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EXCESS INTERSECTIONS IN PROJECTIVE SPACE

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Several approaches to attach invariants to an intersection in projective space with components having excess dimension are compared, among which (1) the cone construction of Severi, leading via work of Samuel and Behrens to an approach by Stückrad and Vogel, extended by Kirby and the author, (2) Fulton and MacPherson's intersection theory and (3) an approach based on a theorem by Pieri on correspondences. The relations between them are explained and it is shown that they are just different aspects of the fundamental notion of the normal cone and its Segre class. The different approaches come each with a different generalization of Bézout's theorem, however, again there is just one fundamental relation in the background, which also gives rise to the double point formula for projections, Holme's imbedding obstructions, Ådlandsvik's join-, Peters and Simonis' secant- and Johnson's connecting formula, and a new principle for fixed points of correspondences.

Introduction

In most intersection or homology theories, one attaches to the intersection of two subvarieties X and Y of a non-singular variety P, a rational equivalence class or homology class $X \cdot Y$ of the expected dimension

$$\dim X \cdot Y = \dim X + \dim Y - \dim P$$

supported on the ambient space. This class is stable under deformations of X and Y.

Opposed to this there have been several approaches which instead relate the geometry of $X \cap Y$ to that of X and Y, even if the intersection has excess

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dimension, i.e.

$$e = \dim X \cap Y - \dim X - \dim Y + \dim P > 0,$$

and which remains interesting if the expected dimension is negative. Several lines of thought can be traced since the past century and we will describe some of these approaches.

Perhaps the oldest one stems from a theorem of Pieri on correspondences, which recently was generalized by Fulton. Severi initiated two other ways, a static and a dynamic one. The first is the cone construction. It inspired several authors to define intersection multiplicities rigorously and canonically. Along the same line, Stückrad and Vogel developed an intersecting algorithm which gives rise to a Bézout type theorem. The dynamic way Severi proposes, is based on moving the varieties. It is very subtle what deformations should be allowed to do this correctly and Lazarsfeld works this out rigorously. A variation is given by Murre. Quite sophisticated is the intersection theory of Fulton and MacPherson. They give not only an intersection product with functorial properties, but also develop a theory of Segre classes of cone bundles.

Our aim is to relate the various approaches with the Segre classes as our guideline. Straightforwardly we obtain also a new insight into well-known results as the double point-, join-, and secant formulas and Johnson's connecting formula. Furthermore, a principle for correspondences comes out, generalizing Pieri's theorem considerably. We conclude with some examples, showing aspects of the relations.

1. Severi's cone construction, Stückrad and Vogel's approach

The idea of Severi's cone construction for projective space is to enlarge the varieties to make the intersection proper. Let Z be an irreducible component of $X \cap Y$ with excess dimension e, let L_X and L_Y be two linear spaces of dimensions $e_1 - 1$ and $e_2 - 1$ in general position such that

$$e_1 + e_2 = e, \quad e_1 \geqslant 0, \quad e_2 \geqslant 0.$$

Let C_X be the cone on X with vertex L_X, i.e.

$$C_X = \bigcup_{\substack{x \in X \\ a \in L_X}} \overline{xa}$$

(and $C_X = X$ if $e_1 = 0$) and let C_Y be defined analogously. Then C_X and C_Y intersect properly along Z and the idea is to take as intersection multiplicity of X and Y along Z, that of C_X and C_Y along Z

$$i(Z; X \cdot Y) = i(Z; C_X \cdot C_Y).$$

This method in particular can be used to assign multiplicities to a proper intersection on a projective variety $V \subsetneq \mathbf{P}^n$ viewing it as an intersection on $\mathbf{P}^n$, or to prove a moving lemma for V: suppose X and Y intersect improperly on V, then $[X] = \sum n_i [X_i] \in A_* V$, for some X_i which intersect Y properly. Indeed, choose L_X of dimension $\operatorname{codim}_{\mathbf{P}^n} V - 1$ generally, and consider the proper intersection of C_X and V. It has X as a component, the rest α of the intersection will intersect Y with smaller excess dimension. The cone C_X can be moved to, say, C_X^t using projective transformations, and will then intersect Y properly. As

$$[X] = \sum_Z i(Z; C_X^t \cdot V)[Z] - \alpha \in A_* V,$$

the moving lemma follows with induction on the excess dimension (see [6] Ex. 11.4.1, p. 206).

After Weil [25] introduced the diagonal construction (the intersection of some X and Y in $\mathbf{A}^n$ can be constructed as the intersection in $\mathbf{A}^{2n}$ of $X \times Y$ with the diagonal Δ, which is a *linear* space) to define intersection multiplicities rigorously, Behrens [2] took up both ideas and defined

$$i(Z; X \cdot Y) = i(Z; C_\Delta \cdot X \times Y),$$

where the last is Weil's intersection multiplicity, here L_Δ is thought at infinity and C_Δ is just a general linear space of dimension $n + e$ containing Δ. He shows that this equals Severi's number. Just before, Samuel proposed in his fundamental paper [20], the multiplicity of the primary ideal of the diagonal in the local ring $\mathcal{O}_{X \times Y, Z}$ as intersection multiplicity and showed it satisfies some natural properties, among which the coincidence with Severi's definition. Achilles, Tworzewski and Winiarski [23] recently used the same idea and described in an analytic context what conditions L_Δ should satisfy to be general enough.

