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Mixed Hodge structures and singularities : a survey

Introduction. In the last ten years a link has been laid between two rather remote parts of geometry: the theory of mixed Hodge structures and the study of isolated singularities. At the base of this development stands P. Deligne, who in a talk at the monodromy colloquium in Metz 1974 gave a first indication of possible developments.

The first publications in this direction deal with mixed Hodge structures on the vanishing cohomology of isolated hypersurface singularities [21,22]. The description of the Hodge filtration, however, was complicated and depends on resolution of singularities. A. Varchenko [30] gave a direct description of this Hodge filtration connecting the mixed Hodge structure with asymptotic developments of integrals of holomorphic forms over vanishing cycles. F. Pham [16] and M. Saito [17,18] gave a description in terms of D-modules, after previous attempts by J. Scherk and the author (see [19]).

These two descriptions of the Hodge filtration on the vanishing cohomology have led to a very useful invariant for hypersurfaces: the spectrum, which is semicontinuous under deformation in a certain sense. We treat this development in §§1-6.

The local cohomology groups of isolated singularities also carry a mixed Hodge structure. Their study was started in [10] and [24], see §8. Together with Du Bois' result about the filtered De Rham complex [1] and vanishing results about its cohomology sheaves (§7) this leads to very general statements about extendability of differential forms near singularities [28]. In the same spirit we describe an application to the relation between Milnor number and Tjurina number of complete intersections in §9.

We have not treated all known results about mixed Hodge structures and singularities. We omitted the D-module theory and its applications. Also we focussed on the local rather than the global aspects of the theory. However, we hope to give a good impression of the fascinating and unexpected patterns which came out.

1. What is a mixed Hodge structure?

If X is a compact Kähler manifold, its complex cohomology groups admit a Hodge-decomposition

$$H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X)$$

with $\overline{H^{p,q}(X)} = H^{q,p}(X)$. This can be understood in the following way. The Kähler metric on X determines a Riemannian metric and hence a Laplace operator

$$\Delta = d\delta + \delta d$$

where δ is the adjoint of d. A differential form ω on X is called harmonic if $\Delta\omega = 0$. It was shown by Hodge that each cohomology class of X contains precisely one harmonic form.

On the other hand, on a complex manifold each complex-valued differential form of degree m admits a canonical decomposition

$$\omega = \Sigma_{p+q=m}\, \omega^{p,q}$$

where each $\omega^{p,q}$ is of type (p,q), i.e. in local holomorphic coordinates $(z_1,\ldots,z_n)$ on X we have

$$\omega^{p,q} = \Sigma\, a_{i_1\ldots i_p j_1\ldots j_q}\, dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge \overline{dz}_{j_1}\wedge\ldots\wedge\overline{dz}_{j_q} .$$

The main point of Hodge theory for Kähler manifolds is that ω is harmonic if and only if all $\omega^{p,q}$ are harmonic, and also ω is harmonic if and only if $\bar{\omega}$ is. Thus it makes sense to speak about cohomology classes of type (p,q) and the Hodge decomposition is a corollary of these facts if we let $H^{p,q}(X)$ be the space of cohomology classes of type (p,q).

A crucial step towards mixed Hodge theory was the idea to replace the Hodge decomposition by the Hodge filtration $F^{\bullet}$: one poses

$$F^pH^m(X,\mathbb{C}) = H^{p,m-p} \oplus H^{p+1,m-p-1} \oplus \ldots \oplus H^{m,0}$$

i.e. F^p consists of classes represented by forms with at least p dz's. It was introduced by Griffiths, who observed that in a family the spaces $H^{p,q}$ do not form a holomorphic subbundle of the fibre cohomology, whereas the F^p do give holomorphic subbundles.

It is the Hodge filtration which generalizes to part of the structure of the cohomology of singular (or non-compact) complex algebraic varieties. Observe that the Hodge filtration again determines the Hodge decomposition: one has

$$H^{p,q}(X) = F^pH^{p+q}(X,\mathbb{C}) \cap \overline{F^qH^{p+q}(X,\mathbb{C})} .$$

These data give rise to the concept of a Hodge structure. A Hodge structure of weight m is given by a finitely generated abelian group $H_{\mathbb{Z}}$, together with a finite decreasing filtration by $\mathbb{C}$-linear subspaces

$$F = (\dots \supset F^p \supset F^{p+1} \supset \dots)$$

on $H_{\mathbb{Z}} \otimes \mathbb{C}$, in such a way that $H_{\mathbb{Z}} \otimes \mathbb{C} = F^p \oplus \overline{F^{m-p+1}}$ for each p. Putting $H^{p,m-p} = F^p \cap \overline{F^{m-p}}$, one obtains a Hodge decomposition as before. The filtration F is called the Hodge filtration.

To see what happens in the singular case, suppose that X is a compact complex algebraic variety with only one singular point. Suppose that we can resolve the singularity by blowing up the point once and that the exceptional divisor becomes smooth. Let $\pi: Y \to X$ denote this blowing-up and let D be its exceptional divisor. Topologically one may consider X as Y/D so we have the long exact cohomology sequence

$$\dots \to H^q(Y) \xrightarrow{\alpha^q} H^q(D) \to H^{q+1}(X) \to H^{q+1}(Y) \xrightarrow{\alpha^{q+1}} H^{q+1}(D) \to \dots$$

Here α is just the restriction mapping. The idea is to consider $H^{q+1}(X)$ as the extension of $\mathrm{Ker}(\alpha^{q+1})$ by $\mathrm{Coker}(\alpha^q)$. As α^q preserves the Hodge decomposition for all q, it is easily seen that $\mathrm{Coker}(\alpha^q)$ is a Hodge structure of weight q, whereas $\mathrm{Ker}(\alpha^{q+1})$ is a Hodge structure of weight q+1. Hence $H^{q+1}(X)$ carries a <u>weight filtration</u>:

$$W_{q-1} = (0),\ W_q = \mathrm{Coker}(\alpha^q),\ W_{q+1} = H^{q+1}(X)$$

such that each W_i/W_{i-1} carries a natural Hodge structure of weight i. Deligne observed that the Hodge filtrations on these W_i/W_{i-1} are in fact induced by one filtration F on $H^{q+1}(X,\mathbb{C})$, as we will see below. Thus one is led to the following definition, due to Deligne [3]:

