

# Series Solutions of Second Order Linear Equations

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Finding the general solution of a linear differential equation rests on determining a fundamental set of solutions of the homogeneous equation. So far, we have given a systematic procedure for constructing fundamental solutions only if the equation has constant coefficients. To deal with the much larger class of equations having variable coefficients it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to that in the method of undetermined coefficients: We assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation.

## 5.1 Review of Power Series

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In this chapter we discuss the use of power series to construct fundamental sets of solutions of second order linear differential equations whose coefficients are functions of the independent variable. We begin by summarizing very briefly the pertinent results about power series that we need. Readers who are familiar with power series may go on to Section 5.2. Those who need more details than are presented here should consult a book on calculus.

1. A power series  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$  is said to converge at a point  $x$  if

$$\lim_{m \rightarrow \infty} \sum_{n=0}^m a_n(x - x_0)^n$$

exists for that  $x$ . The series certainly converges for  $x = x_0$ ; it may converge for all  $x$ , or it may converge for some values of  $x$  and not for others.

2. The series  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$  is said to converge absolutely at a point  $x$  if the series

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n$$

converges. It can be shown that if the series converges absolutely, then the series also converges; however, the converse is not necessarily true.

3. One of the most useful tests for the absolute convergence of a power series is the ratio test. If  $a_n \neq 0$ , and if for a fixed value of  $x$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = |x - x_0| \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L|x - x_0|,$$

then the power series converges absolutely at that value of  $x$  if  $|x - x_0| < 1/L$ , and diverges if  $|x - x_0| > 1/L$ . If  $|x - x_0| = 1/L$ , the test is inconclusive.

### EXAMPLE 1

For which values of  $x$  does the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n(x - 2)^n$$

converge?

To test for convergence we use the ratio test. We have

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n} \right| = |x-2| \lim_{n \rightarrow \infty} \frac{n+1}{n} = |x-2|.$$

According to statement 3 the series converges absolutely for  $|x - 2| < 1$ , or  $1 < x < 3$ , and diverges for  $|x - 2| > 1$ . The values of  $x$  corresponding to  $|x - 2| = 1$  are  $x = 1$  and  $x = 3$ . The series diverges for each of these values of  $x$  since the  $n$ th term of the series does not approach zero as  $n \rightarrow \infty$ .

4. If the power series  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$  converges at  $x = x_1$ , it converges absolutely for  $|x - x_0| < |x_1 - x_0|$ ; and if it diverges at  $x = x_1$ , it diverges for  $|x - x_0| > |x_1 - x_0|$ .
5. There is a nonnegative number  $\rho$ , called the **radius of convergence**, such that  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$  converges absolutely for  $|x - x_0| < \rho$  and diverges for  $|x - x_0| > \rho$ . For a series that converges only at  $x_0$ , we define  $\rho$  to be zero; for a series that

converges for all  $x$ , we say that  $\rho$  is infinite. If  $\rho > 0$ , then the interval  $|x - x_0| < \rho$  is called the **interval of convergence**; it is indicated by the hatched lines in Figure 5.1.1. The series may either converge or diverge when  $|x - x_0| = \rho$ .

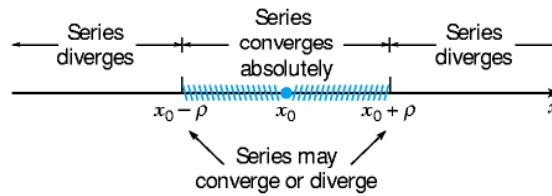


FIGURE 5.1.1 The interval of convergence of a power series.

**EXAMPLE**  
**2**

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}.$$

We apply the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{(x+1)^n} \right| = \frac{|x+1|}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{|x+1|}{2}.$$

Thus the series converges absolutely for  $|x+1| < 2$ , or  $-3 < x < 1$ , and diverges for  $|x+1| > 2$ . The radius of convergence of the power series is  $\rho = 2$ . Finally, we check the endpoints of the interval of convergence. At  $x = 1$  the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. At  $x = -3$  we have

$$\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges, but does not converge absolutely. The series is said to converge conditionally at  $x = -3$ . To summarize, the given power series converges for  $-3 \leq x < 1$ , and diverges otherwise. It converges absolutely for  $-3 < x < 1$ , and has a radius of convergence 2.

If  $\sum_{n=0}^{\infty} a_n(x-x_0)^n$  and  $\sum_{n=0}^{\infty} b_n(x-x_0)^n$  converge to  $f(x)$  and  $g(x)$ , respectively, for  $|x-x_0| < \rho$ ,  $\rho > 0$ , then the following are true for  $|x-x_0| < \rho$ .

6. The series can be added or subtracted termwise and

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n.$$

7. The series can be formally multiplied and

$$f(x)g(x) = \left[ \sum_{n=0}^{\infty} a_n(x-x_0)^n \right] \left[ \sum_{n=0}^{\infty} b_n(x-x_0)^n \right] = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where  $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$ . Further, if  $g(x_0) \neq 0$ , the series can be formally divided and

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x-x_0)^n.$$

In most cases the coefficients  $d_n$  can be most easily obtained by equating coefficients in the equivalent relation

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(x-x_0)^n &= \left[ \sum_{n=0}^{\infty} d_n(x-x_0)^n \right] \left[ \sum_{n=0}^{\infty} b_n(x-x_0)^n \right] \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n d_k b_{n-k} \right) (x-x_0)^n. \end{aligned}$$

Also, in the case of division, the radius of convergence of the resulting power series may be less than  $\rho$ .

8. The function  $f$  is continuous and has derivatives of all orders for  $|x-x_0| < \rho$ . Further,  $f'$ ,  $f''$ ,  $\dots$  can be computed by differentiating the series termwise; that is,

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-x_0) + \cdots + na_n(x-x_0)^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \\ f''(x) &= 2a_2 + 6a_3(x-x_0) + \cdots + n(n-1)a_n(x-x_0)^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}, \end{aligned}$$

and so forth, and each of the series converges absolutely for  $|x-x_0| < \rho$ .

9. The value of  $a_n$  is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The series is called the Taylor<sup>1</sup> series for the function  $f$  about  $x = x_0$ .

10. If  $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$  for each  $x$ , then  $a_n = b_n$  for  $n = 0, 1, 2, 3, \dots$ . In particular, if  $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$  for each  $x$ , then  $a_0 = a_1 = \cdots = a_n = \cdots = 0$ .

<sup>1</sup>Brook Taylor (1685–1731) was the leading English mathematician in the generation following Newton. In 1715 he published a general statement of the expansion theorem that is named for him, a result that is fundamental in all branches of analysis. He was also one of the founders of the calculus of finite differences, and was the first to recognize the existence of singular solutions of differential equations.

A function  $f$  that has a Taylor series expansion about  $x = x_0$ ,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n,$$

with a radius of convergence  $\rho > 0$ , is said to be **analytic** at  $x = x_0$ . According to statements 6 and 7, if  $f$  and  $g$  are analytic at  $x_0$ , then  $f \pm g$ ,  $f \cdot g$ , and  $f/g$  [provided that  $g(x_0) \neq 0$ ] are analytic at  $x = x_0$ .

**Shift of Index of Summation.** The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable. Thus it is immaterial which letter is used for the index of summation. For example,

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{j=0}^{\infty} \frac{2^j x^j}{j!}.$$

Just as we make changes of the variable of integration in a definite integral, we find it convenient to make changes of summation indices in calculating series solutions of differential equations. We illustrate by several examples how to shift the summation index.

**EXAMPLE  
3**

Write  $\sum_{n=2}^{\infty} a_n x^n$  as a series whose first term corresponds to  $n = 0$  rather than  $n = 2$ .

Let  $m = n - 2$ ; then  $n = m + 2$  and  $n = 2$  corresponds to  $m = 0$ . Hence

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2}. \quad (1)$$

By writing out the first few terms of each of these series, you can verify that they contain precisely the same terms. Finally, in the series on the right side of Eq. (1), we can replace the dummy index  $m$  by  $n$ , obtaining

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}. \quad (2)$$

In effect, we have shifted the index upward by 2, and compensated by starting to count at a level 2 lower than originally.

**EXAMPLE  
4**

Write the series

$$\sum_{n=2}^{\infty} (n+2)(n+1)a_n (x-x_0)^{n-2} \quad (3)$$

as a series whose generic term involves  $(x-x_0)^n$  rather than  $(x-x_0)^{n-2}$ .

Again, we shift the index by 2 so that  $n$  is replaced by  $n+2$  and start counting 2 lower. We obtain

$$\sum_{n=0}^{\infty} (n+4)(n+3)a_{n+2} (x-x_0)^n. \quad (4)$$

You can readily verify that the terms in the series (3) and (4) are exactly the same.

**EXAMPLE 5**

Write the expression

$$x^2 \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} \quad (5)$$

as a series whose generic term involves  $x^{r+n}$ .

First take the  $x^2$  inside the summation, obtaining

$$\sum_{n=0}^{\infty} (r+n)a_n x^{r+n+1}. \quad (6)$$

Next, shift the index down by 1 and start counting 1 higher. Thus

$$\sum_{n=0}^{\infty} (r+n)a_n x^{r+n+1} = \sum_{n=1}^{\infty} (r+n-1)a_{n-1} x^{r+n}. \quad (7)$$

Again, you can easily verify that the two series in Eq. (7) are identical, and that both are exactly the same as the expression (5).

**EXAMPLE 6**

Assume that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n \quad (8)$$

for all  $x$ , and determine what this implies about the coefficients  $a_n$ .

We want to use statement 10 to equate corresponding coefficients in the two series. In order to do this, we must first rewrite Eq. (8) so that the series display the same power of  $x$  in their generic terms. For instance, in the series on the left side of Eq. (8), we can replace  $n$  by  $n+1$  and start counting 1 lower. Thus Eq. (8) becomes

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n. \quad (9)$$

According to statement 10 we conclude that

$$(n+1)a_{n+1} = a_n, \quad n = 0, 1, 2, 3, \dots$$

or

$$a_{n+1} = \frac{a_n}{n+1}, \quad n = 0, 1, 2, 3, \dots \quad (10)$$

Hence, choosing successive values of  $n$  in Eq. (10), we have

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!},$$

and so forth. In general,

$$a_n = \frac{a_0}{n!}, \quad n = 1, 2, 3, \dots \quad (11)$$

Thus the relation (8) determines all the following coefficients in terms of  $a_0$ . Finally, using the coefficients given by Eq. (11), we obtain

$$\sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x,$$

where we have followed the usual convention that  $0! = 1$ .

**PROBLEMS**

In each of Problems 1 through 8 determine the radius of convergence of the given power series.

1.  $\sum_{n=0}^{\infty} (x-3)^n$
2.  $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$
3.  $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$
4.  $\sum_{n=0}^{\infty} 2^n x^n$
5.  $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}$
6.  $\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n}$
7.  $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$
8.  $\sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$

In each of Problems 9 through 16 determine the Taylor series about the point  $x_0$  for the given function. Also determine the radius of convergence of the series.

9.  $\sin x$ ,  $x_0 = 0$
10.  $e^x$ ,  $x_0 = 0$
11.  $x$ ,  $x_0 = 1$
12.  $x^2$ ,  $x_0 = -1$
13.  $\ln x$ ,  $x_0 = 1$
14.  $\frac{1}{1+x}$ ,  $x_0 = 0$
15.  $\frac{1}{1-x}$ ,  $x_0 = 0$
16.  $\frac{1}{1-x}$ ,  $x_0 = 2$
17. Given that  $y = \sum_{n=0}^{\infty} nx^n$ , compute  $y'$  and  $y''$  and write out the first four terms of each series as well as the coefficient of  $x^n$  in the general term.
18. Given that  $y = \sum_{n=0}^{\infty} a_n x^n$ , compute  $y'$  and  $y''$  and write out the first four terms of each series as well as the coefficient of  $x^n$  in the general term. Show that if  $y'' = y$ , then the coefficients  $a_0$  and  $a_1$  are arbitrary, and determine  $a_2$  and  $a_3$  in terms of  $a_0$  and  $a_1$ . Show that  $a_{n+2} = a_n / (n+2)(n+1)$ ,  $n = 0, 1, 2, 3, \dots$

In each of Problems 19 and 20 verify the given equation.

19.  $\sum_{n=0}^{\infty} a_n (x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x-1)^n$
20.  $\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^k$

In each of Problems 21 through 27 rewrite the given expression as a sum whose generic term involves  $x^n$ .