This C_Δ is a linear space, so the intersection can be realized as a series of successive hyperplane sections, where the hyperplanes are chosen generically, containing Δ. Stückrad and Vogel in their algebraic approach to intersection theory [22], did not stop at this point where irreducible components of $X \cap Y$ appear. Their idea is to lay these components aside and to continue to intersect the rest of $C_\Delta \cdot X \times Y$ with a generic hyperplane containing the diagonal. The components of this intersection contained in $X \cap Y$, are laid aside and the rest is intersected with the next generic hyperplane containing the diagonal, etc. All components laid aside together with their multiplicities form a cycle to which we refer as the Vogel cycle (a detailed description will follow below) and which we denote by $L \mathbin{\frown} J(X, Y)$. It should be noted that Vogel and Stückrad used the join construction instead of the diagonal construction and they worked in the affine cone over the join space, but this makes no difference, apart from the possible contribution of the origin, which they call the multiplicity $j_\varnothing$ of the empty set. The main result concerning the Vogel cycle is the following.

THEOREM 1.1 (Stückrad, Vogel). *The sum of $j_\varnothing$ and the degree of the Vogel cycle $L \mathbin{\frown} J(X, Y) \mathbin{\frown} J(X, Y)$ equals the Bézout number* $\deg X \cdot \deg Y$,

$$\deg L \mathbin{\frown} J(X, Y) + j_\varnothing = \deg X \cdot \deg Y.$$

This idea was extended by Kirby [14] to a more general situation, the intersection of a graded module over a graded noetherian ring with a so-called g-multiplicity system. Independently a similar generalization was described in a more geometric language in [8], we recall it below. In the meantime Philippon [18] gave a Bézout like statement in multiprojective space, bounding the multidegrees of the components of the intersection of a cycle and a collection of hypersurfaces in terms of the multidegrees of these hypersurfaces and of this cycle (see also Brownawell [4] and Fulton [7]). He used the same series of successive intersection by hypersurfaces as the above authors.

We give here Vogel's algorithm and the definition of the Vogel cycle, as in [9]. Let V be a pure dimensional scheme, $f\colon V\to Y$ a map to a scheme Y, $\mathscr{L}$ a line bundle on Y, and let $\mathscr{D}=\{D_1,\dots,D_d\}$ be a collection of effective Cartier-divisors defined by sections $s_1,\dots,s_d$ of $\mathscr{L}$. Let u_{ij} for $i,j=1,\dots,d$ be indeterminates adjoined to the ground field and call the new ground field $\mathscr{K}$. Define "generic" divisors D_j' by

$$D_j'\stackrel{\text{def}}{=}\Big(\sum_{i=1}^{d}u_{ij}s_i\Big).$$

Denote by W the fibre product $V\times_Y(D_1\cap\ldots\cap D_d)$. By induction we define effective cycles for each codimension j by the following

ALGORITHM. (to start) Decompose $V=\alpha^0+\varrho^0$ where α^0 is the part of the fundamental cycle $[V]$ supported by W, and ϱ^0 is the rest.

(induction step) Decompose

$$f^*D_j'\cdot\varrho^{j-1}=\alpha^j+\varrho^j,$$

where α^j is the part of the intersection supported by $W_{\mathscr{K}}$ and ϱ^j is the rest.

(to finish) After d steps everything will be supported by $W_{\mathscr{K}}$, so $\varrho^d=0$. We put $\alpha^j=\varrho^j=0$ for $j>d$.

Notice that the intersection in the induction step is proper, so we can take for the intersection product Weil's, Severi's, Samuel's, Serre's or Fulton's definition since they are all equivalent here.

DEFINITION. The Vogel cycle of $\mathscr{D}$ and V is

$$\mathscr{D}\cap V\stackrel{\text{def}}{=}\sum_{j\geqslant 0}\alpha^j.$$

For a correspondence T on $\mathbf{P}^n$ (i.e. a closed subscheme of $\mathbf{P}^n\times\mathbf{P}^n$) defined by a bihomogeneous ideal I in $k[x_0,\dots,x_n,y_0,\dots,y_n]$, we consider the scheme $T^*\subset\mathbf{P}^{2n+1}$ defined by the elements of I which are homogeneous in all variables. Let L_i be the hyperplane $x_i=y_i$ and let L be the collection of these hyperplanes. Then the Vogel cycle of T is by definition $L\cap T^*$. In the case

$T=X\times Y$, this T^* is the join $J=J(X,Y)$ of X and Y. Stückrad and Vogel considered the affine cones $\hat{L}_i$ and $\hat{J}$, and one has

$$\hat{L}\cap\hat{J}=\widehat{L\cap J}+j_{\emptyset}\cdot[\mathcal{O}],$$

where $\mathcal{O}$ is the vertex of the cone $\mathbf{A}^{2n+2}$ over $\mathbf{P}^{2n+1}$, so the vertex is interpreted as the cone over the empty set, defined by the irrelevant ideal. Of course, such a $j_{\emptyset}$ exists for any correspondence T.

2. Fulton and MacPherson's intersection theory

Fulton and MacPherson developed an intersection theory [6] which has both dynamic and static aspects. It attaches to an intersection of a variety mapping to an ambient scheme $V\to P$ with a regularly imbedded subscheme $D\subsetneq P$, a rational equivalence class of the expected dimension, supported on the fibred product $W=V\times_P D$. If $V\to P$ is an imbedding then considering the equivalence class on the ambient space, one recovers the usual intersection class. The central idea is to use the deformation to the normal bundle and to move the zero section in order to make the intersection proper, which, passing to rational equivalence, can be done in a canonical way. We will explain this in some more detail.

Recall the notion of the normal cone $C_W V$ of V along W. It is defined by

$$C_W V\stackrel{\text{def}}{=}\mathbf{Spec}\bigoplus_{i=0}^{\infty}\mathfrak{I}^i/\mathfrak{I}^{i+1}$$

where $\mathfrak{I}$ is the sheaf of ideals of W in V. The projectivization $P(C_W V)$ is just the exceptional divisor of the blowing up of V along W. If W is a point, $C_W V$ is the tangent cone and if W is regularly embedded in V it is the normal bundle $N_W V$, see [6] B.6.1, p. 435.

