spanned by global sections and the map $\Phi : X \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma(K_X \otimes L)$ has a $2$ dimension image.

Proof :

If $\Gamma(K_X \otimes L)$ spans $K_X \otimes L$ then $d' > 0$ implies the associated map has a $2$ dimensional image. Thus we can assume by theorem (2.0) that $(X,L) \in \mathcal{R}$ .

If $(X,L) = [\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)]$ for $e = 1, 2$ then $d' = 4$ or $1$ . If $(X,L)$ is a quadric then $d' = 2$ . If $(X,L)$ is geometrically ruled then :

$$d' = K_X \cdot K_X + 4g - 4 - d = 8 - 8h^1(\mathcal{O}_X) + 4g - 4 - d$$

$$= 4 - 4h^1(\mathcal{O}_X) - d .$$

Thus $d' \leq 4$ .

□

(3.2) Theorem :

If $d' \geq 5$ then

a) $d' \geq 2\,(g+h^\circ(K_X)-h^1(\mathcal{O}_X)-2)$ if $X$ has a desingularization of non-negative Kodaira dimension,

b) $d' \geq g + h^\circ(K_X) - h^1(\mathcal{O}_X) - 2$ in general.

Proof :

Let $\Phi$ be the map of theorem (3.1). Since $h^\circ(K_X \otimes L) = g + h^\circ(K_X) - h^1(\mathcal{O}_X) - 1$ . $d' = \text{degree}(\Phi)\cdot \deg\Phi(X) \geq \text{degree}(\Phi)\cdot(g+h^\circ(K_X)-h^1(\mathcal{O}_X)-2)$ .

If degree $(\Phi) \geq 2$ then the bound in a) is true. If degree $(\Phi) = 1$ then the bound in b) is true. If degree $(\Phi) = 1$ and $X$ has a desingularization of non-negative Kodaira dimension then $\Phi(X)$ has a desingularization of non-negative

Kodaira dimension. Now use the standard fact that if a surface $S \subseteq \mathbb{P}^n$ that doesn't belong to a hyperplane has a desingularization of non-negative Kodaira dimension then $\deg(S) \geq 2(n-1)$ .

□

(3.3) <u>Theorem</u> :

<u>If</u> $d' \geq 5$ <u>then</u>

$$d' \geq g = h^1(\mathcal{O}_X) - 2 .$$

<u>Proof</u> :

Let $C$ be a general curve $\in |K_X \otimes L|$ . Since $K_X \otimes L$ is spanned by global sections, $C$ is smooth. Since further $d' > 0$ we conclude that $C$ is connected and that $h^1(\mathcal{O}_X) \leq g(K_X \otimes L)$ . To see this use (0.1) and

$$0 \to K_X \to K_X \otimes (K_X \otimes L) \to K_C \to 0 .$$

Thus :

$$2h^1(\mathcal{O}_X) - 2 \leq 2g(K_X \otimes L) - 2 =$$

$$2d' - 2g + 2 .$$

□

The pairs $(X,L)$ with $g(L) = 0$ are well known and easy to classify, e. g. [F] . If $g(L) = 1$ then either $(X,L)$ is geometrically ruled over an elliptic curve or $K_X \approx L^{-1}$ . To see the latter choose a not everywhere zero section $\sigma$ of $K_X \otimes L$ . Since $(K_X+L) \cdot L = 0$ we conclude $\sigma^{-1}(0) \cap C = \emptyset$ for a generic $C \in |L|$ . Since $L$ is ample $\sigma^{-1}(0)$ is empty. Pairs $(X,L)$ with $K_X \approx L^{-1}$ are called Gorenstein Pezzo surfaces. Much is known about them, e. g. [Br] .

If $(X,L) \in \mathcal{R}$ define $\log(X,L)$ to be $-\infty$ . Otherwise let $\log(X,L) = \dim \Phi(X)$ where $\Phi : X \to \mathbb{P}_{\mathbb{C}}$ is the map associated to $\Phi(K_X \otimes L)$ . We have a good understanding (see [So6]) of the pairs $(X,L)$ with $\log(X,L) \leq 1$ .

(3.4) Theorem :

Assume that $h^\circ(K_X) \neq 0$. Then $g \geq h^\circ(L) + h^1(\mathcal{O}_X) - 1$. If in addition $g \leq 4$ then :

a) $g = 3$ and $|L|$ embeds $X$ as a degree 4 hypersurface of $\mathbb{P}^3$,

b) $g = 4$ and $|L|$ embeds $X$ in $\mathbb{P}^4$ as the intersection of a quadric and a cubic hypersurface.