21.  $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$
22.  $\sum_{n=0}^{\infty} a_n x^{n+2}$
23.  $x \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k$
24.  $(1-x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$
25.  $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} ka_k x^{k-1}$
26.  $\sum_{n=1}^{\infty} na_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$
27.  $x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

28. Determine the  $a_n$  so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series  $\sum_{n=0}^{\infty} a_n x^n$ .

## 5.2 Series Solutions near an Ordinary Point, Part I



In Chapter 3 we described methods of solving second order linear differential equations with constant coefficients. We now consider methods of solving second order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote the independent variable by  $x$ . It is sufficient to consider the homogeneous equation

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0, \quad (1)$$

since the procedure for the corresponding nonhomogeneous equation is similar.

A wide class of problems in mathematical physics leads to equations of the form (1) having polynomial coefficients; for example, the Bessel equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

where  $\nu$  is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where  $\alpha$  is a constant. For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which the functions  $P$ ,  $Q$ , and  $R$  are polynomials. However, as we will see, the method of solution is also applicable when  $P$ ,  $Q$ , and  $R$  are general analytic functions.

For the present, then, suppose that  $P$ ,  $Q$ , and  $R$  are polynomials, and that they have no common factors. Suppose also that we wish to solve Eq. (1) in the neighborhood of a point  $x_0$ . The solution of Eq. (1) in an interval containing  $x_0$  is closely associated with the behavior of  $P$  in that interval.

A point  $x_0$  such that  $P(x_0) \neq 0$  is called an **ordinary point**. Since  $P$  is continuous, it follows that there is an interval about  $x_0$  in which  $P(x)$  is never zero. In that interval we can divide Eq. (1) by  $P(x)$  to obtain

$$y'' + p(x)y' + q(x)y = 0, \quad (2)$$

where  $p(x) = Q(x)/P(x)$  and  $q(x) = R(x)/P(x)$  are continuous functions. Hence, according to the existence and uniqueness Theorem 3.2.1, there exists in that interval a unique solution of Eq. (1) that also satisfies the initial conditions  $y(x_0) = y_0$ ,  $y'(x_0) = y'_0$  for arbitrary values of  $y_0$  and  $y'_0$ . In this and the following section we discuss the solution of Eq. (1) in the neighborhood of an ordinary point.

On the other hand, if  $P(x_0) = 0$ , then  $x_0$  is called a **singular point** of Eq. (1). In this case at least one of  $Q(x_0)$  and  $R(x_0)$  is not zero. Consequently, at least one of the



coefficients  $p$  and  $q$  in Eq. (2) becomes unbounded as  $x \rightarrow x_0$ , and therefore Theorem 3.2.1 does not apply in this case. Sections 5.4 through 5.8 deal with finding solutions of Eq. (1) in the neighborhood of a singular point.

We now take up the problem of solving Eq. (1) in the neighborhood of an ordinary point  $x_0$ . We look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad (3)$$

and assume that the series converges in the interval  $|x - x_0| < \rho$  for some  $\rho > 0$ . While at first sight it may appear unattractive to seek a solution in the form of a power series, this is actually a convenient and useful form for a solution. Within their intervals of convergence power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if we can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, we are likely to need a power series or some equivalent expression if we want to evaluate them numerically or to plot their graphs.

The most practical way to determine the coefficients  $a_n$  is to substitute the series (3) and its derivatives for  $y$ ,  $y'$ , and  $y''$  in Eq. (1). The following examples illustrate this process. The operations, such as differentiation, that are involved in the procedure are justified so long as we stay within the interval of convergence. The differential equations in these examples are also of considerable importance in their own right.

### EXAMPLE 1

Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty. \quad (4)$$

As we know, two linearly independent solutions of this equation are  $\sin x$  and  $\cos x$ , so series methods are not needed to solve this equation. However, this example illustrates the use of power series in a relatively simple case. For Eq. (4),  $P(x) = 1$ ,  $Q(x) = 0$ , and  $R(x) = 1$ ; hence every point is an ordinary point.

We look for a solution in the form of a power series about  $x_0 = 0$ ,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n, \quad (5)$$

and assume that the series converges in some interval  $|x| < \rho$ .

Differentiating term by term yields

$$y' = a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}, \quad (6)$$

$$y'' = 2a_2 + \cdots + n(n-1)a_nx^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}. \quad (7)$$

Substituting the series (5) and (7) for  $y$  and  $y''$  in Eq. (4) gives

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n = 0.$$

To combine the two series we need to rewrite at least one of them so that both series display the same generic term. Thus, in the first sum, we shift the index of summation by replacing  $n$  by  $n + 2$  and starting the sum at 0 rather than 2. We obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n = 0.$$

For this equation to be satisfied for all  $x$ , the coefficient of each power of  $x$  must be zero; hence, we conclude that

$$(n+2)(n+1)a_{n+2} + a_n = 0, \quad n = 0, 1, 2, 3, \dots \quad (8)$$

Equation (8) is referred to as a **recurrence relation**. The successive coefficients can be evaluated one by one by writing the recurrence relation first for  $n = 0$ , then for  $n = 1$ , and so forth. In this example Eq. (8) relates each coefficient to the second one before it. Thus the even-numbered coefficients ( $a_0, a_2, a_4, \dots$ ) and the odd-numbered ones ( $a_1, a_3, a_5, \dots$ ) are determined separately. For the even-numbered coefficients we have

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2!}, \quad a_4 = -\frac{a_2}{4 \cdot 3} = +\frac{a_0}{4!}, \quad a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!}, \dots$$

These results suggest that in general, if  $n = 2k$ , then

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!} a_0, \quad k = 1, 2, 3, \dots \quad (9)$$

We can prove Eq. (9) by mathematical induction. First observe that it is true for  $k = 1$ . Next, assume that it is true for an arbitrary value of  $k$  and consider the case  $k + 1$ . We have

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+1)} = -\frac{(-1)^k}{(2k+2)(2k+1)(2k)!} a_0 = \frac{(-1)^{k+1}}{(2k+2)!} a_0.$$

Hence Eq. (9) is also true for  $k + 1$  and consequently it is true for all positive integers  $k$ .

Similarly, for the odd-numbered coefficients

$$a_3 = -\frac{a_1}{2 \cdot 3} = -\frac{a_1}{3!}, \quad a_5 = -\frac{a_3}{5 \cdot 4} = +\frac{a_1}{5!}, \quad a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{a_1}{7!}, \dots,$$

and in general, if  $n = 2k + 1$ , then<sup>2</sup>

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1, \quad k = 1, 2, 3, \dots \quad (10)$$

Substituting these coefficients into Eq. (5), we have

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 +$$

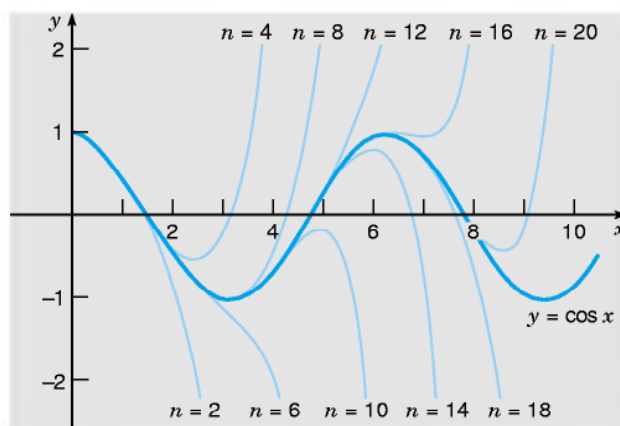
<sup>2</sup>The result given in Eq. (10) and other similar formulas in this chapter can be proved by an induction argument resembling the one just given for Eq. (9). We assume that the results are plausible and omit the inductive argument hereafter.

$$\begin{aligned}
& + \cdots + \frac{(-1)^n a_0}{(2n)!} x^{2n} + \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1} + \cdots \\
& = a_0 \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots \right] \\
& \quad + a_1 \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \right] \\
& = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{11}
\end{aligned}$$

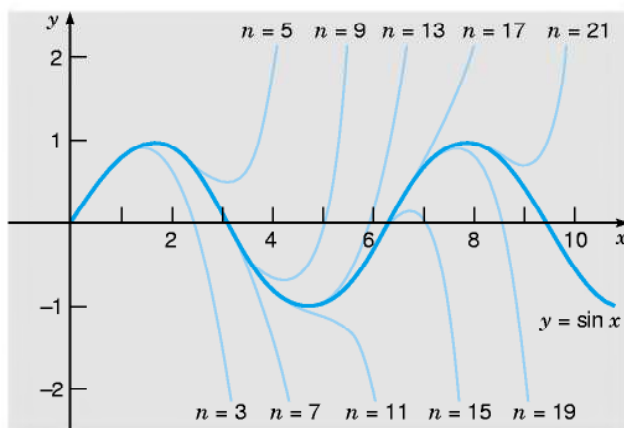
Now that we have formally obtained two series solutions of Eq. (4), we can test them for convergence. Using the ratio test, it is easy to show that each of the series in Eq. (11) converges for all  $x$ , and this justifies retroactively all the steps used in obtaining the solutions. Indeed, we recognize that the first series in Eq. (11) is exactly the Taylor series for  $\cos x$  about  $x = 0$  and the second is the Taylor series for  $\sin x$  about  $x = 0$ . Thus, as expected, we obtain the solution  $y = a_0 \cos x + a_1 \sin x$ .

Notice that no conditions are imposed on  $a_0$  and  $a_1$ ; hence they are arbitrary. From Eqs. (5) and (6) we see that  $y$  and  $y'$  evaluated at  $x = 0$  are  $a_0$  and  $a_1$ , respectively. Since the initial conditions  $y(0)$  and  $y'(0)$  can be chosen arbitrarily, it follows that  $a_0$  and  $a_1$  should be arbitrary until specific initial conditions are stated.

Figures 5.2.1 and 5.2.2 show how the partial sums of the series in Eq. (11) approximate  $\cos x$  and  $\sin x$ . As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each  $x$  in this interval the accuracy of the approximation improves. However, you should always remember that a truncated power series provides only a local approximation of the solution in a neighborhood of the initial point  $x = 0$ ; it cannot adequately represent the solution for large  $|x|$ .



**FIGURE 5.2.1** Polynomial approximations to  $\cos x$ . The value of  $n$  is the degree of the approximating polynomial.



**FIGURE 5.2.2** Polynomial approximations to  $\sin x$ . The value of  $n$  is the degree of the approximating polynomial.

In Example 1 we knew from the start that  $\sin x$  and  $\cos x$  form a fundamental set of solutions of Eq. (4). However, if we had not known this and had simply solved Eq. (4) using series methods, we would still have obtained the solution (11). In recognition of the fact that the differential equation (4) often occurs in applications we might decide to give the two solutions of Eq. (11) special names; perhaps,

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Then we might ask what properties these functions have. For instance, it follows at once from the series expansions that  $C(0) = 1$ ,  $S(0) = 0$ ,  $C(-x) = C(x)$ , and  $S(-x) = -S(x)$ . It is also easy to show that

$$S'(x) = C(x), \quad C'(x) = -S(x).$$

Moreover, by calculating with the infinite series<sup>3</sup> we can show that the functions  $C(x)$  and  $S(x)$  have all the usual analytical and algebraic properties of the cosine and sine functions, respectively.

Although you probably first saw the sine and cosine functions defined in a more elementary manner in terms of right triangles, it is interesting that these functions can be defined as solutions of a certain simple second order linear differential equation. To be precise, the function  $\sin x$  can be defined as the unique solution of the initial value problem  $y'' + y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$ ; similarly,  $\cos x$  can be defined as the unique solution of the initial value problem  $y'' + y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ . Many other functions that are important in mathematical physics are also defined as solutions of certain initial value problems. For most of these functions there is no simpler or more elementary way to approach them.

<sup>3</sup>Such an analysis is given in Section 24 of K. Knopp, *Theory and Applications of Infinite Series* (New York: Hafner, 1951).

