Proceedings of Symposia in Pure Mathematics
Volume 40 (1983), Part 1

GENERICITY AND SMOOTH FINITE DETERMINACY

ANDREW DU PLESSIS

0. Introduction. We will work with three categories C^τ, where

$$\tau = \begin{cases} \omega - \mathbf{C} & \text{the } \mathbf{C}\text{-analytic category,} \\ \omega - \mathbf{R} & \text{the } \mathbf{R}\text{-analytic category,} \\ \infty & \text{the smooth category.} \end{cases}$$

Throughout the article, N, P will be C^τ-manifolds, S a finite subset of N; and $f\colon (N, S) \to (P, y_0)$ a C^τ map-germ.

(0.1) DEFINITION. f is *r-determined* if every C^τ map-germ g with the same r-jet as f is C^τ-equivalent to f (i.e. if there exist C^τ-diffeomorphism germs h of (N, S), k of (P, y_0) such that $f \circ h = k \circ g$).

This is clearly a useful property (and it has been the object of considerable study, e.g. [**2, 4, 5, 6**]); it is of interest to know "how often" it occurs.

To be precise about this, we will need some notation:

(1) $C^\tau(N, P)_{S, y_0}$ is the set of C^τ map-germs $(N, S) \to (P, y_0)$.

(2) $J^r(N, P)_{S, y_0}$ is the set of r-jets of elements of $C^\tau(N, P)_{S, y_0}$; and $j^r\colon C^\tau(N, P)_{S, y_0} \to J^r(N, P)_{S, y_0}$ is the function "r-jet at S".

(3) $\pi^{r, r'}\colon J^r(N, P)_{S, y_0} \to J^{r'}(N, P)_{S, y_0}$ $(r \geqslant r')$ is the function "forgetting derivatives of order $> r'$".

(0.2) DEFINITION (TOUGERON [**9**]). (1) A subset Σ of $C^\tau(N, P)_{S, y_0}$ is *pro-algebraic* if

$$\Sigma = \bigcap_{r \geqslant 1} (j^r)^{-1} \Sigma^r$$

where (for each r) $\Sigma^r \subset J^r(N, P)_{S, y_0}$ is an algebraic subvariety (with respect to the coordinates induced by any (and hence all) choices of C^τ coordinate systems at (N, S), (P, y_0)).

1980 *Mathematics Subject Classification*. Primary 58C27; 58A20, 58C10, 58D15.

0082-0717/81/0000-0621/$05.00

It is easy to see that it can be supposed that

$$(\pi^{r,r'})^{-1}\Sigma^{r'} \supset \Sigma^{r} \qquad (r \geq r');$$

we will suppose this henceforth.

(2) The proalgebraic subset Σ of $C^{\tau}(N, P)_{S,y_0}$ is of *infinite codimension* if

$$\lim_{r\to\infty} \operatorname{cod} \Sigma^{r} = \infty.$$

(3) A property of C^{τ} map-germs holds *in general* if the set of germs not satisfying it is contained in a proalgebraic subset of infinite codimension.

(0.3) EXAMPLE (TOUGERON [9]). The property that a map-germ has finite singularity type (see e.g. [3, III, (2.7)], for a definition) holds in general. □

Let us, in passing, observe that discussion of properties holding in general is made easier by the following:

(0.4) LEMMA (TOUGERON [9]). *Suppose $\Sigma \subset C^{\tau}(N, P)_{S,y_0}$ is proalgebraic. Then Σ is of infinite codimension $\Leftrightarrow$ for any r $(1 \leq r < \infty)$ and any $z \in J^{r}(N, P)_{S,y_0}$, there exists $f \in C^{\tau}(N, P)_{S,y_0} - \Sigma$ such that $j^{r}f = z$.*

PROOF. $\Rightarrow$: Suppose there exist r $(1 \leq r < \infty)$ and $z \in J^{r}(N, P)_{S,y_0}$ such that any C^{τ} map-germ $(N, S) \to (P, y_0)$ with r-jet z is contained in Σ.

It follows that $\Sigma^{r'} \supset (\pi^{r'}, r)^{-1}z$ for all r'; so that

$$\operatorname{cod} \Sigma^{r'} \leq \operatorname{cod}(\pi^{r',r})^{-1}z = \operatorname{cod} z = \dim J^{r}(N, P)_{S,y_0}$$

whence

$$\lim_{r} \operatorname{cod} \Sigma^{r} \leq \dim J^{r}(N, P)_{S,y_0} < \infty.$$

$\Leftarrow$: Since, for $r' \geq r$, $(\pi^{r',r})^{-1}\Sigma^{r} \supset \Sigma^{r'}$, codim Σ^{r} is nondecreasing with r. Thus it is sufficient to show that for each r there exists $r' > r$ such that

$$\operatorname{codim} \Sigma^{r'} > \operatorname{codim} \Sigma^{r}.$$

To do this, it is enough to show that for each irreducible component C of Σ^{r} there exists $r_c > r$ such that $(\pi^{r_c,r})^{-1}C - \Sigma^{r_c} \neq \varnothing$ (for $(\pi^{r_c,r})^{-1}C \cap \Sigma^{r_c}$ is a union of irreducible components of Σ^{r_c}, each, then, of codimension greater than $\operatorname{codim}(\pi^{r_c,r})^{-1}C = \operatorname{codim} C$; so, taking $r' = \max_C r_c$ each irreducible component of $\Sigma^{r'}$ is of greater codimension than some irreducible component of Σ^{r}, i.e. $\operatorname{cod} \Sigma^{r'} > \operatorname{cod} \Sigma^{r}$).

It is then enough to show that for any $z \in \Sigma^{r}$ there exist $r' > r$ and $z' \in (\pi^{r',r})^{-1}z - \Sigma^{r'}$ (for then if C is the irreducible component of Σ^{r} containing z, $(\pi^{r',r})^{-1}C - \Sigma^{r'} \neq \varnothing$).

But of course such z' exists if there exists $f \notin \Sigma$ with $j^{r}f = z$ (for $j^{r'}f \notin \Sigma^{r'}$ for some $r' < \infty$; so simply take $z = j^{r'}f$). □

Our result is the following:

(0.5) THEOREM. ($n = \dim N$, $p = \dim P$).

1. *Finite determinacy of C^τ map-germs $(N, S) \to (P, y_0)$ does not hold in general if*:

(a) $\tau = \omega - \mathbf{R}$ *or* ∞ *and* $n > {}^2_{\mathbf{R}}\sigma(n, p)$.

(b) $\tau = \omega - \mathbf{C}$ *and* $n > {}^2_{\mathbf{C}}\sigma(n, p)$.

2. ($\tau = \omega - \mathbf{C}$, $\omega - \mathbf{R}$ *or* ∞).

Finite determinacy of C^τ map-germs $(N, S) \to (P, y_0)$ does hold in general if $n \leq {}^2_{\mathbf{C}}\sigma(n, p)$.

The integer-valued functions ${}^k_E\sigma$ ($1 \leq k \leq \infty$; $E = \mathbf{R}$ or $\mathbf{C}$) are defined as follows:

Let $J^r_E(n, p)$ denote the vector-space of r-jets of analytic mappings $(E^n, 0) \to (E^p, 0)$.

For nonnegative integers d, let

$$_E W^r_d(n, p) = \{z \in J^r_E(n, p) \mid z \text{ has } \mathcal{K}^r\text{-codimension} \geq d\}$$

(the group $\mathcal{K}^r$, and its (algebraic) action on $J^r_E(n, p)$ is defined in [4, III, §1] (or see [2, III, §6])).

It is clear that $_E W^r_d(n, p)$ is an algebraic subvariety of $J^r_E(n, p)$.

Let ${}^k_E W^r_d(n, p)$ denote the union of all irreducible components of $_E W^r_d(n, p)$ whose codimension in $J^r_E(n, p)$ is less than or equal to $d - k$.

Let ${}^k_E W^r(n, p) = \bigcup_{d \geq 0} {}^k_E W^r_d(n, p)$ (so that ${}^k_E W^r(n, p)$ is "the set of r-jets with $\mathcal{K}^r$-modality $\geq k$").