A mixed Hodge structure is a finitely generated abelian group $H_{\mathbb{Z}}$, together with a weight filtration W. (finite and increasing) on $H_{\mathbb{Q}}$, defined over $\mathbb{Q}$, and a Hodge filtration F (finite and decreasing) on $H_{\mathbb{C}}$, such that W_m/W_{m-1} inherits a Hodge structure of weight m for each m. Here the induced Hodge filtration on $W_m/W_{m-1} = Gr_m^W$ is given as

$$F^p Gr_m^W H_{\mathbb{C}} = F^p \cap W_m + W_{m-1}/W_{m-1} .$$

In [3,4] Deligne developed a theory which provides the cohomology of each complex algebraic variety with a canonical mixed Hodge structure which is functorial in the following sense: for each morphism $\phi: X \to Y$ the map $\phi^*: H^m(Y) \to H^m(X)$ maps $W_i H^m(Y)$ to $W_i H^m(X)$ and $F^p H^m(Y)$ to $F^p H^m(X)$, i.e. ϕ^* is a <u>morphism of mixed Hodge structures</u>.

It is known that every morphism of mixed Hodge structures is in fact strictly compatible with the Hodge and weight filtrations, i.e. for ϕ^* as above we will have

$$\begin{cases} \operatorname{Im} \phi^* \cap F^p H^m(X) = \phi^* F^p H^m(Y) , \\ \operatorname{Im} \phi^* \cap W_i H^m(X) = \phi^* W_i H^m(Y) . \end{cases}$$

We will see below that exactly this property makes the theory of mixed Hodge structures very useful: the category of mixed Hodge structures is thus very similar to a category of graded objects.

The main questions concerning mixed Hodge structures and singularities are the following:

1. What is the behaviour of the (mixed) Hodge structure on the cohomology of fibers in a family,when the fibers acquire (more) singularities?
2. How are the singularities of X reflected in the mixed Hodge structure?

Of course these questions are strongly related, as often singularities are studied by deforming them.

2. Mixed Hodge structures for singular varieties.

The construction of a mixed Hodge structure on the cohomology of a complete singular variety is based on the same idea as was used in the preceding section: one expresses this cohomology in terms of cohomology groups of nonsingular varieties which have pure Hodge structures. The general construction uses semisimplicial resolutions.

Let $\Delta^n = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \Sigma_{i=0}^n x_i = 1\}$ be the standard n-simplex and let

$$\varepsilon_{i,n}: \Delta^{n-1} \to \Delta^n \quad (i=0,\ldots,n)$$

be the map given by

$$(x_0,\ldots,x_{n-1}) \to (x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{n-1}).$$

One has the obvious relations

$$\varepsilon_{i,n} \circ \varepsilon_{j,n-1} = \varepsilon_{j,n} \circ \varepsilon_{i-1,n-1} \quad (i > j).$$

A semisimplicial variety is a collection of varieties

$$(X_n, X_{n-1},\ldots,X_1,X_0)$$

together with a collection of morphisms

$$\varepsilon^{i,q}: X_q \to X_{q-1}, \quad q = 1,\ldots,n, \quad i = 0,\ldots,q,$$

satisfying the dual relations

$$\varepsilon^{j,q-1} \circ \varepsilon^{i,q} = \varepsilon^{i-1,q-1} \circ \varepsilon^{j,q} \quad (i > j).$$

In more sophisticated terms, one may define a category consisting of all Δ^i, $0 \leq i \leq n$ with morphisms compositions of the $\varepsilon_{i,q}$, and define a semisimplicial variety as a contravariant functor from this category to the category of (complex) varieties.

Suppose that $X_.$ is a semisimplicial variety such that all X_q are nonsingular and complete and suppose $\varepsilon: X_0 \to X$ is a morphism. We consider on X the following sheaf complex. First observe that by composition of any chain of $\varepsilon^{i,j}$'s with ε we obtain unambiguous maps

$$\varepsilon^j: X_j \to X.$$

Let $A^{\cdot}_{X_j}$ denote the sheaf complex of C^∞ complex valued differential forms on X_j. Let

$$A_X^q = \bigoplus_{r\geq 0} \varepsilon_*^r A_{X_r}^{q-r}$$

and define

$$\underline{d}: A_X^q \to A_X^{q+1}$$

by

$$\underline{d}(\omega_0,\ldots,\omega_q) = (d\omega_0, \varepsilon^*\omega_0+d\omega_1,\ldots,\varepsilon^*\omega_{q-1}+d\omega_q, \varepsilon^*\omega_q)$$

where

$$\varepsilon^*(\omega_r) = \sum_{j=0}^{r+1} (-1)^{r+j}(\varepsilon^{r+1,j})^*(\omega_r)$$

so ε^* is an alternating sum of pull-back mappings. In case the sequence

$$0 \to \mathbb{C}_X \to A_X^0 \to A_X^1 \quad \ldots$$

is exact, we say that $X_\bullet \to X$ is a <u>semisimplicial resolution</u> of X.

<u>Example.</u> If X has only one singular point x, and π: $Y \to X$ is the blowing up, and moreover Y and $D = \pi^{-1}(x)$ are smooth as in section 1, then we can take:

$$X_0 = Y \cup \{x\}$$

$$X_1 = D$$

$$\varepsilon^{0,1}: D \to \{x\}$$

$$\varepsilon^{1,1}: D \to Y \text{ the inclusion}$$

$$\varepsilon: Y \xrightarrow{\pi} X \text{ and } \{x\} \to X .$$

Outside x $A_X^\bullet$ coincides with the usual De Rham complex so the Poincaré lemma guarantees that $\mathbb{C}_X \to A_X^\bullet$ is exact. If U is a small neighborhood of x in X, then exactness on U follows from the fact that $H^i(\pi^{-1}(U)) \xrightarrow{\sim} H^i(D)$ (by a homotopy argument) and some diagram chasing.