**EXAMPLE**  
**2**

Find a series solution in powers of  $x$  of Airy's<sup>4</sup> equation

$$y'' - xy = 0, \quad -\infty < x < \infty. \quad (12)$$

For this equation  $P(x) = 1$ ,  $Q(x) = 0$ , and  $R(x) = -x$ ; hence every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad (13)$$

and that the series converges in some interval  $|x| < \rho$ . The series for  $y''$  is given by Eq. (7); as explained in the preceding example, we can rewrite it as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \quad (14)$$

Substituting the series (13) and (14) for  $y$  and  $y''$  in Eq. (12), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}. \quad (15)$$

Next, we shift the index of summation in the series on the right side of this equation by replacing  $n$  by  $n-1$  and starting the summation at 1 rather than zero. Thus we have

$$2 \cdot 1a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Again, for this equation to be satisfied for all  $x$  it is necessary that the coefficients of like powers of  $x$  be equal; hence  $a_2 = 0$ , and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1} \quad \text{for } n = 1, 2, 3, \dots \quad (16)$$

Since  $a_{n+2}$  is given in terms of  $a_{n-1}$ , the  $a$ 's are determined in steps of three. Thus  $a_0$  determines  $a_3$ , which in turn determines  $a_6, \dots$ ;  $a_1$  determines  $a_4$ , which in turn determines  $a_7, \dots$ ; and  $a_2$  determines  $a_5$ , which in turn determines  $a_8, \dots$ . Since  $a_2 = 0$ , we immediately conclude that  $a_5 = a_8 = a_{11} = \dots = 0$ .

For the sequence  $a_0, a_3, a_6, a_9, \dots$  we set  $n = 1, 4, 7, 10, \dots$  in the recurrence relation:

$$a_3 = \frac{a_0}{2 \cdot 3}, \quad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \quad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \dots$$

For this sequence of coefficients it is convenient to write a formula for  $a_{3n}$ ,  $n = 1, 2, 3, \dots$ . The preceding results suggest the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)(3n-1)(3n)}, \quad n = 1, 2, \dots$$

For the sequence  $a_1, a_4, a_7, a_{10}, \dots$ , we set  $n = 2, 5, 8, 11, \dots$  in the recurrence relation:

$$a_4 = \frac{a_1}{3 \cdot 4}, \quad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, \dots$$

<sup>4</sup>Sir George Airy (1801–1892), an English astronomer and mathematician, was director of the Greenwich Observatory from 1835 to 1881. One reason Airy's equation is of interest is that for  $x$  negative the solutions are oscillatory, similar to trigonometric functions, and for  $x$  positive they are monotonic, similar to hyperbolic functions. Can you explain why it is reasonable to expect such behavior?

In the same way as before we find that

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3)(3n-2)(3n)(3n+1)}, \quad n = 1, 2, 3, \dots$$

The general solution of Airy's equation is

$$\begin{aligned} y &= a_0 \left[ 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right] \\ &\quad + a_1 \left[ x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right] \\ &= a_0 \left[ 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} \right] + a_1 \left[ x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} \right]. \end{aligned} \tag{17}$$

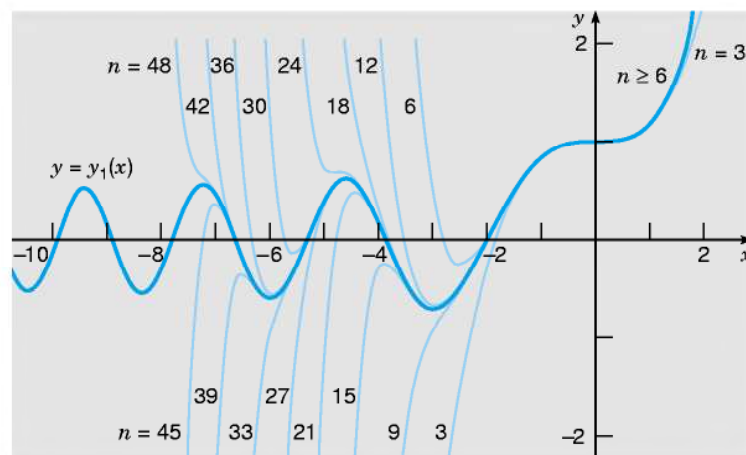
Having obtained these two series solutions, we can now investigate their convergence. Because of the rapid growth of the denominators of the terms in the series (17), we might expect these series to have a large radius of convergence. Indeed, it is easy to use the ratio test to show that both these series converge for all  $x$ ; see Problem 20.

Assuming for the moment that the series do converge for all  $x$ , let  $y_1$  and  $y_2$  denote the functions defined by the expressions in the first and second sets of brackets, respectively, in Eq. (17). Then, by first choosing  $a_0 = 1, a_1 = 0$  and then  $a_0 = 0, a_1 = 1$ , it follows that  $y_1$  and  $y_2$  are individually solutions of Eq. (12). Notice that  $y_1$  satisfies the initial conditions  $y_1(0) = 1, y_1'(0) = 0$  and that  $y_2$  satisfies the initial conditions  $y_2(0) = 0, y_2'(0) = 1$ . Thus  $W(y_1, y_2)(0) = 1 \neq 0$ , and consequently  $y_1$  and  $y_2$  are linearly independent. Hence the general solution of Airy's equation is

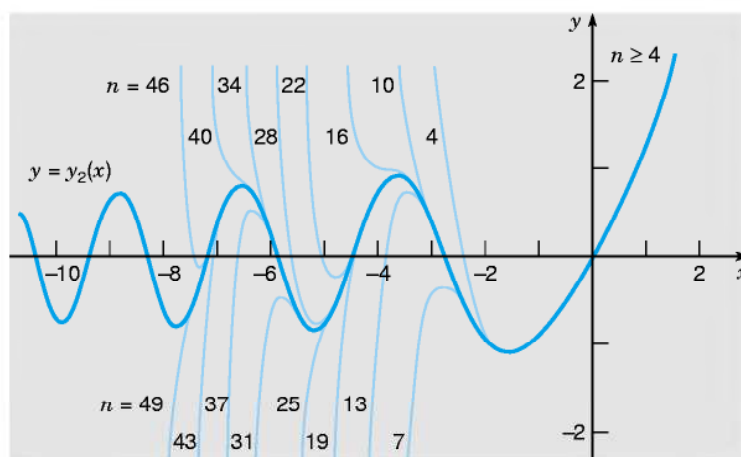
$$y = a_0 y_1(x) + a_1 y_2(x), \quad -\infty < x < \infty.$$

In Figures 5.2.3 and 5.2.4, respectively, we show the graphs of the solutions  $y_1$  and  $y_2$  of Airy's equation, as well as graphs of several partial sums of the two series in Eq. (17). Again, the partial sums provide local approximations to the solutions in a neighborhood of the origin. While the quality of the approximation improves as the number of terms increases, no polynomial can adequately represent  $y_1$  and  $y_2$  for large  $|x|$ . A practical way to estimate the interval in which a given partial sum is reasonably accurate is to compare the graphs of that partial sum and the next one, obtained by including one more term. As soon as the graphs begin to separate noticeably, one can be confident that the original partial sum is no longer accurate. For example, in Figure 5.2.3 the graphs for  $n = 24$  and  $n = 27$  begin to separate at about  $x = -9/2$ . Thus, beyond this point, the partial sum of degree 24 is worthless as an approximation to the solution.

Observe that both  $y_1$  and  $y_2$  are monotone for  $x > 0$  and oscillatory for  $x < 0$ . One can also see from the figures that the oscillations are not uniform, but decay in amplitude and increase in frequency as the distance from the origin increases. In contrast to Example 1, the solutions  $y_1$  and  $y_2$  of Airy's equation are not elementary functions that you have already encountered in calculus. However, because of their importance in some physical applications, these functions have been extensively studied and their properties are well known to applied mathematicians and scientists.



**FIGURE 5.2.3** Polynomial approximations to the solution  $y_1(x)$  of Airy's equation. The value of  $n$  is the degree of the approximating polynomial.



**FIGURE 5.2.4** Polynomial approximations to the solution  $y_2(x)$  of Airy's equation. The value of  $n$  is the degree of the approximating polynomial.

**EXAMPLE**  
**3**

Find a solution of Airy's equation in powers of  $x - 1$ .

The point  $x = 1$  is an ordinary point of Eq. (12), and thus we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - 1)^n,$$

where we assume that the series converges in some interval  $|x - 1| < \rho$ . Then

$$y' = \sum_{n=1}^{\infty} n a_n (x - 1)^{n-1} = \sum_{n=0}^{\infty} (n + 1) a_{n+1} (x - 1)^n,$$



and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n.$$

Substituting for  $y$  and  $y''$  in Eq. (12), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n = x \sum_{n=0}^{\infty} a_n(x-1)^n. \quad (18)$$

Now to equate the coefficients of like powers of  $(x-1)$  we must express  $x$ , the coefficient of  $y$  in Eq. (12), in powers of  $x-1$ ; that is, we write  $x = 1 + (x-1)$ . Note that this is precisely the Taylor series for  $x$  about  $x = 1$ . Then Eq. (18) takes the form

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n &= [1 + (x-1)] \sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1}. \end{aligned}$$

Shifting the index of summation in the second series on the right gives

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n.$$

Equating coefficients of like powers of  $x-1$ , we obtain

$$\begin{aligned} 2a_2 &= a_0, \\ (3 \cdot 2)a_3 &= a_1 + a_0, \\ (4 \cdot 3)a_4 &= a_2 + a_1, \\ (5 \cdot 4)a_5 &= a_3 + a_2, \\ &\vdots \end{aligned}$$

The general recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1} \quad \text{for } n \geq 1. \quad (19)$$

Solving for the first few coefficients  $a_n$  in terms of  $a_0$  and  $a_1$ , we find that

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{6} + \frac{a_0}{6}, \quad a_4 = \frac{a_2}{12} + \frac{a_1}{12} = \frac{a_0}{24} + \frac{a_1}{12}, \quad a_5 = \frac{a_3}{20} + \frac{a_2}{20} = \frac{a_0}{30} + \frac{a_1}{120}.$$

Hence

$$\begin{aligned} y &= a_0 \left[ 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \dots \right] \\ &\quad + a_1 \left[ (x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \dots \right]. \quad (20) \end{aligned}$$

In general, when the recurrence relation has more than two terms, as in Eq. (19), the determination of a formula for  $a_n$  in terms  $a_0$  and  $a_1$  will be fairly complicated, if not impossible. In this example such a formula is not readily apparent. Lacking such a formula, we cannot test the two series in Eq. (20) for convergence by direct methods



such as the ratio test. However, even without knowing the formula for  $a_n$  we shall see in Section 5.3 that it is possible to establish that the series in Eq. (20) converge for all  $x$ , and further define functions  $y_3$  and  $y_4$  that are linearly independent solutions of the Airy equation (12). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy's equation for  $-\infty < x < \infty$ .

It is worth emphasizing, as we saw in Example 3, that if we look for a solution of Eq. (1) of the form  $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ , then the coefficients  $P(x)$ ,  $Q(x)$ , and  $R(x)$  in Eq. (1) must also be expressed in powers of  $x - x_0$ . Alternatively, we can make the change of variable  $x - x_0 = t$ , obtaining a new differential equation for  $y$  as a function of  $t$ , and then look for solutions of this new equation of the form  $\sum_{n=0}^{\infty} a_n t^n$ . When we have finished the calculations, we replace  $t$  by  $x - x_0$  (see Problem 19).

In Examples 2 and 3 we have found two sets of solutions of Airy's equation. The functions  $y_1$  and  $y_2$  defined by the series in Eq. (17) are linearly independent solutions of Eq. (12) for all  $x$ , and this is also true for the functions  $y_3$  and  $y_4$  defined by the series in Eq. (20). According to the general theory of second order linear equations each of the first two functions can be expressed as a linear combination of the latter two functions and vice versa—a result that is certainly not obvious from an examination of the series alone.

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient  $a_n$  in terms of  $a_0$  and  $a_1$ . What is essential is that we can determine *as many coefficients as we want*. Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package one can also produce plots such as those shown in the figures in this section.