${}^k_E W^r(n, p)$ is clearly an algebraic subvariety of $J^r_E(n, p)$; and

$$(\pi^{r', r})^{-1}\, {}^k_E W^r(n, p) \subset {}^k_E W^{r'}(n, p) \quad (\text{for } r' > r).$$

Let

$${}^k_E\sigma^r(n, p) = \operatorname{codim} {}^k_E W^r(n, p)$$

and, finally,

$${}^k_E\sigma(n, p) = \lim_{r \to \infty} {}^k_E\sigma^r(n, p)$$

(so that ${}^k_E\sigma(n, p)$ is "the codimension of the set of analytic map-germs $(E^n, 0) \to (E^p, 0)$ with $\mathcal{K}$-modality $\geq k$").

It is easy to see that ${}^k_{\mathbf{R}}\sigma \geq {}^k_{\mathbf{C}}\sigma$; it seems likely that they are always equal (although this does not, surprisingly, seem to follow from any (known) general principle).

${}^1_E\sigma$ ($E = \mathbf{R}$ or $\mathbf{C}$) has been calculated by Mather, in [4, VI]:

Case I. $n \leq p$.

$${}^1_E\sigma(n, p) = \begin{cases} 6(p - n) + 8 & \text{if } p - n \geq 4 \text{ and } n \geq 4, \\ 6(p - n) + 9 & \text{if } 3 \geq p - n \geq 0 \text{ and } n \geq 4 \text{ or if } n = 3, \\ 7(p - n) + 10 & \text{if } n = 2, \\ \infty & \text{if } n = 1. \end{cases}$$

Case II. $n > p$.

$$ {}^1_E\sigma(n,p) = \begin{cases} 9 & \text{if } n = p+1, \\ 8 & \text{if } n = p+2, \\ n-p+7 & \text{if } n \geq p+3. \end{cases} $$

(The calculation requires, essentially, a classification of all analytic map-germs $(E^n,0) \to (E^p,0)$ of $\mathcal{K}$-modality 0.)

Thus ${}^1_{\mathbf{R}}\sigma = {}^1_{\mathbf{C}}\sigma$; we thus write simply ${}^1\sigma$ for them both.

We have the following information on ${}^2_E\sigma$ ($E = \mathbf{R}$ or $\mathbf{C}$):

Case I. $n \leq p$.

$$ {}^2_E\sigma(n,p) \begin{cases} \leq 7(p-n)+7 & \text{if } p-n \geq 3 \text{ and } n \geq 4, \\ \leq 4(p-n)+16 & \text{if } 2 \geq p-n \geq 0 \text{ and } n \geq 4, \\ \leq 9(p-n)+12 & \text{if } p-n \geq 1 \text{ and } n \geq 2, \\ = 13 & \text{if } p-n = 0 \text{ and } n \geq 2, \\ = \infty & \text{if } n = 1. \end{cases} $$

Case II. $n > p$.

$$ {}^2_E\sigma(n,p) = \begin{cases} 14 & \text{if } p = 1 \text{ and } n = 2,3, \\ 11 & \text{if } n = p+1 \text{ and } n \geq 3, \\ 13 & \text{if } n = p+2 \text{ and } n \geq 4, \\ 2(n-p)+4 & \text{if } 7 \geq n-p \geq 3, \\ n-p+11 & \text{if } n > p+7. \end{cases} $$

The complete calculation of ${}^2_E\sigma$ requires, essentially, a classification of all analytic map-germs $(E^n,0) \to (E^p,0)$ of $\mathcal{K}$-modality ≤ 1.

This classification has not yet been carried out completely in the case $n \leq p$, though some results have been obtained by Dimca and Gibson [1]; their results imply ${}^2_E\sigma(n,n) = 13$ and ${}^2_E\sigma(n,p) \leq 9(p-n)+12$ for $p-n \geq 1$ if $n \geq 2$ ($E = \mathbf{R}$ or $\mathbf{C}$). The other inequalities for ${}^2_E\sigma(n,p)$ in the case $n \leq p$ given above are derived by Mather in [5, p. 240].

The classification has been carried out in the case $n \geq p$, by Wall [11]; the values for ${}^2_E\sigma$ in this case are taken from [11]. Observe in particular that ${}^2_{\mathbf{R}}\sigma = {}^2_{\mathbf{C}}\sigma$ for $n > p$.

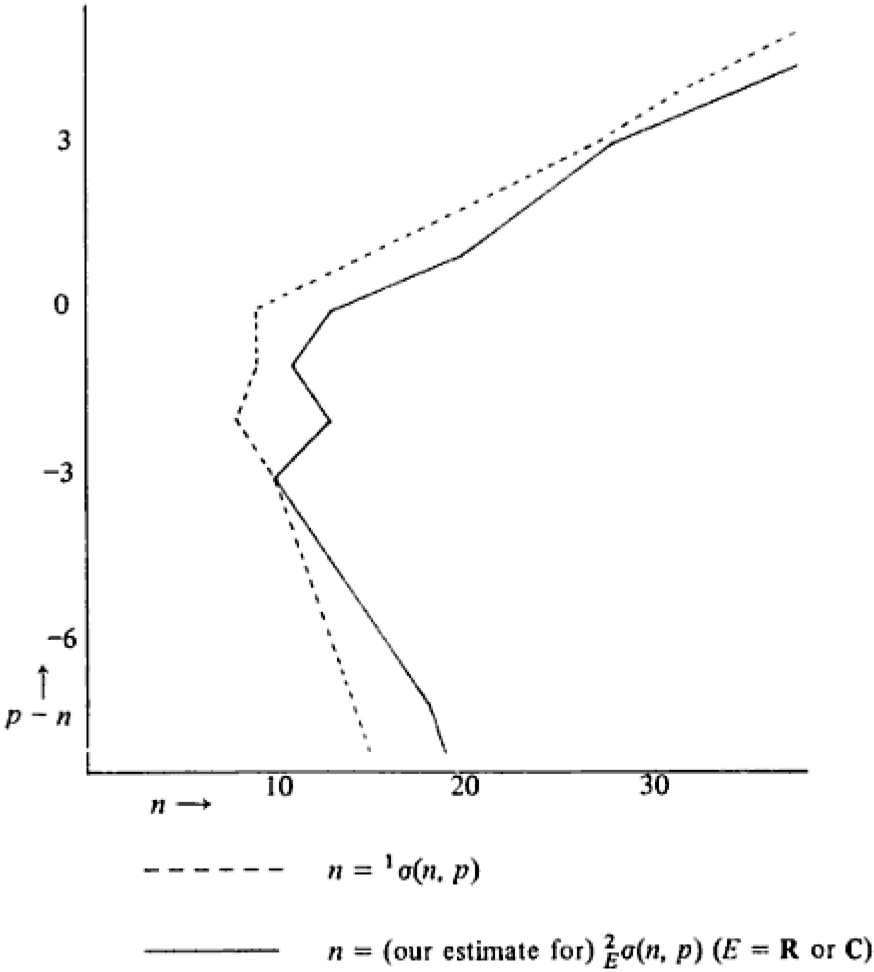
Our direct interest in ${}^1\sigma$ will be the following, which we note for future reference.

(0.6) *Fact*. For all n, p, with $E = \mathbf{R}$ or $\mathbf{C}$

$$ p - n + {}^2_E\sigma(n,p) \leq 2 \cdot \left(p - n + {}^1\sigma(n,p)\right). \quad \square $$

Mather's interest in the values of ${}^1\sigma$ was as the final part of his famous theorem:

MATHER'S THEOREM. *Smoothly stable maps are dense in* $C^\infty(N,P)$ (N *compact*) $\Leftrightarrow n < {}^1\sigma(n,p)$ ($n = \dim N$, $p = \dim P$). $\square$



In fact our theorem has a global variant also, reminiscent of this:

(0.7) COROLLARY (TO (0.5)). {*Maps of finite codimension*} *have complement of infinite codimension in* $C^\infty(N, P)$ (N *compact*; $n = \dim N$, $p = \dim P$) *if* $n \leqslant {}^2_{\mathbf{C}}\sigma(n, p)$; *and have complement of finite codimension if* $n > {}^2_{\mathbf{R}}\sigma(n, p)$.