Consider the fibre diagram

$$\begin{array}{ccc} W & \subsetneq & V\\ \downarrow & & \downarrow\\ D & \subsetneq & P\end{array}$$

where $D\subsetneq P$ is any regular imbedding of codimension d. Applying the deformation to the normal bundle to this diagram, we get

$$\begin{array}{ccc} W & \subsetneq & C_W V\\ \downarrow & & \downarrow\\ D & \overset{s_0}{\subsetneq} & N_D P\end{array}$$

where s_0 is the zero section of the normal bundle $N_D P$, see [6] Ch. 5. Let N be the pullback of $N_D P$ to W with structure map $\pi\colon N\to W$. For any vector bundle E, there is a well known isomorphism

$$\pi_E^*\colon A_* W \xrightarrow{\sim} A_* E,$$

whose inverse can be thought of as intersecting with the deformed zero section (in fact any section) of E ([6] Thm. 3.3.a, p. 64 and Cor. 6.5, p. 111).

The fundamental definition of Fulton and MacPherson is ([6] 6.1, p, 94).

DEFINITION. The intersection product of V with D is

$$D \cdot V = (\pi_N^*)^{-1} [C_W V] \in A_* W.$$

It comes with a canonical decomposition: if $\sum m_i [C_i]$ is the fundamental cycle of $C_W V$, and Z_i is the support of C_i, then the canonical decomposition is

$$\sum m_i (\pi_N^*|_{Z_i})^{-1} [C_i] \in \bigoplus_i A_* Z_i.$$

The Z_i are called the distinguished varieties of the intersection.

A different way of defining the intersection product is by using Segre classes. We recall this concept.

DEFINITION. If $C \to W$ is a cone bundle (the spectrum of a sheaf of graded algebras), and $p\colon \bar{C} \to W$ is its projective closure with tautological line bundle $\mathcal{O}(-1)$, then the Segre class of a subvariety B of $\bar{C}$ is

$$s(B) \stackrel{\text{def}}{=} p_* \Big(\sum_{i=0}^{\infty} c_1 (\mathcal{O}(1))^i \cap [B] \Big) \in A_* W.$$

If $B = \bar{C}$, we write $s(C)$ for $s(\bar{C})$. For the projective closure $\bar{E} \to W$, there is the Grothendieck–Fulton isomorphism ([6] Thm. 3.3.b, p. 64 or [9]): for all k

$$c(E) \cap s(-)\colon A_k \bar{E} \xrightarrow{\sim} \bigoplus_{i=0}^{d} A_{k-i} W$$

where $c(E)$ denotes the total Chern class of E and d is the rank of E.

PROPOSITION 2.1 ([6] Prop. 6.1, p. 94). *The intersection product $D \cdot V$ is the part of $c(N) \cap s(C_W V)$ of expected dimension*

$$D \cdot V = \{c(N) \cap s(C_W V)\}_{\dim V - d}.$$

Later we will see that the normal cone and its Segre class plays also a fundamental role behind the scenes in the other theories.

We turn to a dynamic way Severi proposed to handle excess intersections of n hypersurfaces $X_1, \ldots, X_n$ in $\mathbf{P}^n$ ([21] 6.19, p. 258). The idea is to move the hypersurfaces to make the intersection proper in order to decompose the Bézout number into "numerical equivalences", attached to various subvarieties of the intersection. The numerical equivalence $i(Z)$ of some subvariety Z of $X_1 \cap \ldots \cap X_n$ is determined with induction on $\dim Z$ as follows. Consider a general deformation $X_1^t, \ldots, X_n^t$ of the hypersurfaces, then $i(Z)$ is the degree of the part of the limit intersection cycle $\lim_{t \to 0} X_1^t \cdot \ldots \cdot X_n^t$ supported on Z, diminished by the sum of the numerical equivalences of proper subvarieties of Z. Lazarsfeld [15] made this rigorous (the meaning of general is very subtle here), and explained the relationship with the intersection theory of Fulton and MacPherson, stating that the part of the limit cycle supported on Z is just the part supported on Z of the canonical decomposition of the intersection of $X_1 \times \ldots \times X_n$ and Δ in $\mathbf{P}^n \times \ldots \times \mathbf{P}^n$. Rather than its degree, the equivalence class on Z is the good notion to generalize, Fulton ([6] Remark 11.3, p. 201) shows that this class can be constructed in essentially the same way in the situation of the beginning of this section, as soon as the normal bundle $N_D P$ is generated by sections which are characteristic for some deformation.

Following a suggestion of Weil in [25], Murre [16] defined a multiplicity $i_C(X, Y)$ for a connected component C of the intersection of two subvarieties X and Y of complementary dimension in a projective variety V. He uses the same idea as in the sketch in Section 1 of the proof of the moving lemma for V, defining

$$i_C(X, Y) = \sum_W i(W; C_X^t \cdot Y) - \sum_Z i_Z(\alpha, Y)$$

where W runs through the irreducible components of $C_X^t \cap Y$ and Z runs through the connected components of $\alpha \cap Y$. This definition is only for a connected component, not for an irreducible one, thus problems concerning the choice of L_X and the deformation are avoided. De Boer [3] gave a more intrinsic reformulation of this, using an Euler–Poincaré characteristic. As above, Murre's definition could be extended to arbitrary dimension if the implicitly defined cycle class, instead of its degree would be considered. Fulton ([6] Ch. 8, p. 152) indicates that this is in fact nothing but the part supported on C of the canonical decomposition of $\Delta \cdot X \times Y$.