## PROBLEMS

In each of Problems 1 through 14 solve the given differential equation by means of a power series about the given point  $x_0$ . Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

- |  |   |
|--|---|
| 1. $y'' - y = 0, \quad x_0 = 0$                  | 2. $y'' - xy' - y = 0, \quad x_0 = 0$                               |
| 3. $y'' - xy' - y = 0, \quad x_0 = 1$            | 4. $y'' + k^2 x^2 y = 0, \quad x_0 = 0, \quad k \text{ a constant}$ |
| 5. $(1 - x)y'' + y = 0, \quad x_0 = 0$           | 6. $(2 + x^2)y'' - xy' + 4y = 0, \quad x_0 = 0$                     |
| 7. $y'' + xy' + 2y = 0, \quad x_0 = 0$           | 8. $xy'' + y' + xy = 0, \quad x_0 = 1$                              |
| 9. $(1 + x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0$ | 10. $(4 - x^2)y'' + 2y = 0, \quad x_0 = 0$                          |
| 11. $(3 - x^2)y'' - 3xy' - y = 0, \quad x_0 = 0$ | 12. $(1 - x)y'' + xy' - y = 0, \quad x_0 = 0$                       |
| 13. $2y'' + xy' + 3y = 0, \quad x_0 = 0$         | 14. $2y'' + (x + 1)y' + 3y = 0, \quad x_0 = 2$                      |

In each of Problems 15 through 18:

- Find the first five nonzero terms in the solution of the given initial value problem.
- Plot the four-term and the five-term approximations to the solution on the same axes.
- From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

- ▶ 15.  $y'' - xy' - y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 1$ ; see Problem 2
- ▶ 16.  $(2 + x^2)y'' - xy' + 4y = 0$ ,  $y(0) = -1$ ,  $y'(0) = 3$ ; see Problem 6
- ▶ 17.  $y'' + xy' + 2y = 0$ ,  $y(0) = 4$ ,  $y'(0) = -1$ ; see Problem 7
- ▶ 18.  $(1 - x)y'' + xy' - y = 0$ ,  $y(0) = -3$ ,  $y'(0) = 2$ ; see Problem 12
- 19. By making the change of variable  $x - 1 = t$  and assuming that  $y$  is a power series in  $t$ , find two linearly independent series solutions of

$$y'' + (x - 1)^2y' + (x^2 - 1)y = 0$$

in powers of  $x - 1$ . Show that you obtain the same result directly by assuming that  $y$  is a Taylor series in powers of  $x - 1$  and also expressing the coefficient  $x^2 - 1$  in powers of  $x - 1$ .

- 20. Show directly, using the ratio test, that the two series solutions of Airy's equation about  $x = 0$  converge for all  $x$ ; see Eq. (17) of the text.
- 21. **The Hermite Equation.** The equation

$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty,$$

where  $\lambda$  is a constant, is known as the Hermite<sup>5</sup> equation. It is an important equation in mathematical physics.

- Find the first four terms in each of two linearly independent solutions about  $x = 0$ .
  - Observe that if  $\lambda$  is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for  $\lambda = 0, 2, 4, 6, 8$ , and  $10$ . Note that each polynomial is determined only up to a multiplicative constant.
  - The Hermite polynomial  $H_n(x)$  is defined as the polynomial solution of the Hermite equation with  $\lambda = 2n$  for which the coefficient of  $x^n$  is  $2^n$ . Find  $H_0(x), \dots, H_5(x)$ .
- 22. Consider the initial value problem  $y' = \sqrt{1 - y^2}$ ,  $y(0) = 0$ .
    - Show that  $y = \sin x$  is the solution of this initial value problem.
    - Look for a solution of the initial value problem in the form of a power series about  $x = 0$ . Find the coefficients up to the term in  $x^3$  in this series.

In each of Problems 23 through 28 plot several partial sums in a series solution of the given initial value problem about  $x = 0$ , thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

- ▶ 23.  $y'' - xy' - y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ ; see Problem 2
- ▶ 24.  $(2 + x^2)y'' - xy' + 4y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ ; see Problem 6
- ▶ 25.  $y'' + xy' + 2y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$ ; see Problem 7
- ▶ 26.  $(4 - x^2)y'' + 2y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$ ; see Problem 10
- ▶ 27.  $y'' + x^2y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ ; see Problem 4
- ▶ 28.  $(1 - x)y'' + xy' - 2y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$

<sup>5</sup>Charles Hermite (1822–1901) was an influential French analyst and algebraist. He introduced the Hermite functions in 1864, and showed in 1873 that  $e$  is a transcendental number (that is,  $e$  is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

## 5.3 Series Solutions near an Ordinary Point, Part II



In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad (1)$$

where  $P$ ,  $Q$ , and  $R$  are polynomials, in the neighborhood of an ordinary point  $x_0$ . Assuming that Eq. (1) does have a solution  $y = \phi(x)$ , and that  $\phi$  has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad (2)$$

which converges for  $|x - x_0| < \rho$ , where  $\rho > 0$ , we found that the  $a_n$  can be determined by directly substituting the series (2) for  $y$  in Eq. (1).

Let us now consider how we might justify the statement that if  $x_0$  is an ordinary point of Eq. (1), then there exist solutions of the form (2). We also consider the question of the radius of convergence of such a series. In doing this we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of Eq. (1) of the form (2). By differentiating Eq. (2)  $m$  times and setting  $x$  equal to  $x_0$  it follows that

$$m!a_m = \phi^{(m)}(x_0).$$

Hence, to compute  $a_n$  in the series (2), we must show that we can determine  $\phi^{(n)}(x_0)$  for  $n = 0, 1, 2, \dots$  from the differential equation (1).

Suppose that  $y = \phi(x)$  is a solution of Eq. (1) satisfying the initial conditions  $y(x_0) = y_0$ ,  $y'(x_0) = y'_0$ . Then  $a_0 = y_0$  and  $a_1 = y'_0$ . If we are solely interested in finding a solution of Eq. (1) without specifying any initial conditions, then  $a_0$  and  $a_1$  remain arbitrary. To determine  $\phi^{(n)}(x_0)$  and the corresponding  $a_n$  for  $n = 2, 3, \dots$ , we turn to Eq. (1). Since  $\phi$  is a solution of Eq. (1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about  $x_0$  for which  $P$  is nonvanishing we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \quad (3)$$

where  $p(x) = Q(x)/P(x)$  and  $q(x) = R(x)/P(x)$ . Setting  $x$  equal to  $x_0$  in Eq. (3) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0).$$

Hence  $a_2$  is given by

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \quad (4)$$

To determine  $a_3$  we differentiate Eq. (3) and then set  $x$  equal to  $x_0$ , obtaining

$$\begin{aligned} 3!a_3 = \phi'''(x_0) &= -[p\phi'' + (p' + q)\phi' + q'\phi] \Big|_{x=x_0} \\ &= -2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q'(x_0)a_0. \end{aligned} \quad (5)$$

Substituting for  $a_2$  from Eq. (4) gives  $a_3$  in terms of  $a_1$  and  $a_0$ . Since  $P$ ,  $Q$ , and  $R$  are polynomials and  $P(x_0) \neq 0$ , all of the derivatives of  $p$  and  $q$  exist at  $x_0$ . Hence, we can

continue to differentiate Eq. (3) indefinitely, determining after each differentiation the successive coefficients  $a_4, a_5, \dots$  by setting  $x$  equal to  $x_0$ .

Notice that the important property that we used in determining the  $a_n$  was that we could compute infinitely many derivatives of the functions  $p$  and  $q$ . It might seem reasonable to relax our assumption that the functions  $p$  and  $q$  are ratios of polynomials, and simply require that they be infinitely differentiable in the neighborhood of  $x_0$ . Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for  $y = \phi(x)$ . What is needed is to assume that the functions  $p$  and  $q$  are *analytic* at  $x_0$ ; that is, they have Taylor series expansions that converge to them in some interval about the point  $x_0$ :

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \quad (6)$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} q_n(x - x_0)^n. \quad (7)$$

With this idea in mind we can generalize the definitions of an ordinary point and a singular point of Eq. (1) as follows: If the functions  $p = Q/P$  and  $q = R/P$  are analytic at  $x_0$ , then the point  $x_0$  is said to be an **ordinary point** of the differential equation (1); otherwise, it is a **singular point**.

Now let us turn to the question of the interval of convergence of the series solution. One possibility is actually to compute the series solution for each problem and then to apply one of the tests for convergence of an infinite series to determine its radius of convergence. However, the question can be answered at once for a wide class of problems by the following theorem.

**Theorem 5.3.1** If  $x_0$  is an ordinary point of the differential equation (1),

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if  $p = Q/P$  and  $q = R/P$  are analytic at  $x_0$ , then the general solution of Eq. (1) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x), \quad (8)$$

where  $a_0$  and  $a_1$  are arbitrary, and  $y_1$  and  $y_2$  are linearly independent series solutions that are analytic at  $x_0$ . Further, the radius of convergence for each of the series solutions  $y_1$  and  $y_2$  is at least as large as the minimum of the radii of convergence of the series for  $p$  and  $q$ .

Notice from the form of the series solution that  $y_1(x) = 1 + b_2(x - x_0)^2 + \cdots$  and  $y_2(x) = (x - x_0) + c_2(x - x_0)^2 + \cdots$ . Hence  $y_1$  is the solution satisfying the initial conditions  $y_1(x_0) = 1, y_1'(x_0) = 0$ , and  $y_2$  is the solution satisfying the initial conditions  $y_2(x_0) = 0, y_2'(x_0) = 1$ . Also note that while the calculation of the coefficients by successively differentiating the differential equation is excellent in theory, it is usually not a practical computational procedure. Rather, one should substitute the series (2) for

$y$  in the differential equation (1) and determine the coefficients so that the differential equation is satisfied, as in the examples in the preceding section.

We will not prove this theorem, which in a slightly more general form is due to Fuchs.<sup>6</sup> What is important for our purposes is that there is a series solution of the form (2), and that the radius of convergence of the series solution cannot be less than the smaller of the radii of convergence of the series for  $p$  and  $q$ ; hence we need only determine these.

This can be done in one of two ways. Again, one possibility is simply to compute the power series for  $p$  and  $q$ , and then to determine the radii of convergence by using one of the convergence tests for infinite series. However, there is an easier way when  $P$ ,  $Q$ , and  $R$  are polynomials. It is shown in the theory of functions of a complex variable that the ratio of two polynomials, say  $Q/P$ , has a convergent power series expansion about a point  $x = x_0$  if  $P(x_0) \neq 0$ . Further, if we assume that any factors common to  $Q$  and  $P$  have been canceled, then the radius of convergence of the power series for  $Q/P$  about the point  $x_0$  is precisely the distance from  $x_0$  to the nearest zero of  $P$ . In determining this distance we must remember that  $P(x) = 0$  may have complex roots, and these must also be considered.

**EXAMPLE  
1**

What is the radius of convergence of the Taylor series for  $(1 + x^2)^{-1}$  about  $x = 0$ ?

One way to proceed is to find the Taylor series in question, namely,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

Then it can be verified by the ratio test that  $\rho = 1$ . Another approach is to note that the zeros of  $1 + x^2$  are  $x = \pm i$ . Since the distance from 0 to  $i$  or to  $-i$  in the complex plane is 1, the radius of convergence of the power series about  $x = 0$  is 1.

**EXAMPLE  
2**

What is the radius of convergence of the Taylor series for  $(x^2 - 2x + 2)^{-1}$  about  $x = 0$ ? About  $x = 1$ ?

First notice that

$$x^2 - 2x + 2 = 0$$

has solutions  $x = 1 \pm i$ . The distance from  $x = 0$  to either  $x = 1 + i$  or  $x = 1 - i$  in the complex plane is  $\sqrt{2}$ ; hence the radius of convergence of the Taylor series expansion  $\sum_{n=0}^{\infty} a_n x^n$  about  $x = 0$  is  $\sqrt{2}$ .

The distance from  $x = 1$  to either  $x = 1 + i$  or  $x = 1 - i$  is 1; hence the radius of convergence of the Taylor series expansion  $\sum_{n=0}^{\infty} b_n (x - 1)^n$  about  $x = 1$  is 1.

<sup>6</sup>Immanuel Lazarus Fuchs (1833–1902) was a student and later a professor at the University of Berlin. He proved the result of Theorem 5.3.1 in 1866. His most important research was on singular points of linear differential equations. He recognized the significance of regular singular points (Section 5.4), and equations whose only singularities, including the point at infinity, are regular singular points are known as Fuchsian equations.



According to Theorem 5.3.1 the series solutions of the Airy equation in Examples 2 and 3 of the preceding section converge for all values of  $x$  and  $x - 1$ , respectively, since in each problem  $P(x) = 1$  and hence is never zero.

A series solution may converge for a wider range of  $x$  than indicated by Theorem 5.3.1, so the theorem actually gives only a lower bound on the radius of convergence of the series solution. This is illustrated by the Legendre polynomial solution of the Legendre equation given in the next example.

