משוואת החום

I. משוואת החום ההומוגנית $u_t=\alpha^2 u_{xx}\,,(\alpha\neq 0,t>0,0<x<l)\,,\quad u(x,0)=f(x)$

$u(0,t)=u(l,t)=0$	$\left\{\sin\frac{n\pi x}{l}\right\}$	$u(x,t)=\sum_{n=1}^{\infty}b_n e^{-\left(\alpha\frac{n\pi}{l}\right)^2 t}\sin\frac{n\pi x}{l}$
$u_x(0,t)=u(l,t)=0$	$\left\{\cos\frac{(2n+1)\pi x}{2l}\right\}$	$u(x,t)=\sum_{n=0}^{\infty}b_n e^{-\left(\alpha\frac{(2n+1)\pi}{2l}\right)^2 t}\cos\frac{(2n+1)\pi x}{2l}$
$u(0,t)=u_x(l,t)=0$	$\left\{\sin\frac{(2n+1)\pi x}{2l}\right\}$	$u(x,t)=\sum_{n=0}^{\infty}b_n e^{-\left(\alpha\frac{(2n+1)\pi}{2l}\right)^2 t}\sin\frac{(2n+1)\pi x}{2l}$
$u_x(0,t)=u_x(l,t)=0$	$\left\{1,\cos\frac{n\pi x}{l}\right\}$	$u(x,t)=\frac{b_0}{2}+\sum_{n=1}^{\infty}b_n e^{-\left(\alpha\frac{n\pi}{l}\right)^2 t}\cos\frac{n\pi x}{l}$

II. משוואת חום לא-הומוגנית ותנאי שפה הומוגניים

$u_t=\alpha^2 u_{xx}+g(x,t)\,,(\alpha\neq 0,t>0,0<x<l)\,,\quad u(x,0)=f(x)$

$u(0,t)=u(l,t)=0$	$u(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin\frac{n\pi x}{l}\,,\quad u_n(0)=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ $g(x,t)=\sum_{n=1}^{\infty}g_n(t)\sin\frac{n\pi x}{l}\,,\quad g_n(t)=\frac{2}{l}\int_0^l g(x,t)\sin\frac{n\pi x}{l}dx$
$u_x(0,t)=u(l,t)=0$	$u(x,t)=\sum_{n=0}^{\infty}u_n(t)\cos\frac{(2n+1)\pi x}{2l},u_n(0)=\frac{2}{l}\int_0^l f(x)\cos\frac{(2n+1)\pi x}{2l}dx$ $g(x,t)=\sum_{n=0}^{\infty}g_n(t)\cos\frac{(2n+1)\pi x}{2l},g_n(t)=\frac{2}{l}\int_0^l g(x,t)\cos\frac{(2n+1)\pi x}{2l}dx$
$u(0,t)=u_x(l,t)=0$	$u(x,t)=\sum_{n=0}^{\infty}u_n(t)\sin\frac{(2n+1)\pi x}{2l},\ u_n(0)=\frac{2}{l}\int_0^l f(x)\sin\frac{(2n+1)\pi x}{2l}dx$ $g(x,t)=\sum_{n=0}^{\infty}g_n(t)\sin\frac{(2n+1)\pi x}{2l},\ g_n(t)=\frac{2}{l}\int_0^l g(x,t)\sin\frac{(2n+1)\pi x}{2l}dx$
$u_x(0,t)=u_x(l,t)=0$	$u(x,t)=\frac{u_0(t)}{2}+\sum_{n=1}^{\infty}u_n(t)\cos\frac{n\pi x}{l},\ g(x,t)=\frac{g_0(t)}{2}+\sum_{n=1}^{\infty}g_n(t)\cos\frac{n\pi x}{l}$ $u_0(0)=\frac{2}{l}\int_0^l f(x)dx,\quad u_n(0)=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$ $g_0(t)=\frac{2}{l}\int_0^l g(x,t)dx,\quad g_n(t)=\frac{2}{l}\int_0^l g(x,t)\cos\frac{n\pi x}{l}dx$

III. משוואת חום לא-הומוגנית ותנאי שפה לא-הומוגניים

$u_t = \alpha^2 u_{xx} + g(x,t)$, $(\alpha \neq 0, t \geq 0, 0 \leq x \leq l)$, $u(x,0) = f(x)$

III.1. $u(x,t) = v(x,t) + A(t) + xB(t)$

$u(0,t) = \varphi_1(t)$ $u(l,t) = \varphi_2(t)$	$v(0,t) = v(l,t) = 0$
$u_x(0,t) = \varphi_1(t)$ $u(l,t) = \varphi_2(t)$	$v_x(0,t) = v(l,t) = 0$
$u(0,t) = \varphi_1(t)$ $u_x(l,t) = \varphi_2(t)$	$v(0,t) = v_x(l,t) = 0$