(The point is that, for N compact, maps $N \to P$ of finite codimension (i.e. maps f s.t. $\dim_{\mathbf{R}}(\theta_f/tf(\theta_N) + \omega f(\theta_P)) < \infty$) are precisely those for which the germ of f at $f^{-1}y \cap \Sigma(f)$ is finitely determined for all $y \in P$.

The first statement follows now exactly as in [7, §5]; while the second follows easily from the argument for (1) of (0.5) (see (3.1)).

Mather (essentially) proved (1) of (0.5) (in [**5**, p. 238]); while Wall proved $n \leqslant {}^1\sigma(n, p)$ implies finite determinacy holds in general (in [**10**]; in fact he also claimed there that $n \leqslant {}^1\sigma(n, p)$ is a necessary condition for finite determinacy to hold in general, but this, as we will see, is too pessimistic (the error in Wall's argument in [**10**] is on p. 152, lines −8 to −5)).

The proof of (0.5) makes use of two characterizations of finite determinacy; these are introduced in the expository §§1, 2. The results of §1 on the analytic characterization of finite determinacy are an improvement on Mather's results of [4, III] (though this improvement is not essential for our arguments).

§2 is concerned with the geometrical characterization of finite determinacy; the results of Mather and Gaffney on this are stated, and a variant (well known to the experts) of Mather's geometric characterization of stability proved.

The proof itself is carried out in §3.

I thank Terry Wall for some very helpful remarks.

1. Analytic characterization of finite determinacy.

(1.0) *Notation.*

$$T\mathcal{K}f = tf(m_N\theta_N) + wf(m_P\theta_P).$$

A more precise version of Mather's (analytic) characterization of finite determinacy [4, III, (3.5)], is

(1.1) PROPOSITION. (τ: $\omega - \mathbf{C}$, $\omega - \mathbf{R}$ *or* ∞).

The following are equivalent:

(1) f *is finitely determined.*

(2) *For some* $r < \infty$,

$$m_N^r\theta_f \subset T\mathcal{K}f + m_N^{2r}\theta_f$$

(*in fact, this implies* f *is* $(2r-1)$*-determined*).

(3) *For some* $d < \infty$,

$$\dim\left\{m_N\theta_f/\left(T\mathcal{K}f + m_N^{s(d)}\theta_f\right)\right\} \leq d,$$

where

$$s(d) = \max\left\{s \in \mathbf{Z}: s \leq \tfrac{1}{4}(d+3)^2\right\}$$

(*in fact, this implies* f *is* $(s(d)-1)$*-determined*).

PROOF. (1) $\Leftrightarrow$ (2): See [6, (1.21) and (2.1)].

To prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1), we require the sequence of results (1.2)–(1.5) below.

We will write:

$\mathcal{E}_N = C^\tau_{(N,S)}$; m_N for the ideal of function-germs vanishing at S.

$\mathcal{E}_P = C^\tau_{(P,y_0)}$; m_P for the ideal of function-germs vanishing at y_0.

Let A be a finitely-generated $\mathcal{E}_N$-module, $C \subset f^*m_P \cdot A$ an $f^*\mathcal{E}_P$-module.

(1.2) LEMMA. (r, s *positive integers*). *Suppose*

(1) $$m_N^s A \subset f^*m_P \cdot A + m_N^{s+1}A,$$

(2) $$m_N^r A \subset C + m_N^{r+s}A.$$

Then

$$m_N^r A \subset C.$$

PROOF. Let E be the finitely-generated $\mathcal{E}_N$-module $(f^*m_P + m_N^s)\cdot A/f^*m_P A$. It follows from (1) that $E = m_N \cdot E$; so, by Nakayama's lemma [3, III, (1.10)], $E = 0$; so that

(1)′ $$m_N^s A \subset f^* m_P \cdot A.$$

It follows from this and (2) that

(2)′ $$m_N^r A \subset C + f^* m_P \cdot m_N^r A.$$

Let E' be the $f^*\mathcal{E}_P$-module $(C + m_N^r A)/C$. The proof of the lemma is complete if we show $E' = 0$. Now (2)′ implies $E' = f^*m_P \cdot E'$; so by Nakayama's lemma, it is enough to show E' is a *finitely-generated* $f^*\mathcal{E}_P$-module.

Consider the finitely-generated $\mathcal{E}_N$-module $m_N^r A$.

By the Preparation Theorem [3, III, (1.13)], this is finitely-generated as an $f^*\mathcal{E}_P$-module

$$\Leftrightarrow \dim\{m_N^r A/f^*m_P \cdot m_N^r A\} < \infty.$$

Now (1)′ implies $m_N^{r+s} A \subset f^*m_P \cdot m_N^r A$, so

$$\dim\{m_N^r A/f^*m_P \cdot m_N^r A\} < \dim\{m_N^r A/m_N^{r+s} A\} < \infty,$$

since A is a finitely-generated $\mathcal{E}_N$-module. Thus $m_N^r A$ is finitely generated as an $f^*\mathcal{E}_P$-module.

There is an obvious $f^*\mathcal{E}_P$-module-epimorphism $m_N^r A \to E'$; so E' is also finitely-generated as an $f^*\mathcal{E}_P$-module; and the proof is thus complete. □

(1.3) COROLLARY. *If* $m_N^r A \subset C + m_N^{2r} A$, *then* $m_N^r A \subset C$.

PROOF. (1) of (1.2) holds with $s = r$ (since $C \subset f^*m_P \cdot A$). □

For an integer $d \geq 0$, let

$$r(d) = \text{max integer} \leq \tfrac{1}{4}(d+2)^2,$$
$$s(d) = \text{max integer} \leq \tfrac{1}{4}(d+3)^2.$$

(1.4) LEMMA. (*d is a positive integer*). *If* $\dim\{m_N A/(C + m_N^{s(d)} A)\} \leq d$ $(d \geq 1)$, *then* $m_N^{r(d)} A \subset C$.

PROOF. Let

$$p_k = \dim m_N A/\big(f^*m_P + m_N^k\big)A \qquad (k \geq 1),$$
$$q_k = \dim m_N A/\big(C + m_N^k A\big).$$

Clearly $p_k \leq q_k$; and

$$0 = p_1 \leq p_2 \leq \cdots \leq p_{s(d)} \leq d,$$
$$0 = q_1 \leq q_2 \leq \cdots \leq q_{s(d)} \leq d.$$

Observe that

(∗) $$s(d) \geq d + 2$$

(for $(d+3)^2/4 > d + 2$).

It follows by the pigeonhold principle that for some i, $1 \leqslant i \leqslant d+1$, $p_i = p_{i+1}$. Let s be the least such i. Observe that $p_s \geqslant s-1$. Clearly

(1) $$m_N^s A \subset f^* m_P \cdot A + m_N^{s+1} A.$$

Observe that $s(d) \geqslant (d+3-s)s$

$$\left(\text{for } (d+3-s)\cdot s = ((d+3)/2)^2 - ((d+3)/2-s)^2\right).$$

Consider the increasing sequence of $d+3-s$ integers

$$q_s \leqslant q_{2s} \leqslant \cdots \leqslant q_{(d+3-s)\cdot s}.$$

Since $q_s \geqslant s-1$ (for $q_s \geqslant p_s$), and $q_{(d+3-s)s} \leqslant d$, it follows by the pigeonhole principle that

$$q_{is} = q_{(i+1)s} \quad \text{for some } i,\ 1 \leqslant i \leqslant d+2-s.$$

Thus if r is the least integer s.t. $q_r = q_{r+s}$, then

$$r \leqslant (d+2-s)s \leqslant r(d)$$

(for $(d+2-s)s = ((d+2)/2)^2 - ((d+2)/2-s)^2$). Clearly

(2) $$m_N^r A \subset C + m_N^{r+s} A.$$

(1), (2) and the lemma imply $m_N^r A \subset C$. □

(1.5) COROLLARY (TO THE PROOF OF (1.4)). *If* $\dim\{m_N\theta_f/(T\mathcal{C}f + m_N^{s(d)}\theta_f)\} \leqslant d$, *then f is* $(s(d)-1)$*-determined.*