**EXAMPLE**  
**3**

Determine a lower bound for the radius of convergence of series solutions about  $x = 0$  for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where  $\alpha$  is a constant.

Note that  $P(x) = 1 - x^2$ ,  $Q(x) = -2x$ , and  $R(x) = \alpha(\alpha + 1)$  are polynomials, and that the zeros of  $P$ , namely,  $x = \pm 1$ , are a distance 1 from  $x = 0$ . Hence a series solution of the form  $\sum_{n=0}^{\infty} a_n x^n$  converges for  $|x| < 1$  at least, and possibly for larger values of  $x$ . Indeed, it can be shown that if  $\alpha$  is a positive integer, one of the series solutions terminates after a finite number of terms and hence converges not just for  $|x| < 1$  but for all  $x$ . For example, if  $\alpha = 1$ , the polynomial solution is  $y = x$ . See Problems 22 through 29 at the end of this section for a more complete discussion of the Legendre equation.

**EXAMPLE**  
**4**

Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0 \quad (9)$$

about the point  $x = 0$ ; about the point  $x = -\frac{1}{2}$ .

Again  $P$ ,  $Q$ , and  $R$  are polynomials, and  $P$  has zeros at  $x = \pm i$ . The distance in the complex plane from 0 to  $\pm i$  is 1, and from  $-\frac{1}{2}$  to  $\pm i$  is  $\sqrt{1 + \frac{1}{4}} = \sqrt{5}/2$ . Hence in the first case the series  $\sum_{n=0}^{\infty} a_n x^n$  converges at least for  $|x| < 1$ , and in the second case the series  $\sum_{n=0}^{\infty} b_n (x + \frac{1}{2})^n$  converges at least for  $|x + \frac{1}{2}| < \sqrt{5}/2$ .

An interesting observation that we can make about Eq. (9) follows from Theorems 3.2.1 and 5.3.1. Suppose that initial conditions  $y(0) = y_0$  and  $y'(0) = y'_0$  are given. Since  $1 + x^2 \neq 0$  for all  $x$ , we know from Theorem 3.2.1 that there exists a unique solution of the initial value problem on  $-\infty < x < \infty$ . On the other hand, Theorem 5.3.1 only guarantees a series solution of the form  $\sum_{n=0}^{\infty} a_n x^n$  (with  $a_0 = y_0$ ,  $a_1 = y'_0$ ) for  $-1 < x < 1$ . The unique solution on the interval  $-\infty < x < \infty$  may not have a power series about  $x = 0$  that converges for all  $x$ .

**EXAMPLE**  
**5**

Can we determine a series solution about  $x = 0$  for the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?

For this differential equation,  $p(x) = \sin x$  and  $q(x) = 1 + x^2$ . Recall from calculus that  $\sin x$  has a Taylor series expansion about  $x = 0$  that converges for all  $x$ . Further,  $q$  also has a Taylor series expansion about  $x = 0$ , namely,  $q(x) = 1 + x^2$ , that converges for all  $x$ . Thus there is a series solution of the form  $y = \sum_{n=0}^{\infty} a_n x^n$  with  $a_0$  and  $a_1$  arbitrary, and the series converges for all  $x$ .

## PROBLEMS

In each of Problems 1 through 4 determine  $\phi''(x_0)$ ,  $\phi'''(x_0)$ , and  $\phi^{iv}(x_0)$  for the given point  $x_0$  if  $y = \phi(x)$  is a solution of the given initial value problem.

- $y'' + xy' + y = 0$ ;  $y(0) = 1$ ,  $y'(0) = 0$
- $y'' + (\sin x)y' + (\cos x)y = 0$ ;  $y(0) = 0$ ,  $y'(0) = 1$
- $x^2 y'' + (1+x)y' + 3(\ln x)y = 0$ ;  $y(1) = 2$ ,  $y'(1) = 0$
- $y'' + x^2 y' + (\sin x)y = 0$ ;  $y(0) = a_0$ ,  $y'(0) = a_1$

In each of Problems 5 through 8 determine a lower bound for the radius of convergence of series solutions about each given point  $x_0$  for the given differential equation.

- $y'' + 4y' + 6xy = 0$ ;  $x_0 = 0$ ,  $x_0 = 4$
- $(x^2 - 2x - 3)y'' + xy' + 4y = 0$ ;  $x_0 = 4$ ,  $x_0 = -4$ ,  $x_0 = 0$
- $(1 + x^3)y'' + 4xy' + y = 0$ ;  $x_0 = 0$ ,  $x_0 = 2$
- $xy'' + y = 0$ ;  $x_0 = 1$
- Determine a lower bound for the radius of convergence of series solutions about the given  $x_0$  for each of the differential equations in Problems 1 through 14 of Section 5.2.
- The Chebyshev Equation.** The Chebyshev<sup>7</sup> differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where  $\alpha$  is a constant.

- Determine two linearly independent solutions in powers of  $x$  for  $|x| < 1$ .
- Show that if  $\alpha$  is a nonnegative integer  $n$ , then there is a polynomial solution of degree  $n$ . These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems requiring a polynomial approximation to a function defined on  $-1 \leq x \leq 1$ .
- Find a polynomial solution for each of the cases  $\alpha = n = 0, 1, 2$ , and  $3$ .

For each of the differential equations in Problems 11 through 14 find the first four nonzero terms in each of two linearly independent power series solutions about the origin. What do you expect the radius of convergence to be for each solution?

- $y'' + (\sin x)y = 0$
- $e^x y'' + xy = 0$
- $(\cos x)y'' + xy' - 2y = 0$
- $e^{-x} y'' + \ln(1+x)y' - xy = 0$
- Suppose that you are told that  $x$  and  $x^2$  are solutions of a differential equation  $P(x)y'' + Q(x)y' + R(x)y = 0$ . Can you say whether the point  $x = 0$  is an ordinary point or a singular point?  
*Hint:* Use Theorem 3.2.1, and note the values of  $x$  and  $x^2$  at  $x = 0$ .

<sup>7</sup>Pafnuty L. Chebyshev (1821–1894), professor at Petersburg University for 35 years and the most influential nineteenth century Russian mathematician, founded the so-called “Petersburg school,” which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

**First Order Equations.** The series methods discussed in this section are directly applicable to the first order linear differential equation  $P(x)y' + Q(x)y = 0$  at a point  $x_0$ , if the function  $p = Q/P$  has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series  $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$  is at least as large as the radius of convergence of the series for  $Q/P$ . In each of Problems 16 through 21 solve the given differential equation by a series in powers of  $x$  and verify that  $a_0$  is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

- |  |                       |
|--|-----------------------|
| 16. $y' - y = 0$                       | 17. $y' - xy = 0$     |
| 18. $y' = e^{x^2}y$ , three terms only | 19. $(1 - x)y' = y$   |
| 20. $y' - y = x^2$                     | 21. $y' + xy = 1 + x$ |

**The Legendre Equation.** Problems 22 through 29 deal with the Legendre<sup>8</sup> equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

As indicated in Example 3, the point  $x = 0$  is an ordinary point of this equation, and the distance from the origin to the nearest zero of  $P(x) = 1 - x^2$  is 1. Hence the radius of convergence of series solutions about  $x = 0$  is at least 1. Also notice that it is necessary to consider only  $\alpha > -1$  because if  $\alpha \leq -1$ , then the substitution  $\alpha = -(1 + \gamma)$  where  $\gamma \geq 0$  leads to the Legendre equation  $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$ .

22. Show that two linearly independent solutions of the Legendre equation for  $|x| < 1$  are

$$y_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{\alpha(\alpha - 2)(\alpha - 4) \cdots (\alpha - 2m + 2)(\alpha + 1)(\alpha + 3) \cdots (\alpha + 2m - 1)}{(2m)!} x^{2m},$$

$$y_2(x) = x + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha - 1)(\alpha - 3) \cdots (\alpha - 2m + 1)(\alpha + 2)(\alpha + 4) \cdots (\alpha + 2m)}{(2m + 1)!} x^{2m+1}.$$

23. Show that, if  $\alpha$  is zero or a positive even integer  $2n$ , the series solution  $y_1$  reduces to a polynomial of degree  $2n$  containing only even powers of  $x$ . Find the polynomials corresponding to  $\alpha = 0, 2$ , and  $4$ . Show that, if  $\alpha$  is a positive odd integer  $2n + 1$ , the series solution  $y_2$  reduces to a polynomial of degree  $2n + 1$  containing only odd powers of  $x$ . Find the polynomials corresponding to  $\alpha = 1, 3$ , and  $5$ .
- ▶ 24. The Legendre polynomial  $P_n(x)$  is defined as the polynomial solution of the Legendre equation with  $\alpha = n$  that also satisfies the condition  $P_n(1) = 1$ .
- (a) Using the results of Problem 23, find the Legendre polynomials  $P_0(x), \dots, P_5(x)$ .
- (b) Plot the graphs of  $P_0(x), \dots, P_5(x)$  for  $-1 \leq x \leq 1$ .
- (c) Find the zeros of  $P_0(x), \dots, P_5(x)$ .
25. It can be shown that the general formula for  $P_n(x)$  is

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n - 2k)!}{k!(n - k)!(n - 2k)!} x^{n-2k},$$

<sup>8</sup>Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.



where  $[n/2]$  denotes the greatest integer less than or equal to  $n/2$ . By observing the form of  $P_n(x)$  for  $n$  even and  $n$  odd, show that  $P_n(-1) = (-1)^n$ .

26. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates we encounter the equation

$$\frac{d^2 F(\varphi)}{d\varphi^2} + \cot \varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \quad 0 < \varphi < \pi,$$

where  $n$  is a positive integer. Show that the change of variable  $x = \cos \varphi$  leads to the Legendre equation with  $\alpha = n$  for  $y = f(x) = F(\arccos x)$ .

27. Show that for  $n = 0, 1, 2, 3$  the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

This formula, known as Rodrigues' (1794–1851) formula, is true for all positive integers  $n$ .

28. Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\alpha(\alpha+1)y.$$

Then it follows that  $[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x)$  and  $[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x)$ . By multiplying the first equation by  $P_m(x)$  and the second equation by  $P_n(x)$ , and then integrating by parts, show that

$$\int_{-1}^1 P_n(x)P_m(x) dx = 0 \quad \text{if} \quad n \neq m.$$

This property of the Legendre polynomials is known as the orthogonality property. If  $m = n$ , it can be shown that the value of the preceding integral is  $2/(2n+1)$ .

29. Given a polynomial  $f$  of degree  $n$ , it is possible to express  $f$  as a linear combination of  $P_0, P_1, P_2, \dots, P_n$ :

$$f(x) = \sum_{k=0}^n a_k P_k(x).$$

Using the result of Problem 28, show that

$$a_k = \frac{2k+1}{2} \int_{-1}^1 f(x)P_k(x) dx.$$

## 5.4 Regular Singular Points

ODE

In this section we will consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a singular point  $x_0$ . Recall that if the functions  $P$ ,  $Q$ , and  $R$  are polynomials having no common factors, the singular points of Eq. (1) are the points for which  $P(x) = 0$ .

**EXAMPLE  
1**

Determine the singular points and ordinary points of the Bessel equation of order  $\nu$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \quad (2)$$

The point  $x = 0$  is a singular point since  $P(x) = x^2$  is zero there. All other points are ordinary points of Eq. (2).

**EXAMPLE  
2**

Determine the singular points and ordinary points of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (3)$$

where  $\alpha$  is a constant.

The singular points are the zeros of  $P(x) = 1 - x^2$ , namely, the points  $x = \pm 1$ . All other points are ordinary points.

Unfortunately, if we attempt to use the methods of the preceding two sections to solve Eq. (1) in the neighborhood of a singular point  $x_0$ , we find that these methods fail. This is because the solution of Eq. (1) is often not analytic at  $x_0$ , and consequently cannot be represented by a Taylor series in powers of  $x - x_0$ . Instead, we must use a more general series expansion.

Since the singular points of a differential equation are usually few in number, we might ask whether we can simply ignore them, especially since we already know how to construct solutions about ordinary points. However, this is not feasible because the singular points determine the principal features of the solution to a much larger extent than one might at first suspect. In the neighborhood of a singular point the solution often becomes large in magnitude or experiences rapid changes in magnitude. Thus the behavior of a physical system modeled by a differential equation frequently is most interesting in the neighborhood of a singular point. Often geometrical singularities in a physical problem, such as corners or sharp edges, lead to singular points in the corresponding differential equation. Thus, while at first we might want to avoid the few points where a differential equation is singular, it is precisely at these points that it is necessary to study the solution most carefully.