Proof :

If $h^\circ(K_X) \neq 0$ then we have the standard inequality $h^\circ(K_X) + h^\circ(L) - 1 \leq h^\circ(K_X \otimes L)$. Combining this with $h^\circ(K_X \otimes L) = g + h^\circ(K_X) - h^1(\mathcal{O}_X)$ gives the desired inequality.

If $g = 3$ then $h^\circ(L) + h^1(\mathcal{O}_X) \leq 4$. If $h^\circ(L) = 3$ we get $X \approx \mathbb{P}^2$ which is absurd. If $h^\circ(L) = 4$ we get a).

If $g = 4$ then $h^\circ(L) + h\,(\mathcal{O}_X) \leq 5$. $h^\circ(L) = 3$ or $4$ doesn't occur. Thus we have $h^\circ(L) = 5$ and $h\,(\mathcal{O}_X) = 0$. Considering

*) $$0 \to \mathcal{O}_X \to L \to L_C \to 0$$

for a smooth $C \in |L|$ we conclude that $h^\circ(L_C) = 4$. By Castelnuovo's inequality $d \geq 6$. If $d > 6$ then $K_X \cdot L < 0$ which would imply $h^\circ(K_X) = 0$. Thus $d = 6$. Thus $K_{X,C} \approx \mathcal{O}_C$. Since $C$ is ample and $h^\circ(K_X) \neq 0$ we conclude that $K_X \approx \mathcal{O}_X$. Therefore we can conclude from the Kodaira vanishing theorem that $h^i(L^t) = 0$ for $i > 0$, $t > 0$. Considering *) tensored with $L$ and $L^2$ we conclude that $h^\circ(L^2) = 14$ and $h^\circ(L^3) = 29$.

Consider :

$$\to \mathcal{O}_{\mathbb{P}^4}(2) \otimes J_X \to \mathcal{O}_{\mathbb{P}^4}(2) \to L^2 \to 0$$

where $J_X$ is the ideal sheaf of $X$ in $\mathbb{P}^4$. Since $h^\circ(\mathcal{O}_{\mathbb{P}^4}(2)) = 15$ we conclude there is a quadric $\mathcal{Q}$ containing $X$. Consider :

$$\to \mathcal{O}_{\mathbb{P}^4}(3) \otimes J_X \to \mathcal{O}_{\mathbb{P}^4}(3) \to L^3 \to 0 .$$

Since $h^\circ(\mathcal{O}_{\mathbb{P}^4}(3)) = 35$, $h^\circ(L^3) = 29$ and since $\mathcal{Q} + H$ where $H$ is an arbitrary hyperplane account for only 5 cubics containing $X$ we conclude there is an irreducible cubic $\mathcal{C}$ that contains $X$. $\mathcal{C} \cap \mathcal{Q}$ is codimension 2 (since $\mathcal{Q}$ is irreducible) and degree 6. $X \subseteq \mathcal{C} \cap \mathcal{Q}$ is of degree 6. Thus $X = \mathcal{C} \cap \mathcal{Q}$.

□

(3.5) Theorem :

Let $g = 2$ . Then either $(X,L)$ is geometrically ruled over a genus 2 curve or $\log(X,L) = 1$ and $h^1(\mathcal{O}_X) = 0$ .

Proof :

By theorem (2.0) either $(X,L)$ is geometrically ruled over a genus 2 curve or $K_X \otimes L$ is spanned by global sections. In the latter case $h^\circ(K_X) = 0$ by (3.4). Thus $h^\circ(K_X \otimes L) = 2 - h^1(\mathcal{O}_X)$ . Since $g = 2$ , $K_X \neq L^{-1}$ . Thus $h^1(\mathcal{O}_X) = 0$ and $\dim \Phi(X) = 1$ where $\Phi$ is the map associated to $\Gamma(K_X \otimes L)$ .

□

If $\Phi$ denotes the map associated to $\Gamma(K_X \otimes L)$ and $\dim \Phi(X) = 2$ there it was shown in [So6] that $\Phi$ factors $s \circ r$ where $r : X \to X'$ maps $X$ into a normal Gorenstein surface $X'$ with the positive dimensional fibers are lines and $s : X' \to \mathbb{P}_{\mathbb{C}}$ is finite to one. Further there is an ample line bundle $L'$ on $X'$ with

$$K_X \otimes L = \pi^*(K_{X'} \otimes L') .$$

(3.6) Theorem :

If $g = 3$ then either $(X,L)$ is geometrically ruled over a genus 3 curve, or $(X,L)$ is as in (3.4) or $\log(X,L) = 1$ or $(X',L') = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ , or $X'$ is a double cover of $\mathbb{P}^2$ branched on a degree 4 curve with $L' = s^*\mathcal{O}_{\mathbb{P}^2}(2)$ . In the last case $d = 8$ or 7 depending on whether $r$ is a biholomorphism or not.