PROOF. Let $A = \theta_f/tf(m_N\theta_N)$; $C = T\mathcal{C}f/tf(m_N\theta_N)$. The hypothesis implies

$$\dim\left\{m_N A/\left(C + m_N^{s(d)} A\right)\right\} \leqslant d$$

and, following the proof of (1.4), we obtain

(1) $$m_N^s\theta_f \subset tf(m_N\theta_N) + f^* m_P \cdot \theta_f + m_N^{s+1}\cdot\theta_f,$$

(2) $$m_N^r\theta_f \subset T\mathcal{C}f + m_N^{r+s}\cdot\theta_f.$$

Multiplying (1) through by m_N^r gives

(3) $$\begin{aligned} m_N^{r+s}\cdot\theta_f &\subset tf\left(m_N^{r+1}\theta_N\right) + f^* m_P \cdot m_N^r\theta_f + m_N^{r+s+1}\cdot\theta_f \\ &\subset tf\left(m_N^{r+1}\theta_N\right) + f^* m_P \cdot \left\{T\mathcal{C}f + m_N^{r+s}\cdot\theta_f\right\} + m_N^{r+s+1}\cdot\theta_f \end{aligned}$$

(using (2)). Multiplying (1) through by m_N^{r+1} gives

(4) $$m_N^{r+s+1}\cdot\theta_f \subset tf\left(m_N^{r+2}\theta_N\right) + f^* m_P \cdot m_N^{r+1}\theta_f + m_N^{r+s+2}\cdot\theta_f.$$

(3) and (4) imply, by [6, (3.19)] (with $D := m_N^{r+1}\theta_f$, $s := r+s$, $k := 1$), that f is $(r+s-1)$-determined.

But, as in the proof of (1.4), $r \leqslant (d+2-s)s$ so

$$r + s \leqslant (d+3-s)\cdot s \leqslant s(d)$$

(as in the proof of (1.4)) and the proof is complete. □

Returning to the proof of (1.1): (2) ⇒ (3) follows from (1.3) (for (2) implies, by (1.3), that

$$m'_N \cdot \left(\theta_f / tf(m_N \theta_N)\right) \subset T\mathcal{G}f / tf(m_N \theta_N)$$

so $m'_N \theta_f \subset T\mathcal{G}f$; so that

$$\dim\{m_N \theta_f / T\mathcal{G}f\} \leq \dim\{m_N \theta_f / m'_N \theta_f\} < \infty).$$

(3) ⇒ (1) is (1.5). □

We are now in a position to see that the set of non-finitely-determined germs is pro-algebraic:

(1.6) DEFINITION. (1) For $z \in J^r(N, P)_{s, y_0}$, define

$$d_{\mathcal{G}}(z) = \dim\left\{m_N \theta_f / \left(T\mathcal{G}f + m_N^{r+1} \theta_f\right)\right\},$$

where f is any germ with r-jet z (it is easy to see that the RHS is independent of which representation of z is chosen).

(2) Let $d(r) = \min\{d \in \mathbf{Z}: \frac{1}{4}(d+3)^2 \geq r+1\}$. Define

$$X^r(N, P)_{S, y_0} = \{z \in J^r(N, P)_{S, y_0} \mid d_{\mathcal{G}}(z) \geq d(r)\}.$$

(1.7) PROPOSITION. (1) *$X^r(N, P)_{S, y_0}$ is as algebraic subvariety of $J^r(N, P)_{S, y_0}$.*

(2) *For $r' > r$, $X^{r'} \subset (\pi^{r', r})^{-1} X^r$.*

(3) *If $j^r f \in J^r(N, P)_{S, y_0} - X^r(N, P)_{S, y_0}$, then f is r-determined.*

(4) *If f is not finitely-determined, then $j^r f \in X^r$ for all r.*

PROOF. (1) X^r is defined by the vanishing of determinants whose entries are polynomials on $J^r(N, P)_{S, y_0}$.

(2) This follows at once from the definitions.

(3) This follows from (1.5).

(4) This follows from (3). □

Thus:

(1.8) COROLLARY. *The set of nonfinitely-determined germs in $C^\tau(N, P)_{S, y_0}$ is proalgebraic.* □

As to the relations amongst finite determinancy for the different τ, we have:

(1.9) PROPOSITION. (1) *If f is finitely determined, it is polynomial in some coordinates.*

(2) *If f is* **R**-*analytic, then f is finitely determined in the* **R**-*analytic category ⇔ it is finitely determined when considered as a smooth germ.*

(3) *If f is* **R**-*analytic, then f is finitely determined ⇔ $f_{\mathbf{C}}$ (the complexification of f) is finitely determined.*

(4) *If finite determinacy holds in general in $C^{\omega-\mathbf{C}}(N_{\mathbf{C}}, P_{\mathbf{C}})_{S, y_0}$, then finite determinacy holds in general in $C^{\omega-\mathbf{R}}(N, P)_{S, y_0}$.*

PROOF. (1) Of course!

(2), (3) follow at once from (1.1), (2).

(4) With the obvious identifications, $X^r(N, P)_{S, y_0}$ is the set of real points of $X^r(N_{\mathbf{C}}, P_{\mathbf{C}})_{S, y_0}$; and so has codimension at least as great. □

It follows easily from this that

(1.10) COROLLARY. *To prove* (1)(a) *of* (0.5), *it is enough to consider* $\tau = \infty$. *To prove* (2) *of* (0.5), *it is enough to consider* $\tau = \omega - \mathbf{C}$. □

2. Geometric characterization of finite determinacy.

(2.1) DEFINITION ($\tau = \omega - \mathbf{C}$, $\omega - \mathbf{R}$ or ∞). The C^τ map-germ $f\colon (N, S) \to (P, y)$ is *stable off* y_0 if f has a representative $\tilde{f}\colon U \to V$ such that the germ of $\tilde{f}$ at $\tilde{f}^{-1}(y) \cap \Sigma(\tilde{f})$ is stable for all $y \in V - \{y_0\}$.

(2.2) PROPOSITION (GAFFNEY, MATHER).
(1) ($\tau = \omega - \mathbf{C}$) *$f$ is finitely determined $\Leftrightarrow$ f is stable off* y_0.
(2) ($\tau = \omega - \mathbf{R}$) *$f$ is finitely determined $\Leftrightarrow$ $f_{\mathbf{C}}$ is stable off* y_0.
(3) ($\tau = \infty$) *If f is finitely determined, then f is stable off* y_0.

PROOF. See [2, §3]. □

This is "the geometric characterization of finite determinacy". To make it even more geometric, we give two geometric characterizations of stability.

(2.3) PROPOSITION (MATHER). ($\tau = \omega - \mathbf{C}$, $\omega - \mathbf{R}$ *or* ∞). *f is stable $\Leftrightarrow$ for any (and hence all) $r \geq p$, $j^r f$ is multitransverse to all $\mathcal{K}^r$-classes in $(J^r(N, P), j^r f)$.*
Moreover, whether or not f is stable depends only on its $(p+1)$-jet at S.

PROOF. See [4, V, (4.1) and 4, IV (Corollary to (1.1))]. □

Suppose that the C^τ map-germ $f\colon (N, S) \to (P, y_0)$ is of finite singularity type. Then [3, III, (2.8)] f has a stable unfolding:

$$\begin{array}{ccc} (N', S') & \xrightarrow{F} & (P', y_0') \\ i \uparrow & & \uparrow j \\ (N, S) & \xrightarrow{f} & (P, y_0) \end{array}$$

Let $\tilde{F}\colon U' \to V'$ be a representative for F with $S \subset \tilde{F}^{-1}(y_0') \cap \Sigma(F)$. Define

$$L = \left\{ y \in V' \mid \tilde{F}_{\tilde{F}^{-1}(y) \cap \Sigma(\tilde{F})} \text{ is contact-equivalent to } F \right\}.$$

It is easy to see that the germ of L at y_0' is independent of the choice of representative.