As an alternative to analytical methods, one can consider the use of numerical methods, which are discussed in Chapter 8. However, these methods are ill-suited for the study of solutions near a singular point. Thus, even if one adopts a numerical approach, it is advantageous to combine it with the analytical methods of this chapter in order to examine the behavior of solutions near singular points.

Without any additional information about the behavior of  $Q/P$  and  $R/P$  in the neighborhood of the singular point, it is impossible to describe the behavior of the solutions of Eq. (1) near  $x = x_0$ . It may be that there are two linearly independent solutions of Eq. (1) that remain bounded as  $x \rightarrow x_0$ , or there may be only one with the other becoming unbounded as  $x \rightarrow x_0$ , or they may both become unbounded as  $x \rightarrow x_0$ . To illustrate these possibilities consider the following examples.

**EXAMPLE  
3**

The differential equation

$$x^2 y'' - 2y = 0 \quad (4)$$

has a singular point at  $x = 0$ . It can be easily verified by direct substitution that  $y_1(x) = x^2$  and  $y_2(x) = 1/x$  are linearly independent solutions of Eq. (4) for  $x > 0$  or

$x < 0$ . Thus in any interval not containing the origin the general solution of Eq. (4) is  $y = c_1x^2 + c_2x^{-1}$ . The only solution of Eq. (4) that is bounded as  $x \rightarrow 0$  is  $y = c_1x^2$ . Indeed, this solution is analytic at the origin even though if Eq. (4) is put in the standard form,  $y'' - (2/x^2)y = 0$ , the function  $q(x) = -2/x^2$  is not analytic at  $x = 0$ , and Theorem 5.3.1 is not applicable. On the other hand, notice that the solution  $y_2(x) = x^{-1}$  does not have a Taylor series expansion about the origin (is not analytic at  $x = 0$ ); therefore, the method of Section 5.2 would fail in this case.

## EXAMPLE

## 4

The differential equation

$$x^2y'' - 2xy' + 2y = 0 \quad (5)$$

also has a singular point at  $x = 0$ . It can be verified that  $y_1(x) = x$  and  $y_2(x) = x^2$  are linearly independent solutions of Eq. (5), and both are analytic at  $x = 0$ . Nevertheless, it is still not proper to pose an initial value problem with initial conditions at  $x = 0$ . It is impossible to satisfy arbitrary initial conditions at  $x = 0$ , since any linear combination of  $x$  and  $x^2$  is zero at  $x = 0$ .

It is also possible to construct a differential equation with a singular point  $x_0$  such that every (nonzero) solution of the differential equation becomes unbounded as  $x \rightarrow x_0$ . Even if Eq. (1) does not have a solution that remains bounded as  $x \rightarrow x_0$ , it is often important to determine how the solutions of Eq. (1) behave as  $x \rightarrow x_0$ ; for example, does  $y \rightarrow \infty$  in the same manner as  $(x - x_0)^{-1}$  or  $|x - x_0|^{-1/2}$ , or in some other manner?

Our goal is to extend the method already developed for solving Eq. (1) near an ordinary point so that it also applies to the neighborhood of a singular point  $x_0$ . To do this in a reasonably simple manner, it is necessary to restrict ourselves to cases in which the singularities in the functions  $Q/P$  and  $R/P$  at  $x = x_0$  are not too severe, that is, to what we might call “weak singularities.” At this point it is not clear exactly what is an acceptable singularity. However, as we develop the method of solution you will see that the appropriate conditions (see also Section 5.7, Problem 21) to distinguish “weak singularities” are

$$\lim_{x \rightarrow x_0} (x - x_0) \frac{Q(x)}{P(x)} \text{ is finite} \quad (6)$$

and

$$\lim_{x \rightarrow x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \text{ is finite.} \quad (7)$$

This means that the singularity in  $Q/P$  can be no worse than  $(x - x_0)^{-1}$  and the singularity in  $R/P$  can be no worse than  $(x - x_0)^{-2}$ . Such a point is called a **regular singular point** of Eq. (1). For more general functions than polynomials,  $x_0$  is a regular singular point of Eq. (1) if it is a singular point, and if both<sup>9</sup>

$$(x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2 \frac{R(x)}{P(x)} \quad (8)$$

<sup>9</sup>The functions given in Eq. (8) may not be defined at  $x_0$ , in which case their values at  $x_0$  are to be assigned as their limits as  $x \rightarrow x_0$ .

have convergent Taylor series about  $x_0$ , that is, if the functions in Eq. (8) are analytic at  $x = x_0$ . Equations (6) and (7) imply that this will be the case when  $P$ ,  $Q$ , and  $R$  are polynomials. Any singular point of Eq. (1) that is not a regular singular point is called an **irregular singular point** of Eq. (1).

In the following sections we discuss how to solve Eq. (1) in the neighborhood of a regular singular point. A discussion of the solutions of differential equations in the neighborhood of irregular singular points is more complicated and may be found in more advanced books.

**EXAMPLE**  
**5**

In Example 2 we observed that the singular points of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

are  $x = \pm 1$ . Determine whether these singular points are regular or irregular singular points.

We consider the point  $x = 1$  first and also observe that on dividing by  $1 - x^2$  the coefficients of  $y'$  and  $y$  are  $-2x/(1 - x^2)$  and  $\alpha(\alpha + 1)/(1 - x^2)$ , respectively. Thus we calculate

$$\lim_{x \rightarrow 1} (x - 1) \frac{-2x}{1 - x^2} = \lim_{x \rightarrow 1} \frac{(x - 1)(-2x)}{(1 - x)(1 + x)} = \lim_{x \rightarrow 1} \frac{2x}{1 + x} = 1$$

and

$$\begin{aligned} \lim_{x \rightarrow 1} (x - 1)^2 \frac{\alpha(\alpha + 1)}{1 - x^2} &= \lim_{x \rightarrow 1} \frac{(x - 1)^2 \alpha(\alpha + 1)}{(1 - x)(1 + x)} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)(-\alpha)(\alpha + 1)}{1 + x} = 0. \end{aligned}$$

Since these limits are finite, the point  $x = 1$  is a regular singular point. It can be shown in a similar manner that  $x = -1$  is also a regular singular point.

**EXAMPLE**  
**6**

Determine the singular points of the differential equation

$$2x(x - 2)^2 y'' + 3xy' + (x - 2)y = 0$$

and classify them as regular or irregular.

Dividing the differential equation by  $2x(x - 2)^2$ , we have

$$y'' + \frac{3}{2(x - 2)^2} y' + \frac{1}{2x(x - 2)} y = 0,$$

so  $p(x) = Q(x)/P(x) = 3/2(x - 2)^2$  and  $q(x) = R(x)/P(x) = 1/2x(x - 2)$ . The singular points are  $x = 0$  and  $x = 2$ . Consider  $x = 0$ . We have

$$\begin{aligned} \lim_{x \rightarrow 0} x p(x) &= \lim_{x \rightarrow 0} x \frac{3}{2(x - 2)^2} = 0, \\ \lim_{x \rightarrow 0} x^2 q(x) &= \lim_{x \rightarrow 0} x^2 \frac{1}{2x(x - 2)} = 0. \end{aligned}$$

Since these limits are finite,  $x = 0$  is a regular singular point. For  $x = 2$  we have

$$\lim_{x \rightarrow 2} (x-2)p(x) = \lim_{x \rightarrow 2} (x-2) \frac{3}{2(x-2)^2} = \lim_{x \rightarrow 2} \frac{3}{2(x-2)},$$

so the limit does not exist; hence  $x = 2$  is an irregular singular point.

**EXAMPLE**  
**7**

Determine the singular points of

$$\left(x - \frac{\pi}{2}\right)^2 y'' + (\cos x)y' + (\sin x)y = 0$$

and classify them as regular or irregular.

The only singular point is  $x = \pi/2$ . To study it we consider the functions

$$\left(x - \frac{\pi}{2}\right) p(x) = \left(x - \frac{\pi}{2}\right) \frac{Q(x)}{P(x)} = \frac{\cos x}{x - \pi/2}$$

and

$$\left(x - \frac{\pi}{2}\right)^2 q(x) = \left(x - \frac{\pi}{2}\right)^2 \frac{R(x)}{P(x)} = \sin x.$$

Starting from the Taylor series for  $\cos x$  about  $x = \pi/2$ , we find that

$$\frac{\cos x}{x - \pi/2} = -1 + \frac{(x - \pi/2)^2}{3!} - \frac{(x - \pi/2)^4}{5!} + \dots,$$

which converges for all  $x$ . Similarly,  $\sin x$  is analytic at  $x = \pi/2$ . Therefore, we conclude that  $\pi/2$  is a regular singular point for this equation.

## PROBLEMS

In each of Problems 1 through 18 find all singular points of the given equation and determine whether each one is regular or irregular.

- $xy'' + (1-x)y' + xy = 0$
- $x^2(1-x)y'' + 2xy' + 4y = 0$
- $x^2(1-x)y'' + (x-2)y' - 3xy = 0$
- $x^2(1-x^2)y'' + (2/x)y' + 4y = 0$
- $(1-x^2)^2 y'' + x(1-x)y' + (1+x)y = 0$
- $x^2 y'' + xy' + (x^2 - v^2)y = 0$ , Bessel equation
- $(x+3)y'' - 2xy' + (1-x^2)y = 0$
- $x(1-x^2)^3 y'' + (1-x^2)^2 y' + 2(1+x)y = 0$
- $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$
- $x(3-x)y'' + (x+1)y' - 2y = 0$
- $(x^2 + x - 2)y'' + (x+1)y' + 2y = 0$
- $xy'' + e^x y' + (3 \cos x)y = 0$
- $y'' + (\ln|x|)y' + 3xy = 0$
- $x^2 y'' + 2(e^x - 1)y' + (e^{-x} \cos x)y = 0$
- $x^2 y'' - 3(\sin x)y' + (1+x^2)y = 0$
- $xy'' + y' + (\cot x)y = 0$
- $(\sin x)y'' + xy' + 4y = 0$
- $(x \sin x)y'' + 3y' + xy = 0$

In each of Problems 19 and 20 show that the point  $x = 0$  is a regular singular point. In each problem try to find solutions of the form  $\sum_{n=0}^{\infty} a_n x^n$ . Show that there is only one nonzero solution of this form in Problem 19 and that there are no nonzero solutions of this form in Problem 20. Thus in neither case can the general solution be found in this manner. This is typical of equations with singular points.

- $2xy'' + 3y' + xy = 0$
- $2x^2 y'' + 3xy' - (1+x)y = 0$

21. **Singularities at Infinity.** The definitions of an ordinary point and a regular singular point given in the preceding sections apply only if the point  $x_0$  is finite. In more advanced work in differential equations it is often necessary to discuss the point at infinity. This is done by making the change of variable  $\xi = 1/x$  and studying the resulting equation at  $\xi = 0$ . Show that for the differential equation  $P(x)y'' + Q(x)y' + R(x)y = 0$  the point at infinity is an ordinary point if

$$\frac{1}{P(1/\xi)} \left[ \frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2} \right] \quad \text{and} \quad \frac{R(1/\xi)}{\xi^4 P(1/\xi)}$$

have Taylor series expansions about  $\xi = 0$ . Show also that the point at infinity is a regular singular point if at least one of the above functions does not have a Taylor series expansion, but both

$$\frac{\xi}{P(1/\xi)} \left[ \frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2} \right] \quad \text{and} \quad \frac{R(1/\xi)}{\xi^2 P(1/\xi)}$$

do have such expansions.

In each of Problems 22 through 27 use the results of Problem 21 to determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

22.  $y'' + y = 0$   
 23.  $x^2 y'' + xy' - 4y = 0$   
 24.  $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ , Legendre equation  
 25.  $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ , Bessel equation  
 26.  $y'' - 2xy' + \lambda y = 0$ , Hermite equation  
 27.  $y'' - xy = 0$ , Airy equation

## 5.5 Euler Equations

The simplest example of a differential equation having a regular singular point is the **Euler equation**, or the equidimensional equation,

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0, \quad (1)$$

where  $\alpha$  and  $\beta$  are real constants. It is easy to show that  $x = 0$  is a regular singular point of Eq. (1). Because the solution of the Euler equation is typical of the solutions of all differential equations with a regular singular point, it is worthwhile considering this equation in detail before discussing the more general problem.