<u>Example</u>. Let $D = D_1 \cup \ldots \cup D_k$ be a variety with normal crossings: locally analytically D is isomorphic to a union of coordinate hyperplanes in $\mathbb{C}^{n+1}$ where n= dim D. Let

$$X_j = \coprod_{i_1<\dots<i_j} D_{i_1} \cap \dots \cap D_{i_j} \quad \text{(disjoint union)}$$

and let $\varepsilon^{q,j}: X_j \to X_{j-1}$ be given by inclusions

$$D_{i_1} \cap \dots \cap D_{i_j} \to D_{i_1} \cap \dots \cap D_{i_{q-1}} \cap D_{i_{q+1}} \cap \dots \cap D_{i_j} .$$

Then $X_\bullet$ is a semisimplicial resolution of D.

If $X_\bullet \to X$ is a semisimplicial resolution of X, we have isomorphisms

$$H^m(X,\mathbb{C}) \to H^m(\Gamma(X,A^\bullet_X))$$

because $A^\bullet_X$ is a flabby resolution of X. We will use this to define the Hodge and weight filtrations on $H^m(X,\mathbb{C})$. We define

$$\begin{cases} F^p A^\bullet_X = \bigoplus_{r\geq p} \bigoplus_{p'\geq p} \varepsilon^r_* A^{p'\bullet}_{X_r} \\ W_k A^\bullet_X = \bigoplus_{r\geq -k} \varepsilon^r_* A^\bullet_{X_r} . \end{cases}$$

These are subcomplexes of $A^\bullet_X$, satisfying

$$A^\bullet_X = F^0 \supset F^1 \supset \dots$$
$$A^\bullet_X = W_0 \supset W_{-1} \supset \dots$$

We let

$$\begin{cases} F^p H^m(X,\mathbb{C}) = \text{Image of } H^m(\Gamma(X,F^p A^\bullet_X)) \text{ in } H^m(X,\mathbb{C}); \\ W_k H^m(X,\mathbb{C}) = \text{Image of } H^m(\Gamma(X,W_{k-m} A^\bullet_X)) \text{ in } H^m(X,\mathbb{C}). \end{cases}$$

The main facts which make this idea work are:

1) For every complete complex algebraic variety X there exists a semisimplicial resolution (even one such that $\dim X_r \leq \dim X-r$);
2) The resulting filtrations W and F on $H^m(X,\mathbb{C})$ do not depend on the choice of the semisimplicial resolution and determine a mixed

Hodge structure.

Proofs of these facts can be found in [13]. Deligne's original approach is by simplicial schemes which give necessarily an infinite tower of spaces, and is more convenient for proving functorial properties.

3. Variation of mixed Hodge structure.

Suppose that we have a projective morphism $f : Y \to U$ where U is a complex manifold and Y is a reduced analytic space. Assume that f is surjective with connected fibres. Let $Y_\cdot \to Y$ be a semisimplicial resolution of Y, and for $u \in U$ let $Y_i(u)$ be the fibre of $f_i := f\circ\varepsilon^i : Y_i \to U$ over u. Then we have the following

LEMMA. *There exists a Zariski-dense open subset* U' *of* U *such that* $Y_\cdot(u) \to Y(u)$ *is a semisimplicial resolution of* Y(u) *for all* $u \in U'$.

Proof. This is a simple consequence of Sard's theorem; it suffices to take for U' the intersection of the sets of regular values for all f_i. □

Let us take U' as in the lemma. Then $R^m f_* \mathbb{C}_{Y|U'}$ will be a locally constant sheaf on U' and its sheaf of holomorphic sections

$$\mathcal{H}^m_{Y'/U'} = R^m f_* \mathbb{C}_{Y|U'} \otimes_{\mathbb{C}} \mathcal{O}_{U'}$$

admits a filtration by holomorphic subbundles $\mathcal{F}^p$, where

$$\mathcal{F}^p(u) = F^p H^m(Y(u),\mathbb{C}) .$$

Moreover we have a Griffiths transversality property:

$$\nabla\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_{U'}$$

where $\nabla : \mathcal{H}^m_{Y'/U'} \to \mathcal{H}^m_{Y'/U} \otimes \Omega^1_{U'}$ is the connection given by

$$\nabla(v \otimes g) = v \otimes dg$$

for v,g local sections of $R^m f_* \mathbb{C}_{Y/U'}$ and $\mathcal{O}_{U'}$ respectively. Moreover the weight filtration on each $H^m(Y(u),\mathbb{C})$ induces a filtration $\mathcal{W}_.$ of $\mathcal{H}^m_{Y'/U'}$ which is horizontal, i.e.

$$\nabla(\mathcal{W}_r) \subset \mathcal{W}_r \otimes \Omega^1_{U'}$$

because it comes from a filtration on the underlying local system. In other words:

$$(R^m f_* \mathbb{C}_{Y/U'}, \mathcal{F}^\cdot, \mathcal{W}_.)$$

is a variation of mixed Hodge structure on U' (see [9,27,29]).

This concept generalizes the variations of Hodge structure, introduced by Ph. Griffiths, in an obvious way. We will need a study of its asymptotic behaviour near the boundary of U'. To do this, we specialize to the case that dim U=1.

So let S be a disc in $\mathbb{C}$, centered at 0 and let $S^* = S \setminus \{0\}$. Let $\mathbb{V}$ be a local system on S^* and assume for simplicity that the monodromy transformation T of $\mathbb{V}$ is unipotent. (In the geometric case, the monodromy theorem assures that we can always achieve this situation by a finite base change).