Since F is a stable germ, $\tilde{F}$ may be chosen as a stable map; and then the germs $\tilde{F}_{\tilde{F}^{-1}(y) \cap \Sigma(\tilde{F})}$ are stable for all $j \in V'$. Hence (by [4, IV, Theorem A]) F and $\tilde{F}_{\tilde{F}^{-1}(y) \cap \Sigma(\tilde{F})}$ are contact-equivalent if and only if they are equivalent.

According to [4, IV, p. 227], then, (L, y_0') is a germ of submanifold of (P', y_0'); with tangent-space at y_0' given by

$$TL_{y_0'} = ev_{y_0}\left((\omega F)^{-1}\left(tF(\theta_{N', S'}) + F^* m_{P', y_0'} \cdot \theta_F\right)\right)$$

(where $ev_{y_0'}\colon \theta_{P', y_0'} \to TP_{y_0'}$ is given by $\eta \mapsto \eta(y_0')$).

(2.4) PROPOSITION. *f is stable $\Leftrightarrow$ j is transverse to L at y_0'.*

PROOF. Choosing coordinates as in [3, III, p. 68], we may identify $(F; (i, j))$ with the diagram

$$\begin{array}{ccc} (N \times U, S \times 0) & \xrightarrow{F} & (P \times U, y_0 \times 0) \\ \uparrow 1_N \times 0 & & \uparrow 1_P \times 0 \\ (N, S) & \xrightarrow{f} & (P, y_0) \end{array}$$

(where U is a neighborhood of 0 in $\mathbf{R}^k$, for some $k < \infty$), where F preserves U-levels.

Now, j transverse to L at y_0'

$$\Leftrightarrow T(P \times U)_{y_0 \times 0} = TP_{y_0} + ev_{y_0 \times 0}\left((\omega F)^{-1}\left(tF(\theta_{N \times U}) + F^* m_{P \times U} \cdot \theta_F\right)\right)$$

$$\Leftrightarrow \omega F\left(\frac{\partial}{\partial u_i}\right) \in tF(\theta_{N \times U}) + \omega F(\psi_{P \times U}) + F^* m_{P \times U} \cdot \theta_F \qquad (i = 1, \dots, k)$$

($\psi_{P \times U}$ are vector fields on $P \times U$ with no TU-component)

$$\Leftrightarrow \frac{\partial F}{\partial u_i} \in tF(\theta_{N \times U}) + \omega F(\psi_{P \times U}) + F^* m_{P \times U} \cdot \theta_F \qquad (i = 1, \dots, k)$$

$$\left(\frac{\partial F}{\partial u_i} = tF\left(\frac{\partial}{\partial u_i}\right) - \omega F\left(\frac{\partial}{\partial u_i}\right)\right)$$

$$\Leftrightarrow \frac{\partial F}{\partial u_i} \in tF(\psi_{N \times U}) + \omega F(\psi_{P \times U}) + F^* m_{P \times U} \cdot \psi_F \qquad (i = 1, \dots, k)$$

($\psi_{N \times U}$, ψ_F are vector fields with no TU-component)

($\Leftarrow$ is trivial; for $\Rightarrow$ write

$$\frac{\partial F}{\partial u_i} = \alpha + tF\left(\sum_{j=1}^{k} \beta_{ij} \frac{\partial}{\partial u_j}\right) + \sum_{j=1}^{k} \gamma_{ij} \cdot \omega F\left(\frac{\partial}{\partial u_j}\right),$$

where $\alpha \in tF(\psi_{N \times U}) + \omega F(\psi_{P \times U}) + F^* m_{P \times U} \cdot \psi_F$, $\beta_{ij} \in \xi_{N \times U}$, $\gamma_{ij} \in F^* m_{P \times U} \cdot \xi_{N \times U}$. Since $\partial F/\partial u_i \in \psi_F$, taking TU-components gives $\beta_{ij} = \gamma_{ij}$, so that $\partial f/\partial u_i = \alpha + \sum_{j=1}^{k} \gamma_{ij} \cdot \partial F/\partial u_j$)

$$\Leftrightarrow \left.\frac{\partial F}{\partial u_i}\right|_{u=0} \in tf(\theta_{N,S}) + \omega f(\theta_{P,y_0}) + f^* m_{P,y_0} \cdot \theta_f \qquad (i = 1, \dots, k).$$

($\Rightarrow$ is just restriction to $u = 0$; $\Leftarrow$ follows because the previous inclusion differs from this one by terms vanishing on $\{u = 0\}$; i.e. by terms in $m_U \cdot \psi_F \subset F^* m_{P \times U} \cdot \psi_F$.)

$$\Leftrightarrow \theta_f = tf(\theta_{N,S}) + \omega f(\theta_{P,y_0}) + f^* m_{P,y_0} \cdot \theta_f.$$

(Since F is stable, $\{\partial F/\partial u_i|_{u=0}\}$ is a vector-space basis for

$$\theta_f/\left(tf(\theta_{N,S}) + f(\theta_{p,y_0}) + f^*m_{P,y_0}\cdot\theta_f\right)$$

[3, III, (2.6)]).

$\Leftrightarrow f$ is stable (by [4, IV, (1.1)]). □

3. Proof of (0.5).

(3.1) Proof of (0.5)(1): (a) We suppose $n >^2_{\mathbf{R}} \sigma(n, p)$ and must show that in this range of dimensions, the set of nonfinitely-determined germs does *not* have infinite codimension.

According to (1.10), it is enough to consider the C^∞ case.

According to (0.4), it is enough to exhibit an $r > 0$ and a jet $z \in J^r(n, p)$ which has no finitely-determined representative.

Let $\Sigma^* \subset J^{p+1}(n, p)$ be the set of $(p + 1)$-jets of unstable map-germs; this is an algebraic subvariety of $J^{p+1}(n, p)$ (for a germ f is stable $\Leftrightarrow$ $\theta_f = tf(\theta_N) + wf(\theta_P) + m_N^{p+1}\theta_f$; so Σ^* is defined by the vanishing of determinants whose entries are polynomials on $J^{p+1}(n, p)$).

According to Mather [4, V, (10.5)]:

$$(*)\qquad \text{if } n >^2_{\mathbf{R}} \sigma(n, p), \quad \text{then cod } \Sigma^* < n.$$

Let Σ_0^* be the set of nonsingular points of Σ^*; a smooth (indeed algebraic) submanifold of $J^{p+1}(n, p)$.

Let $\Sigma_0^*(\mathbf{R}^n, \mathbf{R}^p) \subset J^r(\mathbf{R}^n, \mathbf{R}^p)$ be the fibred submanifold with fibre Σ_0^*.

According to Mather [4, V, (6.1)], there exists a smooth map-germ f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ such that:

(1) $j^{p+1}f \in \Sigma_0^*$,

(2) the germ $J^{p+1}f$: $(\mathbf{R}^n, 0) \to (J^{p+1}(\mathbf{R}^n, \mathbf{R}^p), j^{p+1}f)$ is transverse to $\Sigma_0^*(\mathbf{R}^n, \mathbf{R}^p)$ at $j^{p+1}f$.

Take $r = p + 2$; $z = j^{p+2}f$. We claim z has no finitely-determined representative.

To see this, let g have r-jet z. Then:

(1) $j^{p+1}g = j^{p+1}f \in \Sigma_0^*$,

(2) $J^{p+1}g$: $(\mathbf{R}^n, 0) \to (J^{p+1}(\mathbf{R}^n, \mathbf{R}^p), j^{p+1}g)$ is transverse to $\Sigma_0^*(\mathbf{R}^n, \mathbf{R}^p)$ at $j^{p+1}g$.

((2) is a property of the 1-jet of $J^{p+1}g$ at 0; so is determined by the $(p + 2)$-jet of g at 0.)

Let $\tilde{g}$: $U \to V$ be any representative of g with $\tilde{g}^{-1}(0) \cap \Sigma(\tilde{g}) = \{0\}$. $\Sigma_0(\tilde{g}) = (J^{p+1}\tilde{g})^{-1}\Sigma_0^*(\mathbf{R}^n, \mathbf{R}^p)$ is a nonempty submanifold (at least near 0; but, shrinking U, we can suppose this on U), and cod $\Sigma_0^*(g) =$ cod $\Sigma_0^*(\mathbf{R}^n, \mathbf{R}^p) =$ cod $\Sigma_0 < n$ (by $(*)$).