3. Pieri's theorem on correspondences

Implicitly an approach can be found in Pieri's paper on correspondences [19]. Let $T \subsetneq \mathbf{P}^n \times \mathbf{P}^n$ be a correspondence on $\mathbf{P}^n$. Let Δ be the diagonal, the fixed point locus is given by $\Delta \cap T$. We denote the excess dimension by e. Let

$$\mathscr{A}\colon A_0 \subset \ldots \subset A_n$$

be a flag on $\mathbf{P}^n$; it defines special Schubert cycles on $G_1 \mathbf{P}^n$:

$$\sigma(A_i) = \{l \in G_1 \mathbf{P}^n \mid l \cap A_i \neq \emptyset\}.$$

The projectivized normal cone $P(C_{\Delta \cap T} T)$ is called the *scheme of principal tangents*. It has a structure map

$$p\colon P(C_{\Delta \cap T} T) \to \Delta \cap T$$

and it is naturally imbedded in $P(T_{\mathbf{P}^n})$. There is a natural map

$$\mathbf{P}^n \times \mathbf{P}^n - - \to G_1 \mathbf{P}^n \colon (x, y) \mapsto \overline{xy},$$

which can be extended to a map φ from the blowing up $(\mathbf{P}^n \times \mathbf{P}^n)^\sim$ of $\mathbf{P}^n \times \mathbf{P}^n$ along Δ to $G_1 \mathbf{P}^n$. On the exceptional divisor $P(T_{\mathbf{P}^n})$ it sends a tangent line to the imbedded tangent line. So a principal tangent of T is a limit of lines $\overline{xy}$ with $(x, y) \in T$ specializing to some $(z, z) \in \Delta \cap T$. Define

$$v^j = p_*(\varphi^* \sigma(A_{e-j}) \cdot [P(C_{\Delta \cap T} T)]) \in Z_* \Delta \cap T,$$

so set-theoretically

$$v^j = \{z \in \Delta \cap T \mid \exists [l] \in P(C_{\Delta \cap T} T) \text{ s.t. } z \in l \text{ and } l \cap A_{e-j} \neq \emptyset\}.$$

It is not very hard to see that the irreducible components of $\Delta \cap T$ appear with nonzero multiplicity in $v(T, \mathscr{A}) \stackrel{\text{def}}{=} \sum_{j=0}^{e} v^j$. Pieri proved in 1891 the following

THEOREM 3.1 (Pieri). *For a correspondence T of dimension n and a general flag $\mathscr{A}$*

$$\text{totalbidegree } T = \deg v(T, \mathscr{A}).$$

For $T = X \times Y$, the totalbidegree of T is $\deg X \cdot \deg Y$ and $\Delta \cap T$ is $X \cap Y$, so we obtain a generalization of Bézout's theorem valid for improper intersection. Fulton gives a modern proof and a generalization to higher dimension of T ([6] Ex. 16.2.2, p. 316). In the extreme case of self-intersection $T = X \times X$, these cycles are the polar varieties of X with respect to $\mathscr{A}$, obtained as ramification loci of the restriction to X of the linear projection π_i: $\mathbf{P}^n - A_{n-i-1} \to \mathbf{P}^i$. The degree of v^j is called the j-th rank of X. They were extensively studied for a non-singular variety X as they give an extrinsic way to handle the Chern classes of X (see Section 4).

4. The relations

The relationship between the Severi–Lazarsfeld, Murre–De Boer line and Fulton–MacPherson's theory was already discussed in Section 2. Fulton relates Pieri's cycle to their intersection product by intersecting with hyperplanes (cf. [6], Ex. 16.2.1, p. 315)

$$\Delta \cdot T = \sum_{j=0}^{e} c_1(\mathcal{O}(1))^{e-j} \cap v^j \in A_* \Delta \cap T.$$

This could be done as follows. The Schubert cycles $\sigma(A_{e-j})$ represent the Chern classes of the tautological quotient bundle Q on $G_1 \mathbf{P}^n$. The bundle $\varphi^* Q$ differs by a twist with $\mathcal{O}_{\mathbf{P}^n}(-1)$ from the quotient bundle $\xi = N_\Delta \mathbf{P}^n \times \mathbf{P}^n / \mathcal{O}_N(-1)$. This

enables us to express $v(T, \mathscr{A})$ as the Segre class of the twisted cone[1] $s((C \oplus 1) \otimes \mathcal{O}(-1))$, abbreviating $C = C_{\Delta \cap T} T$, $\mathcal{O}(-1) = \mathcal{O}_{\mathbf{P}^n}(-1)$ and $N = N_\Delta \mathbf{P}^n \times \mathbf{P}^n$. Indeed

$$\begin{aligned}
v(T, \mathscr{A}) &= p_* c(\varphi^* Q) \cap [P(C)] \\
&= p_* c(\xi \otimes p^* \mathcal{O}(-1)) \cap [P(C)] \\
&= p_* c(N \otimes p^* \mathcal{O}(-1)) \cap s(\mathcal{O}_N(-1) \otimes p^* \mathcal{O}(-1)) \cap [P(C)] \\
&= p_* c(\mathcal{O}^{\oplus n+1} / p^* \mathcal{O}(-1)) \cap s(\mathcal{O}_N(-1) \otimes p^* \mathcal{O}(-1)) \cap [P(C)] \\
&= p_* s(p^* \mathcal{O}(-1)) \cap s(\mathcal{O}_{C \otimes \mathcal{O}(-1)}(-1)) \cap [P(C \otimes \mathcal{O}(-1))] \\
&= s(\mathcal{O}(-1)) \cap s(C \otimes \mathcal{O}(-1)) \\
&= s((C \otimes \mathcal{O}(-1) \oplus \mathcal{O}(-1)) \\
&= s((C \oplus 1) \otimes \mathcal{O}(-1)).
\end{aligned}$$