Let V be the space of multivalued horizontal sections of $\mathbb{V}$. If S_∞ is the upper half plane in $\mathbb{C}$ and $e : S_\infty \to S^*$, $e(u) = \exp 2\pi i u$, then

$$V = \Gamma(S_\infty, e^*\mathbb{V}) .$$

Let $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_{S^*}$. We define an extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ to a locally free $\mathcal{O}_S$-module as follows. Define a linear map

$$\Phi : V \to \Gamma(S_\infty, e^*\mathcal{V})$$

by

$$\Phi(v)(u) = \exp(u \log T) v(u) .$$

As $\Phi(v)(u+1) = \Phi(v)(u)$ for $v \in V$, Φ maps V isomorphically to a subspace of $\Gamma(S^*, \mathcal{V})$, which generates $\mathcal{V}$ at each point as an $\mathcal{O}_{S^*}$-module. We let

$$\tilde{\mathcal{V}} = \Phi(V) \otimes_{\mathbb{C}} \mathcal{O}_S .$$

Then ∇ extends to a connection with logarithmic pole on $\tilde{\mathcal{V}}$. We can apply this construction to $\mathcal{H}^m_{Y/S^*}$ as well as to its subsheaves $\mathcal{W}_i$, if $f : Y \to S$ is a proper morphism which is locally trivial over S^*.

In the case of a (pure) variation of Hodge structure (with polarization) W. Schmid has proved that the bundles F^p also extend to subbundles $\tilde{F}^p$ of $\tilde{\mathcal{V}}$ and that the pure Hodge structures on each $\tilde{\mathcal{V}}(s)$, $s \neq 0$, tend to a mixed Hodge structure on $\tilde{\mathcal{V}}(0)$ with integral lattice $\Phi(V_{\mathbb{Z}})$ and Hodge filtration $\tilde{F}^{\cdot}(0)$. The weight filtration is determined by the monodromy T on $\tilde{\mathcal{V}}(0)$ in the following way. Let $N = \log T = -2\pi i\ \mathrm{Res}_0(\nabla) \in \mathrm{End}(\tilde{\mathcal{V}}(0))$. Then N is nilpotent. It is a simple exercise in linear algebra to show that there exists a filtration L on $\tilde{\mathcal{V}}(0)$ which satisfies

- L is a finite increasing filtration;
- $N : L_i \to L_{i-2}$ for all i
- $N^k : L_k/L_{k-1} \xrightarrow{\sim} L_{-k}/L_{-k-1}$ for $k \geq 0$.

Moreover L is uniquely determined by these conditions. We let $M_i = L_{i-m}$ where m is the weight of $\mathbb{V}$. Then

$$(\Phi(V_{\mathbb{Z}}) \subset \tilde{\mathcal{V}}(0), M_{\cdot}, \tilde{F}^{\cdot}(0))$$

is a mixed Hodge structure which is called the limit mixed Hodge structure of V. See [20] for much more precise statements.

In [21], the author proved these results in the geometric case by an explicit construction of the limit mixed Hodge structure. If $f : Y \to S$ is smooth and proper over S^* and $D = f^{-1}(0)$ is a union of smooth divisors which is reduced and has normal crossings, then

$$\tilde{\mathcal{V}} = \mathbb{R}^m f_*(\Omega^{\cdot}_{Y/S}(\log D))$$

and the mixed Hodge structure on

$$\tilde{\mathcal{V}}(0) = \mathbb{H}^m(D, \Omega^{\cdot}_{Y/S}(\log D) \otimes \mathcal{O}_D)$$

is obtained by an explicit resolution

$$\Omega^{\cdot}_{Y/S}(\log D) \otimes \mathcal{O}_D \to A^{\cdot\cdot}$$

and by lifting of F,M and N to filtrations and an endomorphism of $A^{\cdot\cdot}$.

In [5] P. Deligne sketched how the situation should look like in tne mixed case. There should exist a limit Hodge filtration as before which on the quotients $Gr_m^{\widetilde{W}}(0)$ induces the limit mixed Hodge structure of W. Schmid. Moreover the weight filtrations of these graded quotients should be induced by a filtration M on $\widetilde{V}(0)$ satisfying $N(M_i) \subset M_{i-2}$ for all i.

Deligne's conjecture, which arose from analogy with the ℓ-adic case, has been verified by several people since: F. El Zein [7,8,9], S. Zucker and the author [27,36], F. Guillén, V. Navarro Aznar and F. Puerta [13] and Ph. du Bois [2]. All of their proofs use some generalization of the complex $A^{\cdot\cdot}$ with suitable filtrations and endomorphisms.

It should be remarked that the existence of the filtration M already gives restrictions for the filtered local system $(\mathbb{V},W)$ without any reference to a Hodge filtration. See [27,§§ 1-3] for a detailed discussion.

4. Vanishing cycles.

If $f : Y \to S$ is a proper morphism which is locally trivial over the punctured disc S^*, the construction of §3 provides one with a complex $K^\cdot(Y_\infty)$ on Y_0, equipped with filtrations F,W and M such that

$$\mathbb{H}^m(Y_0,K^\cdot(Y_\infty)) \cong H^m(Y_\infty,\mathbb{C})$$

where $Y_\infty = Y \times_S S_\infty$ which is homotopy equivalent to the general fibre of f. Moreover F and M induce the limit mixed Hodge structure on $H^m(Y_\infty,\mathbb{C})$.

On the other hand, the construction of §2 gives a complex $K^\cdot(Y_0)$ on Y_0, equipped with filtrations F and W, such that

$$\mathbb{H}^m(Y_0,K^\cdot(Y_0)) \cong H^m(Y_0,\mathbb{C})$$

and F and W induce the usual mixed Hodge structure on $H^m(Y_0,\mathbb{C})$. The relation between these two mixed Hodge structures is the following. There exists a morphism of complexes

$$sp^*: K^\cdot(Y_0) \to K^\cdot(Y_\infty)$$

which maps F^p to F^p and W_j to M_j. We let $R\Phi$ denote the mixed cone (in the sense of El Zein [6]) of sp^*. It carries also filtrations F and W which define a mixed Hodge structure on $\mathbb{H}^m(Y_0, R\Phi)$. One calls $R\Phi$ the <u>sheaf of vanishing cycles</u> of f. It measures the difference of special and general fibre locally. If Y is smooth and f has an isolated critical point, $\mathbb{H}^m(Y_0, R\Phi)$ is isomorphic to the m-th reduced cohomology of the Milnor fibre of f. By construction one obtains a long exact sequence of mixed Hodge structures

$$\ldots \to H^m(Y_0) \to H^m(Y_\infty) \to \mathbb{H}^m(R\Phi) \to H^{m+1}(Y_0) \to \ldots$$

5. <u>Application to deformations of complete intersections</u>.