The germ of $\tilde{g}$ at any point of $\Sigma_0(\tilde{g})$ is unstable (for its $(p + 1)$-jet there is the $(p + 1)$-jet of an unstable germ; but whether or not a germ is stable is determined by its $(p + 1)$-jet (2.3)).

Thus g is not stable off 0; so, by (2.2)(3), g is not finitely determined. □

(b) This proceeds exactly as above; $\omega - \mathbf{C}$-versions of the results of Mather quoted above are proved exactly as the $\mathbf{R}$-versions. □

(3.2) Proof of (0.5)(2). We suppose $n \leqslant^2_{\mathbf{C}} \sigma(n, p)$; and must show that in this range of dimensions, the set of nonfinitely-determined germs has infinite codimension.

According to (1.10), it is enough to consider the $\omega - \mathbf{C}$ case.

According to (0.4) and (1.8), it is enough to exhibit, for any $z \in J^r(N, P)_{S,y_0}$, a finitely-determined representative f for z.

So: choose any $z \in J^r(N, P)_{S,y_0}$.

By (0.3) z has a representative f of finite singularity type. Let $(F; (i, j))$ be a stable unfolding of f. It is well known that we can choose representatives so that

$$(1) \qquad \begin{array}{ccc} U' & \xrightarrow{F} & V' \\ i \uparrow & & \uparrow j \\ U & \xrightarrow{f} & V \end{array}$$

is cartesian and i, j are embeddings.

(2) $F \mid \Sigma(F)$ is finite-to-one and proper.

(3) $F^{-1}j(y_0) \cap \Sigma(F) \subset i(S)$.

(4) $F_{F^{-1}(y')\cap\Sigma(F)}$ is stable for all $y' \in V'$.

From the discussion preceding (2.4) follows:

(5) $L_{y'} = \{y'' \in V' \mid F_{F^{-1}y''\cap\Sigma(F)}, F_{F^{-1}y'\cap\Sigma(F)}$ are contact-equivalent$\}$ is a submanifold of V'.

Let $\Sigma^* \subset J^1(V, V')$ be the set of 1-jets z such that, if $z \in \Sigma^* \cap J^1(V, V')_{y,y'}$, then some (and hence any) representative of z is not transverse to $L_{y'}$.

The proof will proceed as follows.

A. We will show Σ^* is a constructible set.

B. We will show that $n \leqslant^2_{\mathbf{C}} \sigma(n, p)$ implies cod $\Sigma^* \geqslant p$.

C. Let $\mathbf{S}$ be a locally finite manifold partition of Σ^*.

We will show that there exist j': $V \to V'$ with the same r-jet as j at y_0, and a neighbourhood $\tilde{V}$ of y_0 in V, such that $J^1(j') \mid \tilde{V} - \{y_0\}$ is transverse to $\mathbf{S}$.

Proving these three claims will allow us to complete the proof, as follows:

B, C imply that $J^1(j')^{-1}\Sigma^* \cap (\tilde{V} - \{y_0\})$ is constructible of codimension $\geqslant p$; so either empty or a set of isolated points in $\tilde{V} - \{y_0\}$.

Now $(J^1(j'))^{-1}\Sigma^* \cap \tilde{V}$ is constructible, and thus has only finitely many connected components. So $J^1(j')^{-1}\Sigma^* \cap \tilde{V}$ consists of at most finitely many points; so that, by shrinking $\tilde{V}$, we may suppose $J^1(j')^{-1}\Sigma \cap \tilde{V} \subset \{y_0\}$ so

$$(*) \qquad j'_y \text{ is transverse to } L_{j'(y)} \text{ for any } y \in \tilde{V} - \{y_0\}.$$

Let

$$\begin{cases} Q \subset U' \times \tilde{V} \text{ be the fibre-product of } j \mid \tilde{V} \text{ and } F, \\ Q' \subset U' \times \tilde{V} \text{ be the fibre-product of } j' \mid \tilde{V} \text{ and } F. \end{cases}$$

Since j, j' have the same r-jet at y_0, Q, Q' have rth-order contact at $(i(S), y_0)$. Let ρ: $Q \to Q'$ be restriction to Q of the orthogonal projection (w.r.t. some co-ordinates) to Q'; shrinking U', $\tilde{V}$ we can suppose ρ is (well defined and) an analytic diffeomorphism; whose composition with the inclusion $Q' \subset U' \times \tilde{V}$ has the same r-jet at $(i(S), y_0)$ as the inclusion $Q \subset U' \times \tilde{V}$.

Let $\tilde{U}$ be an open neighbourhood of S in $f^{-1}(\tilde{V})$ and define:

(i) f: $\tilde{U} \to \tilde{V}$ by $f' = \pi_{\tilde{V}} \circ \rho \circ (i, f)$,

(ii) i': $\tilde{U} \to U'$ by $i' = \pi_{U'}\pi \circ \rho \circ (i, f)$ (where $\pi_{U'}$, $\pi_{\tilde{V}}$ are the natural projections of $U' \times \tilde{V}$ to U', V respectively).

Shrinking $\tilde{U}$ if necessary, we may suppose i' is an embedding.

Clearly f' has the same r-jet as f at x_0 and

$$\begin{array}{ccc} U' & \xrightarrow{F} & V' \\ i' \uparrow & & \uparrow j' \\ \tilde{U} & \xrightarrow{f'} & \tilde{V} \end{array}$$

is a cartesian square.

For any $y \in \tilde{V}$, $(F_{i'(f'^{-1}(y) \cap \Sigma(f'))};\ i'_{f'^{-1}(y) \cap \Sigma(f')}, j'_y)$ is a stable unfolding of $f'_{f'^{-1}(y) \cap \Sigma(f')}$.

Thus, by (*) and (2.4), $f'_{f'^{-1}(y) \cap \Sigma(f')}$ is stable for any $y \in \tilde{V} - \{y_0\}$. Thus, by (2.1)(1), the germ of f' at S is finitely determined. But this germ has r-jet z.

It thus remains to prove the claims A, B, C:

(3.3) Lemma. *By shrinking U, U'; V, V' if necessary, we can suppose that F has a constructible regular statification* (**A**, **A**′) *such that each stratum S of* **A**′ *is analytically foliated by* $\{L_{y'} \mid y' \in S\}$.

Proof. It is easy to construct a fibred-over- $U' \times V'$ regular stratification $\mathbf{S}$ of $J^{p'+1}(U', V')$ (where $p' = \dim V'$) whose strata are in $J^{p'+1}(n', p')$ smoothly foliated by $\mathcal{K}^{p'+1}$-orbits. (This uses little more than:

(a) The sets $W^{p'+1}(n', p')$ are $\mathcal{K}^{p'+1}$-invariant algebraic varieties.

(b) A G-invariant algebraic variety admits a G-invariant regular stratification.)

Since F has stable germs at every point of U', $\mathbf{B} = (J^{p'+1}F)^{-1}\mathbf{S}$ is a regular stratification.

Using the methods of [3, I, (3.5)], it is easy to construct a (canonical) regular stratification $\mathbf{C}$ of $F\Sigma(F)$ satisfying, for any $U, V \in \mathbf{C}$, $S, T \in \mathbf{B}$,

$CVS_{\mathbf{B}}1$. $F^{-1}U \cap \Sigma(F \mid S)$ is an analytic manifold.

$CVS_{\mathbf{B}}2$. $F|: F^{-1}U \cap \Sigma(F \mid S) \to U$ is a local diffeomorphism.

$CVS_{\mathbf{B}}3$. $F^{-1}U \cap \Sigma(F \mid S)$ is regular over $F^{-1}V \cap \Sigma(F \mid T)$.

$CVS_{\mathbf{B}}4$. $F^{-1}U \cap \{S - \Sigma(F \mid S)\}$ is regular over $F^{-1}V \cap \Sigma(F \mid T)$.