In any interval not including the origin, Eq. (1) has a general solution of the form  $y = c_1 y_1(x) + c_2 y_2(x)$ , where  $y_1$  and  $y_2$  are linearly independent. For convenience we first consider the interval  $x > 0$ , extending our results later to the interval  $x < 0$ . First, we note that  $(x^r)' = r x^{r-1}$  and  $(x^r)'' = r(r-1)x^{r-2}$ . Hence, if we assume that Eq. (1) has a solution of the form

$$y = x^r, \quad (2)$$

then we obtain

$$\begin{aligned} L[x^r] &= x^2 (x^r)'' + \alpha x (x^r)' + \beta x^r \\ &= x^r [r(r-1) + \alpha r + \beta]. \end{aligned} \quad (3)$$



If  $r$  is a root of the quadratic equation

$$F(r) = r(r - 1) + \alpha r + \beta = 0, \quad (4)$$

then  $L[x^r]$  is zero, and  $y = x^r$  is a solution of Eq. (1). The roots of Eq. (4) are

$$r_1, r_2 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}, \quad (5)$$

and  $F(r) = (r - r_1)(r - r_2)$ . As for second order linear equations with constant coefficients, it is necessary to consider separately the cases in which the roots are real and different, real but equal, and complex conjugates. Indeed, the entire discussion in this section is similar to the treatment of second order linear equations with constant coefficients in Chapter 3 with  $e^{rx}$  replaced by  $x^r$ ; see also Problem 23.

**Real, Distinct Roots.** If  $F(r) = 0$  has real roots  $r_1$  and  $r_2$ , with  $r_1 \neq r_2$ , then  $y_1(x) = x^{r_1}$  and  $y_2(x) = x^{r_2}$  are solutions of Eq. (1). Since  $W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2-1}$  is nonvanishing for  $r_1 \neq r_2$  and  $x > 0$ , it follows that the general solution of Eq. (1) is

$$y = c_1 x^{r_1} + c_2 x^{r_2}, \quad x > 0. \quad (6)$$

Note that if  $r$  is not a rational number, then  $x^r$  is defined by  $x^r = e^{r \ln x}$ .

**EXAMPLE  
1**

Solve

$$2x^2 y'' + 3xy' - y = 0, \quad x > 0. \quad (7)$$

Substituting  $y = x^r$  in Eq. (7) gives

$$x^r [2r(r - 1) + 3r - 1] = x^r (2r^2 + r - 1) = x^r (2r - 1)(r + 1) = 0.$$

Hence  $r_1 = \frac{1}{2}$  and  $r_2 = -1$ , so the general solution of Eq. (7) is

$$y = c_1 x^{1/2} + c_2 x^{-1}, \quad x > 0. \quad (8)$$

**Equal Roots.** If the roots  $r_1$  and  $r_2$  are equal, then we obtain only one solution  $y_1(x) = x^{r_1}$  of the assumed form. A second solution can be obtained by the method of reduction of order, but for the purpose of our future discussion we consider an alternative method. Since  $r_1 = r_2$ , it follows that  $F(r) = (r - r_1)^2$ . Thus in this case not only does  $F(r_1) = 0$  but also  $F'(r_1) = 0$ . This suggests differentiating Eq. (3) with respect to  $r$  and then setting  $r$  equal to  $r_1$ . Differentiating Eq. (3) with respect to  $r$  gives

$$\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r} [x^r F(r)].$$

Substituting for  $F(r)$ , interchanging differentiation with respect to  $x$  and with respect to  $r$ , and noting that  $\partial(x^r)/\partial r = x^r \ln x$ , we obtain

$$L[x^r \ln x] = (r - r_1)^2 x^r \ln x + 2(r - r_1)x^r. \quad (9)$$

The right side of Eq. (9) is zero for  $r = r_1$ ; consequently,

$$y_2(x) = x^{r_1} \ln x, \quad x > 0 \quad (10)$$

is a second solution of Eq. (1). It is easy to show that  $W(x^{r_1}, x^{r_1} \ln x) = x^{2r_1-1}$ . Hence  $x^{r_1}$  and  $x^{r_1} \ln x$  are linearly independent for  $x > 0$ , and the general solution of Eq. (1) is

$$y = (c_1 + c_2 \ln x)x^{r_1}, \quad x > 0. \quad (11)$$

**EXAMPLE  
2**

Solve

$$x^2 y'' + 5x y' + 4y = 0, \quad x > 0. \quad (12)$$

Substituting  $y = x^r$  in Eq. (12) gives

$$x^r [r(r-1) + 5r + 4] = x^r (r^2 + 4r + 4) = 0.$$

Hence  $r_1 = r_2 = -2$ , and

$$y = x^{-2}(c_1 + c_2 \ln x), \quad x > 0. \quad (13)$$

**Complex Roots.** Finally, suppose that the roots  $r_1$  and  $r_2$  are complex conjugates, say,  $r_1 = \lambda + i\mu$  and  $r_2 = \lambda - i\mu$ , with  $\mu \neq 0$ . We must now explain what is meant by  $x^r$  when  $r$  is complex. Remembering that

$$x^r = e^{r \ln x} \quad (14)$$

when  $x > 0$  and  $r$  is real, we can use this equation to *define*  $x^r$  when  $r$  is complex. Then

$$\begin{aligned} x^{\lambda+i\mu} &= e^{(\lambda+i\mu)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x} = x^\lambda e^{i\mu \ln x} \\ &= x^\lambda [\cos(\mu \ln x) + i \sin(\mu \ln x)], \quad x > 0. \end{aligned} \quad (15)$$

With this definition of  $x^r$  for complex values of  $r$ , it can be verified that the usual laws of algebra and the differential calculus hold, and hence  $x^{r_1}$  and  $x^{r_2}$  are indeed solutions of Eq. (1). The general solution of Eq. (1) is

$$y = c_1 x^{\lambda+i\mu} + c_2 x^{\lambda-i\mu}. \quad (16)$$

The disadvantage of this expression is that the functions  $x^{\lambda+i\mu}$  and  $x^{\lambda-i\mu}$  are complex-valued. Recall that we had a similar situation for the second order differential equation with constant coefficients when the roots of the characteristic equation were complex. In the same way as we did then we observe that the real and imaginary parts of  $x^{\lambda+i\mu}$ , namely,

$$x^\lambda \cos(\mu \ln x) \quad \text{and} \quad x^\lambda \sin(\mu \ln x), \quad (17)$$

are also solutions of Eq. (1). A straightforward calculation shows that

$$W[x^\lambda \cos(\mu \ln x), x^\lambda \sin(\mu \ln x)] = \mu x^{2\lambda-1}.$$

Hence these solutions are also linearly independent for  $x > 0$ , and the general solution of Eq. (1) is

$$y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x), \quad x > 0. \quad (18)$$



**EXAMPLE**  
**3**

Solve

$$x^2 y'' + x y' + y = 0. \quad (19)$$

Substituting  $y = x^r$  in Eq. (19) gives

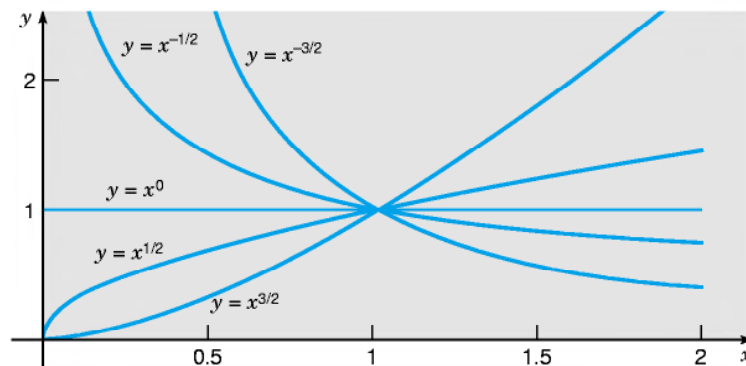
$$x^r [r(r-1) + r + 1] = x^r (r^2 + 1) = 0.$$

Hence  $r = \pm i$ , and the general solution is

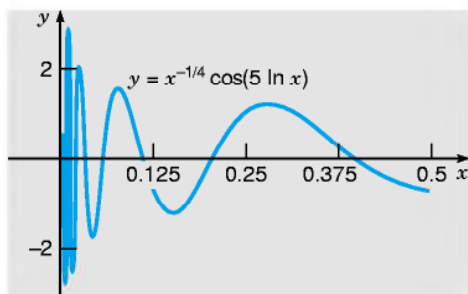
$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x), \quad x > 0. \quad (20)$$

Now let us consider the qualitative behavior of the solutions of Eq. (1) near the singular point  $x = 0$ . This depends entirely on the nature of the exponents  $r_1$  and  $r_2$ . First, if  $r$  is real and positive, then  $x^r \rightarrow 0$  as  $x$  tends to zero through positive values. On the other hand, if  $r$  is real and negative, then  $x^r$  becomes unbounded, while if  $r = 0$ , then  $x^r = 1$ . These possibilities are shown in Figure 5.5.1 for various values of  $r$ . If  $r$  is complex, then a typical solution is  $x^\lambda \cos(\mu \ln x)$ . This function becomes unbounded or approaches zero if  $\lambda$  is negative or positive, respectively, and also oscillates more and more rapidly as  $x \rightarrow 0$ . This behavior is shown in Figures 5.5.2 and 5.5.3 for selected values of  $\lambda$  and  $\mu$ . If  $\lambda = 0$ , the oscillation is of constant amplitude. Finally, if there are repeated roots, then one solution is of the form  $x^r \ln x$ , which tends to zero if  $r > 0$  and becomes unbounded if  $r \leq 0$ . An example of each case is shown in Figure 5.5.4.

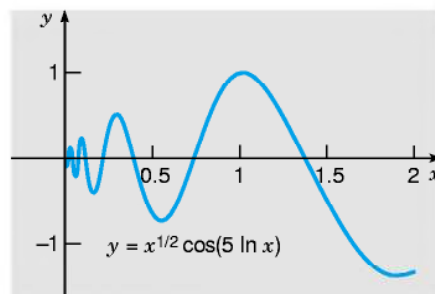
The extension of the solutions of Eq. (1) into the interval  $x < 0$  can be carried out in a relatively straightforward manner. The difficulty lies in understanding what is meant by  $x^r$  when  $x$  is negative and  $r$  is not an integer; similarly,  $\ln x$  has not been defined for  $x < 0$ . The solutions of the Euler equation that we have given for  $x > 0$  can be shown to be valid for  $x < 0$ , but in general they are complex-valued. Thus in Example 1 the solution  $x^{1/2}$  is imaginary for  $x < 0$ .



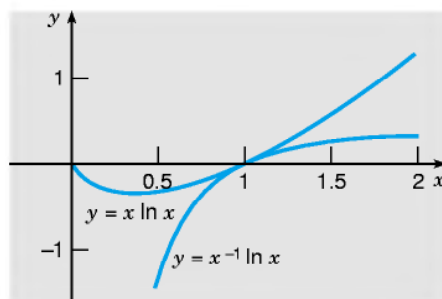
**FIGURE 5.5.1** Solutions of an Euler equation; real roots.



**FIGURE 5.5.2** Solution of an Euler equation; complex roots with negative real part.



**FIGURE 5.5.3** Solution of an Euler equation; complex roots with positive real part.



**FIGURE 5.5.4** Solutions of an Euler equation; repeated roots.

It is always possible to obtain real-valued solutions of the Euler equation (1) in the interval  $x < 0$  by making the following change of variable. Let  $x = -\xi$ , where  $\xi > 0$ , and let  $y = u(\xi)$ . Then we have

$$\frac{dy}{dx} = \frac{du}{d\xi} \frac{d\xi}{dx} = -\frac{du}{d\xi}, \quad \frac{d^2y}{dx^2} = \frac{d}{d\xi} \left( -\frac{du}{d\xi} \right) \frac{d\xi}{dx} = \frac{d^2u}{d\xi^2}. \quad (21)$$

Thus Eq. (1), for  $x < 0$ , takes the form

$$\xi^2 \frac{d^2u}{d\xi^2} + \alpha \xi \frac{du}{d\xi} + \beta u = 0, \quad \xi > 0. \quad (22)$$

But this is exactly the problem that we have just solved; from Eqs. (6), (11), and (18) we have

$$u(\xi) = \begin{cases} c_1 \xi^{r_1} + c_2 \xi^{r_2} \\ (c_1 + c_2 \ln \xi) \xi^{r_1} \\ c_1 \xi^\lambda \cos(\mu \ln \xi) + c_2 \xi^\lambda \sin(\mu \ln \xi), \end{cases} \quad (23)$$

depending on whether the zeros of  $F(r) = r(r-1) + \alpha r + \beta$  are real and different, real and equal, or complex conjugates. To obtain  $u$  in terms of  $x$  we replace  $\xi$  by  $-x$  in Eqs. (23).