Let (X_0, x) be an isolated complete intersection singularity. Then there exists a miniversal deformation

$$f : X \to U$$

of (X_0, x) where U is smooth (as well as X). It is not hard to show that there exists a projective map

$$F : Y \to U$$

where Y is a connected complex manifold containing X as an open subset, such that $f = F_{|X}$ and F has no critical points outside X ([26], Lemma 2.5). We are going to apply the constructions of §4 by choosing arcs $h : S \to U$ and considering the induced deformation $h^*(X)$ over S.

First observe that there exists a complex analytic stratification of U such that the map F is locally trivial over each stratum of U. The arcs we consider will map the punctured disc S^* to one single stratum, so the induced family over S will be locally trivial over S^*.

In general the mixed Hodge structure on the vanishing cycles of $h^*(X)$ will not only depend on the type of the fibres, i.e. the two strata in the image of h, but also on the way $h(S^*)$ approaches the lower stratum.

<u>Example</u>. Consider two smoothings of the cusp

$$X_1 = \mathbb{C}^2 \qquad f_1(x,y) = x^2-y^3$$

and

$$X_2 = V(x^2-y^3-t^7-txy) \subset \mathbb{C}^3, f_2(x,y,t) = t.$$

The mixed Hodge structure on $\mathbb{H}^1(R\Phi)$ is purely of weight one in the case of X_1, but in the case of X_2 it will be an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}$ (here $\mathbb{Q}(-i)$ is the 1-dimensional Hodge structure which is purely of type (i,i)). This can be concluded from the fact that X_2 has a singular point of type $T_{2,3,7}$ and this forces the monodromy of f_2 to be of infinite order.

To obtain numbers which are independent of the choice of the arc connecting two given strata we must forget about the weight filtration on $\mathbb{H}^*(R\Phi)$ and look at the Hodge filtration alone.

For any $u \in U$, consider an arc $h : (S,0) \to (U,u)$ such that $h(S^*)$ consists of regular values of F. The fibre $F^{-1}(u) = Y_u$ will have only isolated complete intersection singularities and the general fibre of $h^*(Y)$ will be smooth. We define for $p \in \mathbb{N}$

$$s_p(u) = \dim Gr_F^p \mathbb{H}^n(R\Phi_h).$$

By the exact sequence at the end of §4 and the fact that Milnor fibres of isolated complete intersections have only cohomology in dimensions 0 and n we obtain that

$$s_p(u) = \Sigma_{i=-n}^{n}(-1)^i\{\dim Gr_F^p H^{n+i}(Y_v) - \dim Gr_F^p H^{n+i}(Y_u)\}$$

if v is any regular value of F. Hence the functions $s_p : U \to \mathbb{Z}$ do not depend on the choice of h so they are well-defined and constant on each stratum of U.

<u>Lemma</u>. The functions s_p are upper semicontinuous on U.

<u>Proof</u>. Suppose $u_1, u_2 \in U$ and u_2 is in the closure of the stratum of u_1. Choose an arc $h: (S,0) \to (U,u_2)$ such that $u_1 \in h(S^*)$. Then one computes that

$$s_p(u_1) - s_p(u_2) = \dim Gr_F^p \mathbb{H}^n(R\Phi_h)$$

because $\mathbb{H}^r(R\Phi_h) = 0$ for $r \neq n$ by a result of G.-M. Greuel [12, Lemma 3.2].

□

6. The spectrum of isolated hypersurface singularities.

In the isolated hypersurface singularity case the preceding lemma can be refined as follows. First, the exact sequence of the end of §4 can be simplified by choosing a suitable globalization. In fact, J. Scherk has observed that, if deg(F) is suitably large, then the map

$$H^n(Y_\infty) \to \mathbb{H}^n(R\Phi)$$

is surjective (for the "standard" smoothing of the hypersurface). As $\mathbb{H}^m(R\Phi) = 0$ for $m \neq n$, we obtain the short exact sequence

$$(*) \qquad 0 \to H^n(Y_0) \xrightarrow{sp^*} H^n(Y_\infty) \to \mathbb{H}^n(R\Phi) \to 0 .$$

The monodromy T acts on this sequence. There are two fundamental properties:

(1) as Y is nonsingular for the standard smoothing, the invariant cycle theorem can be applied, which states that

$$\operatorname{Image}(sp^*) = \operatorname{Ker}(T-I; H^n(Y_\infty)) ;$$

(2) the semisimple part T_s of T preserves the limit mixed Hodge structure.