Define then

$$\mathbf{A}' = \mathbf{C} \cup \{V' - F\Sigma(F)\},$$

$$\mathbf{A} = \bigcup_{\substack{U \in \mathbf{C} \\ S \in \mathbf{B}}} \{F^{-1}U \cap \Sigma(F \mid S)\}$$

$$\cup \bigcup_{\substack{U \in \mathbf{C} \\ S \in \mathbf{B}}} \{F^{-1}U \cap \{S - \Sigma(F \mid S)\}\} \cup \{U' - F^{-1}F\Sigma(F)\}.$$

By arguments similar to those of [3, I, (3.1)], (**A**, **A**′) is a regular stratification for F.

Since a canonical regular stratification is defined independently of co-ordinate choice, and so is invariant under the action of analytic diffeomorphisms, $L_{y'} \subset S$ for all $y' \in S$ (for stable germs are contact-equivalent if and only if they are equivalent).

Now

$$L_{y'} = \bigcap_{x \in F^{-1}(y') \cap \Sigma(F)} F(K_{x'}) \quad \text{(regular intersection)},$$

where $K_{x'} = \{x'' \in U' \mid F_{x''} \text{ is contact-equivalent to } F_{x'}\}$ and a stratum T of **B** is foliated by $\{K_{x'} \mid x' \in T\}$ (for $K_{x'} = (J^{p'+1}F)^{-1}\{\mathcal{K}^{p'+1}\text{-class of } j^{p'+1}F_x\}$).

It follows easily that a stratum S of **A**′ is foliated by $\{L_{y'} \mid y' \in S\}$. □

(3.4) Definition. Let W be a constructible submanifold of the algebraic manifold M; N an algebraic manifold.

Let $\sigma \subset TW$ be an algebraic sub-bundle.

Define ${}^{N,M}\Sigma^*_{W,\sigma} \subset \operatorname{Hom}(TN, TM) \mid N \times W$ as having fibre over $(x, w) \in N \times W$ the linear maps $TN_x \to TM_w$ which are *not* transverse to σ_w.

It is clear that ${}^{N,M}\Sigma^*_{W,\sigma}$ is a constructible subset of $\operatorname{Hom}(TN, TM) \mid N \times W \subset \operatorname{Hom}(TN, TM)$.

(3.5) Lemma. *The codimension of* ${}^{N,M}\Sigma^*_{W,\sigma}$ *in* $\operatorname{Hom}(TN, TM)$ *is*

$$\begin{cases} n - (w - e) + 1 & (n \geq m - e), \\ m - w & (n < m - e) \end{cases}$$

($n = \dim N$, $m = \dim M$; $w = \dim W$, $e = $ fibre $\dim \sigma$).

Proof. Let ρ: $\operatorname{Hom}(TN_x, TM_w) \to \operatorname{Hom}(TN_x, TM_w/\sigma_w)$ be induced by the projection $TM_w \to TM_w/\sigma_w$.

Clearly

$$\left({}^{N,M}\Sigma^*_{W,\sigma}\right)_{(x,w)} = \rho^{-1}\{\phi \in \operatorname{Hom}(TN_x, TM_w/\sigma_w) \mid \phi \text{ is not an epimorphism}\}$$

ρ is a submersion; so

$$\operatorname{cod}\left({}^{N,M}\Sigma^*_{W,\sigma}\right)_{(x,w)} = \operatorname{cod}\{\phi \in \operatorname{Hom}(TN_x, TM_w/\sigma_w) \mid \phi \text{ is not an epimorphism}\}$$

$$= \begin{cases} n - (m - e) + 1, & n \geq m - e, \\ 0, & n < m - e. \end{cases}$$

This is also the codimension of ${}^{N,M}\Sigma^*_{W,\sigma}$ in $\operatorname{Hom}(TN, TM) \mid N \times W$; since $\operatorname{Hom}(TN, TM) \mid N \times W$ has codimension $m - w$ in $\operatorname{Hom}(TN, TM)$, the result follows at once. □

(3.6) Lemma (Claims A, B). A. $\Sigma^* \subset J^1(V, V')$ *is constructible.*

B. *If* $n \leq^2_{\mathbf{C}} \sigma(n, p)$, *then* $\operatorname{cod} \Sigma^* \geq p$.

PROOF. A. Clearly

$$\Sigma^* = \bigcup_{S \in \mathbf{A}'} {}^{V,V'}\Sigma^*_{S,\sigma_S},$$

where σ_S is the leaf-tangent-bundle of the foliation $\{L_{y'} \mid y' \in S\}$ of $S \in \mathbf{A}'$.

It follows at once that Σ^* is constructible.

B. $\operatorname{cod} \Sigma^* = \min_{S \in \mathbf{A}'} \operatorname{cod} {}^{V,V'}\Sigma^*_{S,\sigma_S}$.

By (3.5), $\operatorname{cod} {}^{V,V'}\Sigma^*_{S,\sigma_S} \geqslant p$ if σ_S is of codimension $\leqslant 1$ in TS.

σ_S is of codimension > 1 in $TS \Leftrightarrow$ either (1) for each $y' \in S$, there exists

$$x' \in F^{-1}y' \cap (J^{p'+1}F)^{-1}({}^2_{\mathrm{C}}W^{p'+1}(U', V'))$$

or (2) for each $y' \in S$, there exist

$$x'_1, x'_2 \in F^{-1}y' \cap (J^{p'+1}F)^{-1}({}^1_{\mathrm{C}}W^{p'+1}(U', V')) \qquad (x'_1 \neq x'_2).$$

In case (1):

$$\operatorname{cod} {}^2_{\mathrm{C}}W^{p'+1}(n', p')/\Sigma^{n+1}(n', p') = \operatorname{cod} {}^2_{\mathrm{C}}W^{p'+1}(n, p)$$

(for $\mathcal{K}$-classes of map-germs $(\mathbf{C}^{n'}, 0) \to (\mathbf{C}^{p'}, 0)$ of kernel-rank $\leqslant n$ are in (1-1)-correspondence with $\mathcal{K}$-classes of map-germs $(\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, the correspondence (given by local algebras) preserving codimension)

$$= {}^2_{\mathrm{C}}\sigma^{p'+1}(n, p).$$

Now the stratum T of $\mathbf{B}$ (notation of (3.3)) containing x' has at most this codimension; and S has at most the dimension of T (for $F \mid \Sigma(F)$ is finite-to-one) so

$$\begin{aligned} \operatorname{codim} S &\geqslant p' - n' + {}^2_{\mathrm{C}}\sigma^{p'+1}(n, p) \\ &\geqslant p - n + {}^2_{\mathrm{C}}\sigma(n, p) \quad (\text{for } p' - n' = p - n). \end{aligned}$$

In case (2):

$$\begin{aligned} \operatorname{cod} {}^1_{\mathrm{C}}W^{p'+1}(n', p')/\Sigma^{n+1}(n', p') &= \operatorname{cod} {}^1_{\mathrm{C}}W^{p'+1}(n, p) \\ &= {}^1_{\mathrm{C}}\sigma^{p'+1}(n, p). \end{aligned}$$

Now the strata T_1, T_2 of $\mathbf{B}$ containing x_1, x_2 respectively have at most this codimension and $S \subset F(T_1) \cap F(T_2)$, this intersection being regular, so:

$$\begin{aligned} \operatorname{codim} S &\geqslant \operatorname{codim} F(T_1) + \operatorname{codim} F(T_2) \\ &\geqslant 2(p' - n' + {}^1_{\mathrm{C}}\sigma^{p'+1}(n, p)) \\ &\geqslant 2(p - n + {}^1\sigma(n, p)) \\ &\geqslant p - n + {}^2_{\mathrm{C}}\sigma(n, p) \quad (\text{by } (0.6)). \end{aligned}$$

Thus, since we are in the range of dimensions $n \leqslant {}^2_{\mathrm{C}}\sigma(n, p)$, $\operatorname{codim} S \geqslant p$ (for σ_S of codimension > 1 in TS), and so $\operatorname{cod} \Sigma_{S,\sigma_S} \geqslant p$ for such S.