Proof :

We can assume that $K_X \otimes L$ is spanned by global sections with $\dim \Phi(X) = 2$ and $h^\circ(L) \geq 5$ . Let $\Phi$ factor $r \circ s$ as in the paragraph before the theorem. Since $h^\circ(K_X \otimes L) = 3 - h^1(\mathcal{O}_X)$ we concluded that $h^1(\mathcal{O}_X) = 0$ and $\Phi(X) = \mathbb{P}^2$ .

Since $h^\circ(L) \geq 5$ , it follows that $d \geq 6$ by Castelnuovo's inequality. From the Hodge index theorem we get :

$$d \cdot d' \leq 4(g-1)^2 = 16 .$$

Since $d \geq 6$ , $d' \leq 2$ .

If $d' = 1$ then $X' \approx \mathbb{P}^2$ and $L' \approx \mathcal{O}_{\mathbb{P}^2}(4)$ . If $d' = 2$ then $s$ is a double cover and a direct calculation shows that the branch locus is degree 4 and $L' = s^*\mathcal{O}_{\mathbb{P}^2}(2)$ .

□

## References

[A+K] A. ALTMAN and S. KLEIMAN, Introduction to Grothendieck duality theory .
Springer Lecture Notes 146 (1970).

[B+P] M. BELTRAMETTI and M. PALLESCHI, Threefolds of low sectional genus.
Preprint.

[Br] L. BRENTON, On singular complex surfaces with negative canonical bundle, with applications to singular compactifications of $\mathbb{C}^2$ and 3 dimensional rational singularities.
Math. Ann. 248 (1980), 117-124.

[(Fa+So)1 M.L. FANIA and A.J. SOMMESE, On varieties with $\mathbb{P}^k$ bundles as hyperplane sections.
Preprint.

[(Fa+So)2] M.L. FANIA and A.J. SOMMESE, On the minimality of hyperplane sections of Gorenstein threefolds.
To appear in Proceedings of the 1984 Notre Dame Several Complex Variable Conference in honor of Wilhelm Stoll, Vieweg.

[F] T. FUJITA, On the structure of polarized varieties of $\Delta$ genus zero.
J. Fac. Sci. Univ. of Tokyo, 22 (1975), 103-115.

[I] P. IONESCU, Embedded projective varieties of small invariants.
Springer Lecture Notes 1056 (1984), 142-186.

[K+O] S. KOBAYASHI and T. OCHIAI, Characterization of the complex projective space and hyperquadrics.
J. Math. Kyoto Univ. 13 (1972), 31-47.

[Li+So] J. LIPMAN and A.J. SOMMESE, On blowing down projective spaces in singular varieties.
To appear Jour. für die reine und angewandte Mathematik.

[Ma] H. MATSUMURA, Geometric structure of the cohomology rings in abstract algebraic geometry.
Mem. Coll. Sci. Univ. Kyoto (A) 32 (1959), 33-84.

[Sa] F. SAKAI, Ample Cartier divisors on normal surfaces.
Preprint.

[So1] A.J. SOMMESE, Hyperplane sections of projective surfaces :I-the adjunction mapping.
Duke Math. J. 46 (1979), 377-401.

[So2] A.J. SOMMESE, On the minimality of hyperplane sections of projective threefolds.
Journal für die reine une angewandte Mathematik, 329 (1981), 16-41.

[So3] A.J. SOMMESE, Ample divisors on 3-folds, Algebraic Threefolds. Springer Lecture Notes in Math 947 (1982), 229-240.

[So4] A.J. SOMMESE, On the birational theory of hyperplane sections of projective threefolds. Unpublished 1981 manuscript.

[So5] A.J. SOMMESE, Configurations of -2 rational curves on hyperplane sections of projective threefolds. Classification of Algebraic and Analytic Manifolds. Ed. by K. Ueno, Progress in Mathematics 39 (1983), Birkhauser Boston.

[So6] A.J. SOMMESE, Ample divisors on normal Gorenstein surfaces. To appear Abh. Math. Hamburg.

[So7] A.J. SOMMESE, Hyperplane sections. Springer Lecture Notes 862 (1981), 232-271.

Department of Mathematics
University of Notre Dame
Notre Dame, Indiana, 46556
U. S. A.