It follows that all data of the sequence (*) are determined by the pair $(H^n(Y_\infty), T)$. Let $N = \log T_u$ where T_u is the unipotent part of T. Moreover let

$$H^n(Y_\infty)_\alpha = \operatorname{Ker}(T_s - e^{-2\pi i\alpha}) \text{ for } -1 < \alpha \leq 0 .$$

Then

$$H^n(Y_\infty) = \bigoplus_{-1<\alpha\leq 0} H^n(Y_\infty)_\alpha .$$

We have a similar decomposition for $\mathbb{H}^n(R\Phi)$ and it follows from the above that

$$\mathbb{H}^n(R\Phi)_\alpha \cong H^n(Y_\infty)_\alpha \ , \ -1 < \alpha < 0 \ ;$$
$$\mathbb{H}^n(R\Phi)_0 \cong H^n(Y_\infty)_0/\mathrm{Ker}(N).$$

For $\alpha \neq 0$ we have

$$N^k : Gr^M_{n+k}\, \mathbb{H}^n(R\Phi)_\alpha \xrightarrow{\sim} Gr^M_{n-k}\, \mathbb{H}^m(R\Phi)_\alpha$$

but for $\alpha = 0$:

$$N^k : Gr^M_{n+1+k}\, \mathbb{H}^n(R\Phi)_0 \xrightarrow{\sim} Gr^M_{n+1-k}\, \mathbb{H}^n(R\Phi)_0 .$$

To define the spectrum of the isolated hypersurface singularity $(X_0,0)$ we consider its standard smoothing and take for the spectrum the unordered μ-tuple $\alpha_1,\ldots,\alpha_\mu$ of rational numbers such that the frequency of a number α in the spectrum is equal to the multiplicity of $e^{-2\pi i\alpha}$ as an eigenvalue of T_s acting on $Gr^p_F\, \mathbb{H}^n(R\Phi)$, where $p = [n-\alpha]$.

Some properties of the spectrum are immediate.

(1) As $Gr^p_F\, \mathbb{H}^n(R\Phi) = 0$ for $p < 0$ or $p > n$, each spectrum number α satisfies $-1 < \alpha < n$;

(2) The two-fold symmetry on $\mathbb{H}^n(R\Phi)$ by N^k as above and by complex conjugation can be used to show that the frequency of α in the spectrum equals the frequency of $n-1-\alpha$.

A third important property is the following.

(3) Let $f \in \mathbb{C}\{z_0,\ldots,z_n\}$, $g \in \mathbb{C}\{y_0,\ldots,y_m\}$ have isolated critical points at 0 with spectra $\{\alpha_1,\ldots,\alpha_\mu\}$ and $\{\beta_1,\ldots,\beta_\nu\}$ respectively. Then the function germ $f+g \in \mathbb{C}\{z_0,\ldots,z_n,y_0,\ldots,y_m\}$ also has an isolated critical point with spectrum numbers $\alpha_i + \beta_j + 1$ (with their obvious multiplicities).

The method of proof of this property, however, is completely different. It uses Varchenko's description of the Hodge filtration, see [19] or [31].

As a consequence of (3), the spectrum of $f(z_0,\dots,z_n)+z_{n+1}^q$ ($q \in \mathbb{N}$) consists of a sum of copies of the spectrum of f shifted by $1/q$, $2/q,\dots,(q-1)/q$. Each of these copies belongs to some eigenspaces for the action of $\mathbb{Z}/q\mathbb{Z}$ by $z_{n+1} \mapsto \zeta z_{n+1}$, $\zeta^q = 1$. Finally one has the important

(4) semicontinuity of the singularity spectrum.

To formulate what this means we choose a semi-universal deformation $f: X \to U$ of our hypersurface (X_0,x). For $u \in U$ we let Σ_u be the union (with multiplicities) of the spectra of all critical points of f in the fibre $X(u) = f^{-1}(u)$. For $A \subset \mathbb{R}$ and $u \in U$ we let

$s(A,u)$ = sum of the frequencies of the numbers of Σ_u which lie in A .

In particular, for $A = (a,a+1]$ with $a \in \mathbb{Z}$, we recover the function $s_p: U \to \mathbb{Z}$, with $p = n-a-1$.

The subset A of $\mathbb{R}$ is called a semicontinuity domain for deformations of isolated hypersurface singularities if these functions $s(A,-)$ are upper semicontinuous on U. It follows from the lemma in §5 that each interval $(a,a+1]$ with $a \in \mathbb{Z}$ is such a semicontinuity domain.

<u>Theorem.</u> For each $a \in \mathbb{R}$, the interval $(a,a+1]$ is a semicontinuity domain for deformations of isolated hypersurface singularities.

This theorem has been conjectured and partially proved by A. Varchenko [33], and a general proof by the author [26]. In fact it is an easy exercise now: first observe that it suffices to prove the theorem for $a \in \mathbb{Q}$. Then use property (3) above to shift the spectra in a given deformation over a rational distance.

An important consequence of the semicontinuity of the spectrum is the semicontinuity of the lowest spectrum number, which is one less than the complex singularity index of Arnol'd-Malgrange. Again this was proved by Varchenko in the special case where the smallest spectrum number is negative in a deformation [34].

Another consequence: in a μ-constant deformation of a hypersurface singularity the spectrum numbers remain constant. This was proved before by Varchenko [32].

A very nice application of the semi-continuity gives an estimate of the maximal number of singular points that may occur on a projective hypersurface of given degree and dimension [33].

Finally the semicontinuity property gives a priori necessary conditions for adjacency of isolated hypersurface singularities. For certain classes (unimodal singularities) these are even sufficient ([11]).

7. The filtered de Rham complex.

Let us return to the construction of the bifiltered complex $A_X^{\bullet}$ of §2. It is clear that this depends heavily on the choice of a semi-simplicial resolution of the singular space X. However, Ph. du Bois [1] has shown that it is again useful to forget about the weight filtration W and to consider the complex $A_X^{\bullet}$ together with its single filtration F : the Hodge filtration. The filtered complex $(A_X^{\bullet},F)$ is unique in the following sense: suppose one has two semi-simplicial resolutions $X_{\bullet}$ and $X'_{\bullet}$ of X, giving rise to filtered complexes

$$(A_X^{\bullet},F) \quad \text{and} \quad ({}'A_X^{\bullet},F) \ .$$

Then there exists a third filtered complex $({}''A_X^{\bullet},F)$ and a diagram

$$(A_X^{\bullet},F) \xleftarrow{u} ({}''A_X^{\bullet},F) \xrightarrow{v} ({}'A_X^{\bullet},F)$$

where u and v are filtered quasi-isomorphisms. (Such a pair (u,v) corresponds even to a canonical isomorphism in the filtered derived category of sheaves of $\mathbb{C}$-vector spaces on X).