By the formula for the Segre classes of a cone C twisted by a line bundle L (see [9])

$$s(C \otimes L) = \sum_{j \geq 0} s(L)^j \cap s_{k-j}(C)$$

where k is the dimension of C, $s(L)$ is the total Segre class of L and $s_i(C)$ is the ith dimensional Segre class of C, we obtain as $\dim v^j = \dim C - n + e - j$

$$s(C) = \sum_{j \geq 0} s(\mathcal{O}_{\mathbf{P}^n}(1))^{n-e+j+1} \cap v^j,$$

so

$$\Delta \cdot T = \{c(N) \cap s(C)\}_{k-n} = \{\sum_j c(\mathcal{O}(1)^{\oplus e-j}) \cap v^j\}_{k-n} = \sum c_1(\mathcal{O}(1))^{e-j} \cap v^j.$$

The description of the connection between the Severi, Samuel, Behrens, Stückrad–Vogel approach and Fulton–MacPherson theory is the main result of [8] and is stated here. From the definition it is easy to see that

PROPOSITION 4.1. *For* $1 \leq j \leq d$

$$D_j \cdot \ldots \cdot D_1 \cdot V = \sum_{i \leq j} c_1(\mathscr{L})^{j-i} \cap \alpha^i + \varrho^j \in A_{k-j}(f^{-1} D_j \cap \ldots \cap f^{-1} D_1 \cap V),$$

$$\deg_{\mathscr{L}} V = \sum_{i \leq j} \deg_{\mathscr{L}} \alpha^i + \deg_{\mathscr{L}} \varrho^i.$$

In particular, for $j = d$, the relationship between the intersection product and the Vogel cycle follows. The statement about the degrees is a generalized

[1] The twisted cone is defined as follows: if $C = \mathbf{Spec}\, S^{\cdot}$ and $L = \mathbf{Spec}\, \mathbf{Sym}\, R$, then $C \otimes L \stackrel{\text{def}}{=} \mathbf{Spec}(S^{\cdot} * R)^{\cdot}$, where $(S^{\cdot} * R)^{d} = S^d \otimes R^{\otimes d}$. There is an isomorphism β: $P(C \otimes L) \to P(C)$, and we have for the tautological bundles

$$\mathcal{O}_{C \otimes L}(-1) = \beta^* \mathcal{O}_C(-1) \otimes (p \circ \beta)^* L.$$

Bézout theorem and proves Theorem 1.1 of Stückrad and Vogel and the Bézout theorems of Philippon, Kirby. However, there is a deeper lying connection.

THEOREM 4.2. *The equivalence class of the Vogel cycle is the Segre class of the twisted cone* $C_W V \otimes \mathcal{L}^\vee$

$$\mathcal{D} \cap V = s(C_W V \otimes \mathcal{L}^\vee) \in A_* W.$$

So the algorithmic and the Segre classes approach coincide up to a twist. On the one hand, this gives us insight into the geometrical meaning of the Vogel cycle, on the other hand, we have here a rather easy way to compute the Segre class. Apart from reproving the above proposition, it gives also

COROLLARY 4.3. *The coefficient of an irreducible component* Z *of* W *in* $\mathcal{D} \cap V$ *is the multiplicity* $(e_W V)_Z$ *of* V *along* W *at* Z.

The proof of Theorem 4.2 in [9] is based on the invariance of the Vogel cycle under the deformation to the normal bundle. This gives something more, namely: the distinguished varieties are precisely the only components of $\mathcal{D} \cap V$ not depending on the u_{ij}. The main problem Vogel poses in [24], is to describe the contribution of imbedded components of $\Delta \cap X \times Y$ to the Bézout number $\deg X \cdot \deg Y$. If this contribution is to be measured by the Vogel cycle, this question is changed to the characterization of those imbedded components which give rise to distinguished varieties of $\Delta \cap X \times Y$. This is in fact a purely algebraic question: given an ideal I in a ring R, what are the minimal primes of the associated graded ring $\mathrm{gr}_I R$? An example by Flenner [5] shows that there is no hope for bounding the number of imbedded components by the Bézout number.

From the invariance of the Vogel cycle also follows that for a correspondence T on $\mathbf{P}^n$ not only

$$L \cap T^* = v(T, \mathcal{A})$$

as equivalence classes, as both equal

$$s\big(C_{L \cap T^*} T^* \otimes p^* \mathcal{O}(-1)\big) = s\big((C_{\Delta \cap T} T \oplus 1) \otimes p^* \mathcal{O}(-1)\big),$$

but in fact also as cycles, once we choose $\mathcal{A}$ to be the generic flag

$$A_j = V\Big(\sum_{i=0}^{n} u_{i+1,h} x_i \,|\, j+1 \leqslant h \leqslant n\Big).$$

For the self-intersection $T = X \times X$, it follows that the Vogel cycle consists of ramification classes which are just twisted Johnson–Segre classes $s(C_X X \times X)$

$$v(X \times X, \mathcal{A}) = s\big((C_X X \times X \oplus 1) \otimes \mathcal{O}(-1)\big),$$

so if X is non-singular

$$v(X \times X, \mathcal{A}) = s\big((T_X \oplus 1) \otimes \mathcal{O}(-1)\big) \cap [X].$$

If X is singular, Stückrad and Vogel's algorithm enables us to compute the contribution of singularities to the Johnson–Segre classes (see [10]).