III.2. $u(x,t) = v(x,t) + xA(t) + x^2 B(t)$

$u_x(0,t) = \varphi_1(t)$ $u_x(l,t) = \varphi_2(t)$	$v_x(0,t) = v_x(l,t) = 0$

IV. $u_t = \alpha^2 u_{xx} + bu + g(x,t)$, $(\alpha \neq 0, t \geq 0, 0 \leq x \leq l)$, $u(x,0) = f(x)$

$u(x,t) = v(x,t)\, e^{bt}$

דוגמאה 1 :

$$u_t = 16 u_{xx} \, , \, u_x(0,t) = u(\pi,t) = 0 \, , \; t > 0, \; u(x,0) = 4\cos\frac{3x}{2} - 5\cos\frac{7x}{2}, \, 0 < x < \pi$$

פתרון

$$\left.\begin{array}{l} u_t = 16 u_{xx} \\ u_x(0,t) = 0 \\ u(\pi,t) = 0 \end{array}\right\} \Rightarrow u(x,t) = \sum_{n=0}^{\infty} b_n e^{-4(2n+1)^2 t} \cos\frac{(2n+1)x}{2}$$

$$u(x,0) = \sum_{n=0}^{\infty} b_n \cos\frac{(2n+1)x}{2} = 4\cos\frac{3x}{2} - 5\cos\frac{7x}{2} \Rightarrow \begin{cases} b_1 = 4 \\ b_3 = -5 \\ b_n = 0 \quad , \quad n \neq 1, n \neq 3 \end{cases}$$

$$u(x,t) = 4e^{-36t}\cos\frac{3x}{2} - 5e^{-196t}\cos\frac{7x}{2}$$

דוגמאה 2 :

$$u_t = u_{xx} + 3 + t\cos 2x\,,\ u_x(0,t) = u_x(\pi,t) = 0\,,\ t>0,\ u(x,0) = \begin{cases}\pi, 0<x<\pi/2\\ 0, \pi/2<x<\pi\end{cases}$$

<u>פתרון</u>

$$u_x(0,t) = u_x(\pi,t) = 0 \Rightarrow u(x,t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t)\cos nx$$

$$u_0(0) = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} u(x,0)dx = \frac{2}{\pi}\int_0^{\pi/2} \pi\, dx = \pi$$

$$u_n(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi/2} \pi\cos nx\, dx = \frac{2}{n}\sin\frac{n\pi}{2}$$

$$g(x,t) = \frac{g_0(t)}{2} + \sum_{n=1}^{\infty} g_n(t)\cos nx = 3 + t\cos 2x \Rightarrow g_n(t) = \begin{cases}6 & , \quad n=0\\ t & , \quad n=2\\ 0 & , \quad n\neq 0, n\neq 2\end{cases}$$

$$\left.\begin{array}{l} u_t = u_{xx} + g(x,t)\\ u(x,t) = \dfrac{u_0(t)}{2} + \displaystyle\sum_{n=1}^{\infty} u_n(t)\cos nx\\ g(x,t) = \dfrac{g_0(t)}{2} + \displaystyle\sum_{n=1}^{\infty} g_n(t)\cos nx \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{u_0'(t)}{2} + \displaystyle\sum_{n=1}^{\infty} u_n'(t)\cos nx = \sum_{n=1}^{\infty} -n^2 u_n(t)\cos nx +\\ + \dfrac{g_0(t)}{2} + \displaystyle\sum_{n=1}^{\infty} g_n(t)\cos nx \end{array}\right\} \Rightarrow$$

$$\Rightarrow u_0'(t) = g_0(t) \quad , \quad u_n'(t) + n^2 u_n(t) = g_n(t)$$

$$\left.\begin{array}{l} u_0'(t) = g_0(t) \quad , \quad u_n'(t) + n^2 u_n(t) = g_n(t) \quad , \quad u_0(0) = \pi \quad , \quad u_n(0) = \dfrac{2}{n}\sin\dfrac{n\pi}{2}\\ g_n(t) = \begin{cases}6 & , \quad n=0\\ t & , \quad n=2\\ 0 & , \quad n\neq 0, n\neq 2\end{cases} \end{array}\right\} \Rightarrow$$

$$u_0'(t) = 6\,, \quad u_0(0) = \pi \Rightarrow u_0(t) = 6t + \pi$$

$$u_2'(t) + 4u_2(t) = t\,,\ u_2(0) = 0 \Rightarrow u_2(t) = \frac{1}{16}e^{-4t} + \frac{1}{4}t - \frac{1}{16}$$