We call $(A_X^{\bullet},F)$ the filtered de Rham complex of X .It should be remarked that, if X is an algebraic variety, there exists an algebraic version of this construction, which has been considered by Du Bois.

The differentials in the complexes $Gr_F^p A_X^{\bullet}$ are $\mathcal{O}_X$-linear and their <u>cohomology sheaves</u>

$$\mathcal{H}^q(Gr_F^p A_X^{\bullet})$$

are coherent $\mathcal{O}_X$-modules ([1], Prop. 4.4). These have been studied by Guillén, Navarro and Puerta [13] and the author [25]. In particular, if X is itself smooth, then

$$\begin{cases} \mathcal{H}^q(\mathrm{Gr}_F^p A_X^\bullet) = 0 & \text{for } q \neq p ; \\ \mathcal{H}^p(\mathrm{Gr}_F^p A_X^\bullet) = \Omega_X^p . \end{cases}$$

If $\pi: \tilde{X} \to X$ is a resolution of X, then

$$\begin{cases} \mathcal{H}^q(\mathrm{Gr}_F^n A_X^\bullet) = 0 & \text{for } q \neq n \\ \mathcal{H}^n(\mathrm{Gr}_F^n A_X^\bullet) \cong \pi_* \Omega_{\tilde{X}}^n & (n = \dim X) \end{cases}$$

This is a rewritten version of the vanishing theorem of Grauert and Riemenschneider. In general one has

$$\mathcal{H}^q(\mathrm{Gr}_F^p A_X^\bullet) = 0 \quad \text{if} \quad q < p \quad \text{or} \quad q > n$$

([13], §6.7, [25] Main Theorem). If X has normal isolated singularities and $\pi: \tilde{X} \to X$ is a good resolution with exceptional divisor D which has normal crossings, then

$$\begin{cases} \mathcal{H}^0(\mathrm{Gr}_F^0 A_X^\bullet) \cong \mathcal{O}_X \\ \mathcal{H}^q(\mathrm{Gr}_F^p A_X^\bullet) \cong R^{q-p}\pi_* \Omega_{\tilde{X}}^p(\log D)(-D) & \text{for} \quad (p,q) \neq (0,0) . \end{cases}$$

(see [25], Corollary (3.4)).
So we obtain

$$R^q \pi_* \Omega_{\tilde{X}}^p(\log D)(-D) = 0 \quad \text{for} \quad p+q > \dim X .$$

This appears to be a very useful vanishing theorem in the study of isolated singularities. For example, if $(\tilde{X},D) \xrightarrow{\pi} (X,x)$ is a good resolution of an isolated singularity of dimension n and ω is a holomorphic p-form on $X\backslash\{x\}$ where $p < n-1$, then ω extends to a meromorphic form on $\tilde{X}$ which has at most a logarithmic pole along D. In fact one can do better, see the next section.

8. Local cohomology .

Another ingredient of "local Hodge theory" for the study of isolated singularities is the following exact sequence. Let (X,x) be an isolated singularity of dimension $n \geq 2$ and let $\pi : (\tilde{X},D) \to (X,x)$ be a good resolution. Assume that X is a contractible Stein space. Then D is a deformation retract of $\tilde{X}$, so $H^i(\tilde{X}) \cong H^i(D)$. As D is a complete algebraic variety, this provides $H^i(\tilde{X})$ with a mixed Hodge structure.

One always may put X as an open subset inside a projective variety Y with only one singular point x. Then the local cohomology groups $H^i_{\{x\}}(Y) \cong H^i_{\{x\}}(X)$ carry a mixed Hodge structure by the general construction of Deligne [4]. It can be shown that this mixed Hodge strucure on $H^i_{\{x\}}(X)$ does not depend on the choice of Y (see e.g. [24],§1). Because X is contractible, we have

$$H^i_{\{x\}}(X) \xrightarrow{\sim} \tilde{H}^{i-1}(X \setminus \{x\}).$$

Let $M = \partial X$ which can be supposed to be a (2n-1)-manifold, homotopy equivalent to $X\setminus\{x\} = \tilde{X}\setminus D$. The isomorphism above provides $H^i(M)$ with a mixed Hodge structure. Consider the exact cohomology sequence of the pair $(\tilde{X},\tilde{X}\setminus D)$:

$$\begin{array}{lcccccccccc} & \to & H^{i-1}(\tilde{X}\setminus D) & \to & H^i_D(\tilde{X}) & \to & H^i(\tilde{X}) & \to & H^i(\tilde{X}\setminus D) & \to & \dots \\ & & \| & & \| & & \| & & \| & & \\ (*) & \to & H^{i-1}(M) & \to & H^i_D(\tilde{X}) & \xrightarrow{\alpha_i} & H^i(D) & \to & H^i(M) & \to & \dots \end{array}$$

Theorem. (1) The sequence (*) is an exact sequence of mixed Hodge structures (see [24],(1.10)).
(2) (Goresky-MacPherson). The maps α_i are injective for $i \leq n$ and surjective for $i \geq n$. (ibid, (1.11)).

The proof of (2) by Goresky and MacPherson uses the decomposition theorem of intersection homology. One can find a Hodge-theoretic proof of (2) in [15].

The Hodge filtration levels of the mixed Hodge structures in (*) have been computed in [24]. In particular one has

$$F^p H^p(M,\mathbb{C}) \cong H^0(D,\Omega^p_{\tilde{X}}(\log D) \otimes \mathcal{O}_D)$$

and

$$F^p H^p(D,\mathbb{C}) \cong H^0(D,\Omega^p_{\tilde{X}}/\Omega^p_{\tilde{X}}(\log D)(-D)).$$

The fact that α_p is injective for $p \le n$ gives that the natural map

$$H^0(D,\Omega^p_{\tilde{X}}/\Omega^p_{\tilde{X}}(\log D)(-D)) \to H^0(D,\Omega^p_{\tilde{X}}(\log D) \otimes \mathcal{O}_D)$$

is surjective for $p \le n-1$. One can use this to show that every meromorphic p-form on $\tilde{X}$ with logarithmic poles along D is in fact holomorphic on $\tilde{X}$ provided that $p \le n-1$.