5. On Stückrad and Vogel's algorithm

In view of the generalized Bézout theorem and to understand Stückrad and Vogel's algorithm better, it would be interesting to have a geometric interpretation in the case of a correspondence for the rest cycles ϱ^j and the multiplicity of the empty set $j_\emptyset$ as well. We recall the description from [9]. Consider the rational map relating the join and the diagonal space

$$\Phi \colon \mathbf{P}^{2n+1} - - \to \mathbf{P}^n \times \mathbf{P}^n,$$

$$(x_0 : \ldots : x_n : y_0 : \ldots : y_n) \mapsto ((x_0 : \ldots : x_n), (y_0 : \ldots : y_n))$$

which is not defined in $x_0 = \ldots = x_n = 0$ and in $y_0 = \ldots = y_n = 0$, it gives an isomorphism between L and Δ. Denote the strict transform of T under the blowing up $(\mathbf{P}^n \times \mathbf{P}^n)^\sim$ of $\mathbf{P}^n \times \mathbf{P}^n$ along Δ by $p \colon \tilde{T} \to T$. Let $\varphi \colon (\mathbf{P}^n \times \mathbf{P}^n)^\sim \to G_1 \mathbf{P}^n$ and the Schubert cycle $\sigma(A_{n-j-1})$ be as in Section 3, where we choose for $\mathcal{A}$ the generic flag of Section 4.

DEFINITION. The cycle of new fixed points $NF(T, \pi_j)$ of the correspondence T under the projection $\pi_j \colon \mathbf{P}^n - A_{n-j-1} \to \mathbf{P}^j$ is

$$NF(T, \pi_j) \stackrel{\text{def}}{=} \overline{p_*(\varphi^* \sigma(A_{n-j-1}) \cdot \tilde{T})}$$

Indeed, it consists of points of T which are generically not fixed, but become fixed under π_j. In the case $T = X \times X$, the new fixed point cycle is just the double point cycle $\mathbf{D}(\pi_j)$ of π_j restricted to X.

PROPOSITION 5.1. *The push forward under* Φ *of* ϱ^j *is the new fixed point cycle of* T *under* π_{j-1}

$$\Phi_* \varrho^j = NF(T, \pi_{j-1}).$$

If $T = X \times X$, the defining relation

$$L_{j+1} \cdot \varrho^j = \alpha^{j+1} + \varrho^{j+1}$$

pushes forward to

$$c_1(\mathcal{O}(1)) \cdot \mathbf{D}(\pi_{j-1}) = \Phi_* \alpha^{j+1} + \mathbf{D}(\pi_j) \in A_* X,$$

(and $\Phi_* \alpha^{j+1} = v^{e-n+j}(X \times X)$ is the ramification cycle of $\pi_{j-1}|_X$), this is the connecting formula found by Johnson [13] and explained by Hansen using the diagonal construction [11]. Indeed, (forget Φ_* for a moment) eliminating one variable less, i.e. considering π_j instead of π_{j-1} comes down to intersecting the double point cycle $\mathbf{D}(\pi_{j-1})$ with the hyperplane L'_j corresponding to this variable in the join space. The intersection cycle decomposes into a part not contained in the diagonal, consisting of pairs of points (x, y) whose line joining them $\overline{xy}$ hits the smaller projection centre A_{n-j-1}, thus giving $\mathbf{D}(\pi_j)$, and a part contained in the diagonal, consisting of points with tangents hitting A_{n-j}, thus giving the ramification cycle α^{j+1}

The equality of Proposition 4.1

$$L'_j \cdot \ldots \cdot L'_1 \cdot (X \times X)^* = \sum_{k \leq j} c_1(\mathcal{O}(1))^{j-k} \cap \alpha^k + \varrho^j$$

pushes forward via Φ_* to

$$\Delta_{\mathbf{P}^{j-1}} \cdot X \times X = \sum_{k \leq j} c_1(\mathcal{O}(1))^{j-k} \cap \alpha^k + \mathbf{D}(\pi_{j-1})$$

and as

$$\sum_{k \leq j} c_1(\mathcal{O}(1))^{j-k} \cap \alpha^k = \{\pi_{j-1}^* c(T_{\mathbf{P}^{j-1}}) \cap c(T_X)^{-1} \cap [X]\}_{2d+1-j},$$

one recovers the well-known double point formula for linear projections (cf. [6] Th. 9.3, p. 166).

The multiplicity $j_\emptyset$ of the empty set has a similar interpretation. Consider the incidence correspondence $I = \{([l], x) \in G_1 \mathbf{P}^n \times \mathbf{P}^n | x \in l\}$ with natural projections $p_1\colon I \to G_1 \mathbf{P}^n$ and $p_2\colon I \to \mathbf{P}^n$.

Definition. The span T^s of a correspondence T on $\mathbf{P}^n$ is

$$T^s \stackrel{\text{def}}{=} p_2\big(p_1^{-1}(\varphi(\tilde{T}))\big) = \bigcup_{(x,y) \in T} \overline{xy}.$$

The proper part $\bar{T}$ of the span is the part of expected dimension $\dim T + 1$ of the fundamental cycle of T^s

$$\bar{T} \stackrel{\text{def}}{=} (p_2)_* p_1^* \varphi_* \tilde{T} \in Z_{\dim T+1} T^s.$$

For $T = X \times Y$, this T^s is the imbedded join in $\mathbf{P}^n$ of X and Y, for $T = X \times X$, it is the secant variety.

Proposition 5.2. *The multiplicity of the empty set $j_\emptyset$ is the degree of the proper part of the span of the correspondence*

$$j_\emptyset = \deg \bar{T}.$$

6. The generalized Bézout theorem

Now that we have this geometric interpretation of the cycles of Vogel's algorithm and of the multiplicity of the empty set at our disposal, it is worth reconsidering the generalized Bézout theorem for a correspondence. We obtain, following [9], from Proposition 4.1 applied in the cone over the join space the following principle for fixed points.