$$\left.\begin{array}{l} u_n'(t) + n^2 u_n(t) = 0\,,\ u_n(0) = \dfrac{2}{n}\sin\dfrac{n\pi}{2}\\ n\neq 0, n\neq 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} u_n(t) = \dfrac{2}{n}\sin\dfrac{n\pi}{2}e^{-n^2 t}\\ n\neq 0, n\neq 2 \end{array}\right\}$$

$$u(x,t) = \frac{6t+\pi}{2} + \sum_{n=1}^{\infty} u_n(t)\cos nx\,, \quad u_n(t) = \begin{cases}\dfrac{1}{16}e^{-4t} + \dfrac{1}{4}t - \dfrac{1}{16} & , n=2\\ u_n(t) = \dfrac{2}{n}\sin\dfrac{n\pi}{2}e^{-n^2 t} & n\neq 2\end{cases}$$

V. מצא פתרון פרטי עבור משוואת החום הנתונה המקיים את התנאים המצורפים

1) $u_t = u_{xx}$, $u(0,t) = u(1,t) = 0$, $t > 0$, $u(x,0) = x - x^2$, $0 < x < 1$

2) $u_t = 4u_{xx}$, $u(0,t) = u(2,t) = 0$, $t > 0$, $u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

3) $u_t = u_{xx}$, $u_x(0,t) = u(\pi,t) = 0$, $t > 0$, $u(x,0) = 3\cos\frac{5x}{2}$, $0 \le x \le \pi$

4) $u_t = 4u_{xx}$, $u_x(0,t) = u_x(\pi,t) = 0$, $t > 0$, $u(x,0) = \cos^2 x$, $0 < x < \pi$

5) $u_t = u_{xx} + t$, $u(0,t) = u_x(\pi,t) = 0$, $t > 0$, $u(x,0) = x^2 - 2\pi x$, $0 < x < \pi$

6) $u_t = u_{xx} + t\sin x$, $u(0,t) = u(\pi,t) = 0$, $t > 0$, $u(x,0) = 3\sin 5x$, $0 < x < \pi$

7) $u_t = 9u_{xx}$, $u(0,t) = 0$, $u(\pi,t) = e^{-t}$, $t > 0$, $u(x,0) = \frac{x}{\pi}$, $0 < x < \pi$

8) $u_t = 2u_{xx} - 3u + \cos x$, $u_x(0,t) = u_x(\pi,t) = 0$, $t > 0$, $u(x,0) = \cos 2x$, $0 < x < \pi$

9) $u_t = 2u_{xx} - u + x(t+1)$, $u(0,t) = 1$, $u_x(\pi,t) = t$, $t > 0$, $u(x,0) = 1 + 4\sin\frac{3x}{2}$, $0 < x < \pi$

תשובות:

1) $u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\, e^{-n^2\pi^2 t}\sin n\pi x = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\, e^{-(2n-1)^2\pi^2 t}\sin(2n-1)\pi x$

2) $u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\, e^{-n^2\pi^2 t}\sin\frac{n\pi x}{2}$

3) $u(x,t) = 3e^{-\frac{25}{4}t}\cos\frac{5x}{2}$

4) $u(x,t) = 0.5 + 0.5e^{-16t}\cos 2x$

5) $u(x,t) = \sum_{n=0}^{\infty}\frac{16}{\pi(2n+1)^3}\left[\left(-2 + \frac{4}{(2n+1)^2}\right)e^{-\frac{(2n+1)^2}{4}t} + t - \frac{4}{(2n+1)^2}\right]\sin\frac{(2n+1)x}{2}$

6) $u(x,t) = (e^{-t} + t - 1)\sin x + 3e^{-25t}\sin 5x$

7) $u(x,t) = \frac{e^{-t}}{\pi}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(9n^2-1)}\,(e^{-9n^2 t} - e^{-t})\sin nx$

8) $u(x,t) = \frac{1}{5}(1 - e^{-5t})\cos x + e^{-11t}\cos 2x$

9) $$\begin{cases} u(x,t) = 1 + xt + e^{-t}\sum_{n=0}^{\infty}\left[\left(w_n(0) - \frac{\gamma_n}{\alpha_n + 1}\right)e^{-\alpha_n t} + \frac{\gamma_n}{\alpha_n + 1}e^{t}\right]\sin\frac{(2n+1)x}{2} \\ w_n(0) = \begin{cases} 4, & n = 1 \\ 0, & n \ne 1 \end{cases}, \quad \gamma_n = -\frac{4}{\pi(2n+1)}, \quad \alpha_n = \frac{(2n+1)^2}{2} \end{cases}$$