By the result of §7, each p-form on $\tilde{X} \setminus D$ is already logarithmic along D for $p \le n-2$, so these forms extend over $\tilde{X}$. A local computation shows that the same holds for closed (n-1)-forms on $\tilde{X} \setminus D$. See [28] for more details.

9. Milnor number and Tjurina number.

We preserve the notations of the preceding paragraph, but we take (X,x) to be an isolated complete intersection singularity of dimension $n \ge 2$. The dimension τ of the base space of a semi-universal deformation of (X,x) is called the Tjurina number, and the n^{th} Betti number of a nonsingular fibre in this deformation is the Milnor number μ.

It has been proved by J. Wahl for n=2 [35] (even for smoothable Gorenstein surfaces) and by E. Looijenga and the author for $n \ge 2$, [14] that $\mu \ge \tau$. In fact there is a formula

$$\mu-\tau = h^{n-1}(\mathcal{O}_D) - h^0(\Omega^{n-1}_{\tilde{X}}(\log D)(-D)) + a_1+a_2+a_3$$

where a_1,a_2,a_3 are non-negative constants to be defined below. Observe that the other two terms are Hodge numbers

$$h^{n-1}(\mathcal{O}_D) = \dim Gr^0_F\, H^{n-1}(D,\mathbb{C})$$

$$h^0(\Omega^{n-1}_{\tilde{X}}/\Omega^{n-1}_{\tilde{X}}(\log D)(-D)) = \dim Gr^{n-1}_F H^{n-1}(D,\mathbb{C})$$

and the inequality

$$\dim Gr_F^0 H^{n-1}(D,\mathbb{C}) \geq \dim Gr^{n-1}_F H^{n-1}(D,\mathbb{C})$$

follows from the fact that $H^{n-1}(D,\mathbb{C})$ has a mixed Hodge structure with weights $\leq n-1$. Hence, if

$$h^{r,s} = \dim Gr_F^r Gr^W_{r+s} H^{n-1}(D,\mathbb{C}),$$

we have $\dim Gr^{n-1}H^{n-1}(D,\mathbb{C}) = h^{n-1,0}$ but $\dim Gr_F^0 H^{n-1}(D,\mathbb{C}) = h^{0,n-1} + h^{0,n-2} + \ldots + h^{0,0}$. Moreover by complex conjugation $h^{r,s} = h^{s,r}$, so we have our inequality.

Now to our invariants a_1,a_2,a_3. Consider the differentiation map (where $U = X\setminus\{x\}$):

$$d_1 : H^0(\Omega_U^{n-1})/H^0(\Omega_{\tilde{X}}^{n-1}) \to H^0(\Omega_U^n)/H^0(\Omega_{\tilde{X}}^n(D)).$$

Because $H^0(\Omega_{\tilde{X}}^{n-1}) = H^0(\Omega_{\tilde{X}}^{n-1}(\log D))$ and $d : H_D^1(\Omega_{\tilde{X}}^{n-1}(\log D)) \to H_D^1(\Omega_{\tilde{X}}^n(D))$ is injective ([14]) it follows as in [28] that d_1 is injective.
We have

$$a_1 = \dim \operatorname{Coker} d_1.$$

Moreover

$$a_2 = \dim H^0(\Omega_{\tilde{X}}^n)/dH^0(\Omega_{\tilde{X}}^{n-1}),$$

$$a_3 = \dim \operatorname{Coker}[H^0(\Omega^{n-1}_{U,cl}) \to Gr_F^{n-1}H^{n-1}(D,\mathbb{C})]$$

where cl stands for closed forms. (Observe that closed (n-1)-forms on U extend holomorphically on $\tilde{X}$ and thus can be restricted to D to give elements of $Gr_F^{n-1}H^{n-1}(D,\mathbb{C})$).

In the two-dimensional Gorenstein case Wahl has shown that $a_1=a_2=a_3=0$ implies that the singularity is either quasi-homogeneous or a cusp singularity. The key is a result of Scheja and Wiebe: a complete normal local ring of dimension two which admits a non-nilpotent derivation possesses a good $\mathbb{C}^*$-action. Their result also holds for isolated complete intersections of any dimension.

To see how this works, suppose (X,x) is a two dimensional Gorenstein surface singularity with $a_1 = 0$ but $d_1 \neq 0$. Then one has that the Gorenstein 2-form ω_0 on $\tilde{X}$ has a pole of order at least 2 along some component C of D. Wahl shows that there exists a C like this which has either genus > 0 or C is rational with at least 3 intersection points with other components (work on the minimal good resolution).

Because d_1 is bijective, there exists a 1-form η on U such that $\omega_0 \equiv d\eta \bmod H^0(\Omega^2_{\tilde{X}}(D))$. On the other hand, there exists a unique derivation ϑ of $\mathcal{O}_X$ such that $\eta = i_\vartheta \omega_0$, because ω_0 is nowhere vanishing on U. Thus

$$(*) \qquad \omega_0 \equiv d i_\vartheta \omega_0 \bmod H^0(\Omega^2_{\tilde{X}}(D)).$$

Any automorphism of (X,x), hence any derivation of $\mathcal{O}_X$, lifts to the minimal good resolution. So ϑ extends over $\tilde{X}$; it will be tangent to C and vanish at the intersection points of C with the other components of D. The assumptions about C imply that ϑ vanishes identically on C. (Exactly this argument is missing in higher dimension). In local coordinates x,y on $\tilde{X}$, C is given by x=0, and $\omega_0 = x^{-k}dx \wedge dy$, $k \geq 2$, $\vartheta = ax\frac{\partial}{\partial x} + bx\frac{\partial}{\partial y}$. One checks easily that (*) implies that x does not divide a. It follows that ϑ is non-nilpotent.

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