Thus, indeed $\operatorname{cod} \Sigma_{S,\sigma_S} \geqslant p$ for all S; so B is proved. □

C follows at once from the following elementary transversality lemma, essentially due to Thom [8, Theorem 3]:

(3.7) LEMMA ($\tau = \omega - \mathbf{C}$, $\omega - \mathbf{R}$ *or* ∞). *Let* $\mathbf{S}$ *be a countable collection of* C^τ*-submanifolds of* $J^r(N, P)$; *let* $f\colon (N, x_0) \to (P, y_0)$ *be a* C^τ *map-germ.*

Then, for any $r' > 0$, *there exists a* C^τ *map-germ* $g\colon (N, S) \to (P, y_0)$ *with the same* r'*-jet as* f *and such that*

$$J^r g \mid U - \{x_0\} \text{ is transverse to } \mathbf{S}.$$

PROOF. Let $\tilde{f}\colon U \to V$ be a representative for f; we may without loss of generality identify U, V as neighbourhoods of 0 in E^n, E^p respectively ($E = \mathbf{C}$ if $\tau = \omega - \mathbf{C}$, $E = \mathbf{R}$ otherwise).

Let W be an open neighborhood of 0 in the vector space of p-tuples of polynomials in n variables each of whose monomials has order between $r' + 1$ and $r + r' + 1$.

Define $F\colon U \times W \to V$ by

$$F(x, w) = \tilde{f}(x) + w(x)$$

(shrinking U, W if necessary to make this well defined).

F induces a map $JF\colon U \times W \to J^r(N, P)$, given by

$$JF(x, w) = J^r F_w(x)$$

($F_w\colon U \to V$ being defined by $F_w(x) = F(x, w)$).

It is not hard to see that $JF \mid U \times W - x_0 \times W$ is a submersion: it follows that $\{Q_S \mid S \in \mathbf{S}\}$ (where $Q_S = (JF \mid U \times W - x_0 \times W)^{-1} S$) is a manifold partition of $U \times W - x_0 \times W$.

Let $\pi_S\colon Q_S \to W$ (for $S \in \mathbf{S}$) be the restriction to Q_S of the projection $U \times W \to W$.

Let $w^* \in W$ be a regular value of each π_S ($S \in \mathbf{S}$) (such w^* exists by Sard's theorem); and let $g = F_{w^*}$. g clearly has the same r'-jet as f at x_0; and it follows easily from the definition of w^* that $J^r g \mid U - \{x_0\}$ is transverse to $\mathbf{S}$. □

The proof of (3.2); and so of (0.5), is complete. □

Note added in proof. From the point of view of applications, the following generalization of (0.5) and (0.7) seems to be useful:

Let $\Omega \subset J^r_{\mathbf{C}}(n, p)$ be open, constructible and $\mathcal{K}^r$-invariant.

Let $\Omega^\tau \subset C^\tau(n, p)$ be the set of C^τ map-germs $(E^n, 0) \to (E^p, 0)$ with r-jet (or in the $\mathbf{R}$-cases, complexification of r-jet) in $\Omega(n, p)$; let $\Omega^\tau(N, P)$ be the set of C^τ maps $N \to P$ such that for all $x \in N$, $f_x \in \Omega^\tau$ (w.r.t. some local coordinates); and let $\Omega^\tau(N, P)_{S, y_0}$ be germs $(N, S) \to (P, y_0)$ of such maps.

Define ${}^k_E\sigma(\Omega)(n, p)$ to be "the codimension of the set of analytic map-germs in $\Omega^{\omega - E}$ with $\mathcal{K}$-modality $\geq k$" (as in the discussion following (0.5)).

It follows from Mather's results in [4, VI] that

$${}^1_{\mathbf{R}}\sigma(\Omega)(n, p) = {}^1_{\mathbf{C}}\sigma(\Omega)(n, p)$$

and from Wall's results in [11] that

$${}^2_{\mathbf{R}}\sigma(\Omega)(n, p) = {}^2_{\mathbf{C}}\sigma(\Omega)(n, p) \quad \text{for } n > p.$$

(0.5)′ THEOREM ($n = \dim N, p = \dim P$).

1. *Finite determinacy of C^τ map-germs in $\Omega^\tau(N, P)_{S,y_0}$ does not hold in general if*

(a) $\tau = \omega - \mathbf{R}$ *or* ∞ *and* $n >^2_{\mathbf{R}}\sigma(\Omega)(n, p)$,

(b) $\tau = \omega - \mathbf{C}$ *and* $n >^2_{\mathbf{C}}\sigma(\Omega)(n, p)$,

nor, if $|S| \geq 2$, *if* $p > 2(p - n + {}^1\sigma(\Omega)(n, p))$.

2. *Finite determinacy of C^τ map-germs in $\Omega^\tau(N, P)_{S,y_0}$ does hold in general if*

$$p \leq \max\{p - n + {}^2_{\mathbf{C}}\sigma(\Omega)(n, p), 2(p - n + {}^1_{\mathbf{C}}\sigma(\Omega)(n, p))\}.$$

(0.7)′ COROLLARY.

{*Maps of finite codimension*} *have complement of infinite codimension in* $\Omega^\infty(N, P)$ (*N compact*; $n = \dim N, p = \dim P$) *if*

$$p \leq \max\{p - n + {}^2_{\mathbf{C}}\sigma(\Omega)(n, p), 2(p - n + {}^1\sigma(\Omega)(n, p))\};$$

and have complement of finite codimension if

$$p > \max\{p - n + {}^2_{\mathbf{R}}\sigma(\Omega)(n, p), 2(p - n + {}^1\sigma(\Omega)(n, p))\}.$$

The proofs are a completely trivial adaption of the foregoing arguments.

(For the $|S| \geq 2$ clause in (0.5)′, 1, argue in the versal unfolding of a finite singularity type pair f of individually $\mathcal{K}$-unimodular germs in Ω^τ. The set Σ^* (of (3.2)) has codimension $< p$; so, by Mather [4, V, (6.1)] has a transversal point. Any embedding-germ representing this 1-jet yields, via the fibre-product construction, a map-germ f' with the given pair-$\mathcal{K}$-class which is not stable off the base-point, so not finitely-determined; and, via [7, (3.5)], the same is true for any germ with the same r-jet as f' for some finite r.)

REFERENCES

1. A. Dimca and C. G. Gibson, *Classification of equidimensional contact unimodular map germs*. Preprint no. 36 (1982), Nat. Inst. Sci. Tech., Bucharest.

2. T. Gaffney, *Properties of finitely-determined germs*, Thesis, Brandeis Univ., 1975.

3. C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math., vol. 552, Springer-Verlag, Berlin and New York, 1976.

4. J. N. Mather, *Stability of C^∞ mappings*. III, Inst. Hautes Etudes Sci. Publ. Math., **35** (1969), 127–156; IV, **37** (1970), 223–248; V, Adv. in Math. **4** (1970), 301–336; VI, Proc. Liverpool Singularities Sympos. I (C. T. C. Wall, editor), Lecture Notes in Math., vol. 192, Springer-Verlag, Berlin and New York, 1971, pp. 207–253.

5. ______, *Generic projections*, Ann. of Math. (2) **98** (1973), 226–245.

6. A. du Plessis, *On the determinacy of smooth map-germs*, Invent. Math. **58** (1980), 107–160.

7. ______, *On the genericity of topologically finitely-determined map-germs*, Topology **21** (1982), 131–156.

8. R. Thom, *Local topological properties of differentiable mappings*, Colloq. Differential Analysis (Tata Inst.), Oxford Univ. Press, 1964, pp. 191–202.

9. J.-C. Tougeron, *Idéaux de fonctions différentiables*. I, Ann. Inst. Fourier (Grenoble) **18** (1968), 177–240.

10. C. T. C Wall, *Are maps finitely determined in general*?, Bull. London Math. Soc. **11** (1979), 151–154.

11. ______, *Classification of unimodular isolated singularities of complete intersections*, these PROCEEDINGS.

MATEMATISK INSTITUT, AARHUS, DENMARK