Theorem 6.1. *For a correspondence T on $\mathbf{P}^n$ the following relation holds*

$$\deg v(T) + \deg \bar{T} = \text{totalbidegree } T.$$

Corollary 6.2. *For a correspondence T on $\mathbf{P}^n$ the following relations hold*

(1) *T has no fixed points iff* $\deg \bar{T} = \text{totalbidegree } T$.

(2) *T has fixed points if* $\dim T^s < \dim T + 1$.

(3) *Let $\mathcal{T} \to B$ be a flat family of correspondences on $\mathbf{P}^n$. The change in the degree of $\bar{T}$ is compensated by that of $v(T)$*

$$\lim_{b \to 0} \deg \bar{\mathcal{T}}_b - \deg \bar{\mathcal{T}}_0 + \lim_{b \to 0} \deg v(\mathcal{T}_b) - \deg v(\mathcal{T}_0) = 0.$$

If $\dim T \geq n$, then $\bar{T} = 0$ as $\dim T^s \leq n < \dim T + 1$, so Pieri–Fulton's theorem reappears. If $T = X \times Y$, the above is Stückrad and Vogel's theorem, but it can also be read as

$$\deg \overline{X \times Y} = \deg X \cdot \deg Y - \deg v(X \times Y).$$

Applying the formula for Segre classes for twisted cones and standard arguments for Segre classes of bundles, we get a formula due to Ådlandsvik [1]:

$$\deg \overline{X \times Y} = \deg X \cdot \deg Y - \sum_i \binom{l+m+1}{l+m-i} \deg s_{l+m-i}(C_{X \cap Y} X \times Y)$$

where $l = \dim X$, $m = \dim Y$ and $s_j(C_{X \cap Y} X \times Y)$ denotes the part of the Segre class of dimension j. The cycle $\overline{X \times X}$ is twice the secant variety if this has the right dimension $2 \dim X + 1$, and we get

$$2 \deg \operatorname{Sec} X = (\deg X)^2 - \deg v(X \times X).$$

For non-singular X, expressing $v(X \times X)$ in the Chern classes of X, a formula due to Peters and Simonis [17] reappears.

If the secant variety has not the expected dimension, X might be imbedded in a smaller projective space via a linear projection π_j. The restriction $\pi_j|_X$ is for non-singular X an imbedding iff $\mathbf{D}(\pi_j) = 0$ (indeed, there are no ramification points by the connecting formula), and we see that this is the case iff

$$0 = \deg \varrho^{j+1} = (\deg X)^2 - \sum_{k=0}^{j+1} \deg \alpha^k.$$

Expressing the α^k in the Chern classes of X, we obtain Holme's imbedding obstructions [12]. For X singular, these obstructions are more subtle, however under some conditions, the same result holds (see [12]).

Concluding the comparison of the different approaches to excess intersections, we see that the normal cone $C_{X \cap Y} X \times Y$ plays a fundamental role behind all of them. Different approaches centre about different aspects of the cone. In the Severi, Samuel, Behrens, Stückrad–Vogel line in fact the Segre class of the twisted cone $s(C \otimes \mathcal{O}(-1))$ was studied, first only the top-dimensional components, although the way of arriving at these classes is very different from their definition. In the Pieri–Fulton approach, the focus is on the extrinsic geometry of the cone, the way it is situated in projective space, measured by pulled back Schubert cycles. Fulton–MacPherson's intersection product, and also the Severi, Murre, De Boer, Lazarsfeld multiplicities, describe the stable part under deformation of the Segre class $\{c(N_\Delta)\cap s(C)\}_{\dim X+\dim Y-n}$.

In the same way, the generalized Bézout relation

$$\deg_{\mathscr{S}} V = \deg_{\mathscr{S}} \mathscr{D} \cap V + \deg_{\mathscr{S}} \varrho^{\dim V}$$

gives the Philippon and Kirby versions of Bézout's theorem and gives for a correspondence a fixed point principle

$$\deg v(T) + \deg \bar{T} = \text{totalbidegree } T,$$

which gives rise to statements concerning the intersection product and the extrinsic geometry (Pieri's theorem, Stückrad and Vogel's Bézout theorem and double point-, imbeddings-, secant- and join formulas).

7. Examples

(1) The self-intersection of a hypersurface in $\mathbf{P}^n$

Let X be a reduced hypersurface of degree d defined by $F=0$, let $\mathscr{A}$: $A_0 \subset \ldots \subset A_{n-1} \subset \mathbf{P}^n$ be a flag. Then every tangent space meets A_1, so

$$v^0(X\times X) = p_*\big(\varphi^*\sigma(A_1)\cdot[P(C_X X\times X)]\big) = [X].$$

The zeroes of $\sum_i a_i\,\partial F/\partial x_i$ form the polar variety with respect to $A_0 = (a_0:a_1:\ldots:a_n)$ and

$$v^1(X\times X, \mathscr{A}) = p_*\big(\varphi^*\sigma(A_0)\cdot[P(C_X X\times X)]\big) = V\Big(\sum_i a_i \frac{\partial F}{\partial x_i}\Big)\cap[X].$$

So only the singularities in codimension 1 contribute to the Vogel cycle, with the multiplicity of the Jacobian ideal. We can compute the Johnson–Segre classes of X using the formula for twisting Segre classes (cf. Yokura [26]). Bézout theorem gives indeed

$$d^2 = d + d(d-1) = \deg v^0(X\times X) + \deg v^1(X\times X).$$

(2) The non-singular case

Suppose X, Y and $W = X\cap Y$ are non-singular. The excess normal bundle of the intersection is defined as

$$E \overset{\text{def}}{=} N_Y\mathbf{P}^n|_W / N_W X.$$

Then

$$v(X\times Y) = s\big(N_W J(X, Y)\otimes\mathcal{O}(-1)\big)\cap[W] = c\big(E\otimes\mathcal{O}(-1)\big)\cap[W].$$

This follows from the exact sequence

$$0\to N_W J\to N_L\mathbf{P}^{2n+1}|_W\to E\to 0$$

after tensoring with $\mathcal{O}(-1)$. Fulton ([6] Ex. 12.3.6, p. 225) already gives for the case $\dim X+\dim Y \geqslant n$

$$\deg X\cdot\deg Y = \deg c\big(E\otimes\mathcal{O}(-1)\big)\cap[W].$$

(3) Bilinearity

Let the curves C and D in $\mathbf{P}^2$ be defined by $x^2y=0$ and $xy^2=0$. Note that the Vogel cycle $v(X\times Y)$ is defined on cycles and therefore is bilinear in X and Y and that

$$\begin{aligned}[C\times D] &= \big(2[V(x)]+[V(y)]\big)\times\big([V(x)]+2[V(y)]\big)\\ &= 2[V(x)\times V(x)]+2[V(y)\times V(y)]+5[V(x)\times V(y)].\end{aligned}$$

It is easily seen (e.g. using the Bézout theorem) that $v\big(V(x)\times V(x)\big) = [V(x)]$, hence

$$v(C\times D) = 2[V(x)]+2[V(y)]+5[V(x, y)].$$

Indeed, $2+2+5 = 3\times 3$.

(4) The imbedded join

Let X_b and Y_b be flat families of projective varieties of degrees d_1 and d_2, over a non-singular curve B, disjoint for $b\neq 0$ and having a finite number of points $P_1, \ldots, P_t$ in common for $b=0$, with multiplicities $m_i = (e_{X_0\cap Y_0} X_0\times Y_0)_{P_i}$. Then the imbedded joins satisfy

$$\deg\overline{X_b\times Y_b} = d_1 d_2 \qquad (b\neq 0),$$
$$\deg\overline{X_0\times Y_0} = d_1 d_2 - \sum_i m_i.$$

The limit cycle $\lim_{b\to 0}\overline{X_b\times Y_b}$ falls apart in $\overline{X_0\times Y_0}$ and components of degree m_i for each point of intersection P_i.

References

[1] B. Ådlandsvik, *Joins and higher secant varieties*, Math. Scand. 61 (1987), 213–222.

[2] E.-A. Behrens, *Zur Schnittmultiplizität uneigentlicher Komponenten in der algebraischen Geometrie*, Math. Z. 55 (1952), 199–215.

[3] J. H. de Boer, *The intersection multiplicity of a connected component*, Math. Ann 172 (1967), 238–246.

[4] W. D. Brownawell, *Note on a paper of P. Philippon*, Michigan Math. J. 34 (1987), 461–464.

[5] H. Flenner, *On imbedded components of improper intersections in projective space*, Archiv Math. (6) 48 (1987), 533–537.

[6] W. Fulton, *Intersection Theory*, Ergebnisse der Math. 3, Bd. 2, Springer-Verlag, Berlin Heidelberg 1984.

[7] —, *Algebraic refined intersections*, Notes for Math. 271, Brown University, Providence, 1986.

[8] L. J. van Gastel, *A geometric approach to Vogel's intersection theory*, Preprint 427, University of Utrecht, 1986.

[9] L. J. van Gastel, *Excess intersection and a correspondence principle*, to appear in Invent. Math.
[10] —, *Segre classes and singularities*, to appear.
[11] J. Hansen, *Double points of compositions*, J. Reine Angew. Math. 352 (1984), 71–80.
[12] A. Holme, *Embedding obstructions for singular varieties in* $\mathbf{P}^n$, Acta Math. 135 (1975), 155–185.
[13] K. W. Johnson, *Immersion and embedding of projective varieties*, ibid, 140 (1978), 49–74.
[14] D. Kirby, *On Bézout's theorem*, Quart. J. Math. Oxford (2) 39 (1988), 469–481.
[15] R. Lazarsfeld, *Excess intersections of divisors*, Compositio Math. 43 (1981), 281–296.
[16] J. P. Murre, *Intersection multiplicities of maximal connected bunches*, Amer. J. Math. 80 (1958), 311–330.
[17] C. A. M. Peters and J. Simonis, *A secant formula*, Quart. J. Math. Oxford 27 (2) (1976), 181–189.
[18] P. Philippon, *Lemmes de zéros dans les groupes algébriques commutatifs*, Bull. Soc. Math. France 114 (1986), 355–383.
[19] M. Pieri, *Formule di coincidenza per le serie algebriche* ∞^n *di coppie di punti dello spazio a n dimensioni*, Rend. Circ. Mat. Palermo 5 (1891), 18–29.
[20] P. Samuel, *La notion de multiplicité en algèbre et en géométrie algébrique*, J. Math. Pures Appl. 30 (1951), 159–274.
[21] F. Severi, *Il concetto generale di molteplicità delle soluzioni pei sistemi di equazioni algebriche e la teoria dell'eliminazione*, Ann. Mat. (4) 26 (1947), 221–270.
[22] J. Stückrad and W. Vogel, *An algebraic approach to the intersection theory*, in: Curves Seminar at Queen's University, Vol. II, Queen's Papers Pure Appl. Math. 61, Kingston, Ontario 1982, 1–32.
[23] P. Tworzewski, T. Winiarski, and R. Achilles, *On improper isolated intersection in complex analytic geometry*, to appear in Ann. Polon. Math.
[24] W. Vogel, *Results on Bézout's Theorem*, Tata Lecture Notes 74, Springer-Verlag, Heidelberg 1984.
[25] A. Weil, *Foundations of Algebraic Geometry*, Amer. Math. Soc. Colloq. Publ. 29, 1946.
[26] S. Yokura, *Polar classes and Segre classes on singular projective varieties*, Trans. Amer. Math. Soc. 298 (1986), 169–192.