We can combine the results for  $x > 0$  and  $x < 0$  by recalling that  $|x| = x$  when  $x > 0$  and that  $|x| = -x$  when  $x < 0$ . Thus we need only replace  $x$  by  $|x|$  in Eqs. (6),

(11), and (18) to obtain real-valued solutions valid in any interval not containing the origin (also see Problems 30 and 31). These results are summarized in the following theorem.

**Theorem 5.5.1** The general solution of the Euler equation (1),

$$x^2 y'' + \alpha x y' + \beta y = 0,$$

in any interval not containing the origin is determined by the roots  $r_1$  and  $r_2$  of the equation

$$F(r) = r(r - 1) + \alpha r + \beta = 0.$$

If the roots are real and different, then

$$y = c_1 |x|^{r_1} + c_2 |x|^{r_2}. \quad (24)$$

If the roots are real and equal, then

$$y = (c_1 + c_2 \ln |x|) |x|^{r_1}. \quad (25)$$

If the roots are complex, then

$$y = |x|^\lambda [c_1 \cos(\mu \ln |x|) + c_2 \sin(\mu \ln |x|)], \quad (26)$$

where  $r_1, r_2 = \lambda \pm i\mu$ .

The solutions of an Euler equation of the form

$$(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0 \quad (27)$$

are similar to those given in Theorem 5.5.1. If one looks for solutions of the form  $y = (x - x_0)^r$ , then the general solution is given by either Eq. (24), (25), or (26) with  $x$  replaced by  $x - x_0$ . Alternatively, we can reduce Eq. (27) to the form of Eq. (1) by making the change of independent variable  $t = x - x_0$ .

The situation for a general second order differential equation with a regular singular point is analogous to that for an Euler equation. We consider that problem in the next section.

## PROBLEMS

In each of Problems 1 through 12 determine the general solution of the given differential equation that is valid in any interval not including the singular point.

- |                               |   |
|-------------------------------|---|
| 1. $x^2 y'' + 4xy' + 2y = 0$  | 2. $(x + 1)^2 y'' + 3(x + 1)y' + 0.75y = 0$ |
| 3. $x^2 y'' - 3xy' + 4y = 0$  | 4. $x^2 y'' + 3xy' + 5y = 0$                |
| 5. $x^2 y'' - xy' + y = 0$    | 6. $(x - 1)^2 y'' + 8(x - 1)y' + 12y = 0$   |
| 7. $x^2 y'' + 6xy' - y = 0$   | 8. $2x^2 y'' - 4xy' + 6y = 0$               |
| 9. $x^2 y'' - 5xy' + 9y = 0$  | 10. $(x - 2)^2 y'' + 5(x - 2)y' + 8y = 0$   |
| 11. $x^2 y'' + 2xy' + 4y = 0$ | 12. $x^2 y'' - 4xy' + 4y = 0$               |

In each of Problems 13 through 16 find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as  $x \rightarrow 0$ .

- ▶ 13.  $2x^2 y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$
- ▶ 14.  $4x^2 y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$
- ▶ 15.  $x^2 y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$
- ▶ 16.  $x^2 y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1$

17. Find all values of  $\alpha$  for which all solutions of  $x^2y'' + \alpha xy' + (5/2)y = 0$  approach zero as  $x \rightarrow 0$ .
18. Find all values of  $\beta$  for which all solutions of  $x^2y'' + \beta y = 0$  approach zero as  $x \rightarrow 0$ .
19. Find  $\gamma$  so that the solution of the initial value problem  $x^2y'' - 2y = 0$ ,  $y(1) = 1$ ,  $y'(1) = \gamma$  is bounded as  $x \rightarrow 0$ .
20. Find all values of  $\alpha$  for which all solutions of  $x^2y'' + \alpha xy' + (5/2)y = 0$  approach zero as  $x \rightarrow \infty$ .
21. Consider the Euler equation  $x^2y'' + \alpha xy' + \beta y = 0$ . Find conditions on  $\alpha$  and  $\beta$  so that
- All solutions approach zero as  $x \rightarrow 0$ .
  - All solutions are bounded as  $x \rightarrow 0$ .
  - All solutions approach zero as  $x \rightarrow \infty$ .
  - All solutions are bounded as  $x \rightarrow \infty$ .
  - All solutions are bounded both as  $x \rightarrow 0$  and as  $x \rightarrow \infty$ .
22. Using the method of reduction of order, show that if  $r_1$  is a repeated root of  $r(r-1) + \alpha r + \beta = 0$ , then  $x^{r_1}$  and  $x^{r_1} \ln x$  are solutions of  $x^2y'' + \alpha xy' + \beta y = 0$  for  $x > 0$ .
23. **Transformation to a Constant Coefficient Equation.** The Euler equation  $x^2y'' + \alpha xy' + \beta y = 0$  can be reduced to an equation with constant coefficients by a change of the independent variable. Let  $x = e^z$ , or  $z = \ln x$ , and consider only the interval  $x > 0$ .
- Show that

$$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2} \frac{d^2y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz}.$$

- Show that the Euler equation becomes

$$\frac{d^2y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0.$$

Letting  $r_1$  and  $r_2$  denote the roots of  $r^2 + (\alpha - 1)r + \beta = 0$ , show that

- If  $r_1$  and  $r_2$  are real and different, then

$$y = c_1 e^{r_1 z} + c_2 e^{r_2 z} = c_1 x^{r_1} + c_2 x^{r_2}.$$

- If  $r_1$  and  $r_2$  are real and equal, then

$$y = (c_1 + c_2 z) e^{r_1 z} = (c_1 + c_2 \ln x) x^{r_1}.$$

- If  $r_1$  and  $r_2$  are complex conjugates,  $r_1 = \lambda + i\mu$ , then

$$y = e^{\lambda z} [c_1 \cos(\mu z) + c_2 \sin(\mu z)] = x^\lambda [c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)].$$

In each of Problems 24 through 29 use the method of Problem 23 to solve the given equation for  $x > 0$ .

- |                                       |   |
|---------------------------------------|---|
| 24. $x^2y'' - 2y = 0$                 | 25. $x^2y'' - 3xy' + 4y = \ln x$          |
| 26. $x^2y'' + 7xy' + 5y = x$          | 27. $x^2y'' - 2xy' + 2y = 3x^2 + 2 \ln x$ |
| 28. $x^2y'' + xy' + 4y = \sin(\ln x)$ | 29. $3x^2y'' + 12xy' + 9y = 0$            |
30. Show that if  $L[y] = x^2y'' + \alpha xy' + \beta y$ , then

$$L[(-x)^r] = (-x)^r F(r)$$

for all  $x < 0$ , where  $F(r) = r(r-1) + \alpha r + \beta$ . Hence conclude that if  $r_1 \neq r_2$  are roots of  $F(r) = 0$ , then linearly independent solutions of  $L[y] = 0$  for  $x < 0$  are  $(-x)^{r_1}$  and  $(-x)^{r_2}$ .

31. Suppose that  $x^{r_1}$  and  $x^{r_2}$  are solutions of an Euler equation for  $x > 0$ , where  $r_1 \neq r_2$ , and  $r_1$  is an integer. According to Eq. (24) the general solution in any interval not containing the origin is  $y = c_1|x|^{r_1} + c_2|x|^{r_2}$ . Show that the general solution can also be written as

$$y = k_1 x^{r_1} + k_2 |x|^{r_2}.$$

*Hint:* Show by a proper choice of constants that the expressions are identical for  $x > 0$ , and by a different choice of constants that they are identical for  $x < 0$ .

## 5.6 Series Solutions near a Regular Singular Point, Part I



We now consider the question of solving the general second order linear equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (1)$$

in the neighborhood of a regular singular point  $x = x_0$ . For convenience we assume that  $x_0 = 0$ . If  $x_0 \neq 0$ , the equation can be transformed into one for which the regular singular point is at the origin by letting  $x - x_0$  equal  $t$ .

The fact that  $x = 0$  is a regular singular point of Eq. (1) means that  $xQ(x)/P(x) = xp(x)$  and  $x^2R(x)/P(x) = x^2q(x)$  have finite limits as  $x \rightarrow 0$ , and are analytic at  $x = 0$ . Thus they have convergent power series expansions of the form

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad (2)$$

on some interval  $|x| < \rho$  about the origin, where  $\rho > 0$ . To make the quantities  $xp(x)$  and  $x^2q(x)$  appear in Eq. (1), it is convenient to divide Eq. (1) by  $P(x)$  and then to multiply by  $x^2$ , obtaining

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \quad (3)$$

or

$$x^2y'' + x(p_0 + p_1x + \cdots + p_nx^n + \cdots)y' + (q_0 + q_1x + \cdots + q_nx^n + \cdots)y = 0. \quad (4)$$

If all of the coefficients  $p_n$  and  $q_n$  are zero except possibly

$$p_0 = \lim_{x \rightarrow 0} \frac{xQ(x)}{P(x)} \quad \text{and} \quad q_0 = \lim_{x \rightarrow 0} \frac{x^2R(x)}{P(x)}, \quad (5)$$

then Eq. (4) reduces to the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0, \quad (6)$$

which was discussed in the preceding section. In general, of course, some of the  $p_n$  and  $q_n$ ,  $n \geq 1$ , are not zero. However, the essential character of solutions of Eq. (4) is identical to that of solutions of the Euler equation (6). The presence of the terms  $p_1x + \cdots + p_nx^n + \cdots$  and  $q_1x + \cdots + q_nx^n + \cdots$  merely complicates the calculations.

We restrict our discussion primarily to the interval  $x > 0$ . The interval  $x < 0$  can be treated, just as for the Euler equation, by making the change of variable  $x = -\xi$  and then solving the resulting equation for  $\xi > 0$ .

Since the coefficients in Eq. (4) are “Euler coefficients” times power series, it is natural to seek solutions in the form of “Euler solutions” times power series. Thus we assume that

$$y = x^r (a_0 + a_1 x + \cdots + a_n x^n + \cdots) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad (7)$$

where  $a_0 \neq 0$ . In other words,  $r$  is the exponent of the first term in the series and  $a_0$  is its coefficient. As part of the solution we have to determine:

1. The values of  $r$  for which Eq. (1) has a solution of the form (7).
2. The recurrence relation for the coefficients  $a_n$ .
3. The radius of convergence of the series  $\sum_{n=0}^{\infty} a_n x^n$ .

The general theory is due to Frobenius<sup>10</sup> and is fairly complicated. Rather than trying to present this theory we simply assume in this and the next two sections that there does exist a solution of the stated form. In particular, we assume that any power series in an expression for a solution has a nonzero radius of convergence, and concentrate on showing how to determine the coefficients in such a series. To illustrate the method of Frobenius we first consider an example.

### EXAMPLE 1

Solve the differential equation

$$2x^2 y'' - xy' + (1+x)y = 0. \quad (8)$$

It is easy to show that  $x = 0$  is a regular singular point of Eq. (8). Further,  $xp(x) = -1/2$  and  $x^2q(x) = (1+x)/2$ . Thus  $p_0 = -1/2$ ,  $q_0 = 1/2$ ,  $q_1 = 1/2$ , and all other  $p$ 's and  $q$ 's are zero. Then, from Eq. (6), the Euler equation corresponding to Eq. (8) is

$$2x^2 y'' - xy' + y = 0. \quad (9)$$

To solve Eq. (8) we assume that there is a solution of the form (7). Then  $y'$  and  $y''$  are given by

$$y' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1} \quad (10)$$

and

$$y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}. \quad (11)$$

By substituting the expressions for  $y$ ,  $y'$ , and  $y''$  in Eq. (8) we obtain

$$\begin{aligned} 2x^2 y'' - xy' + (1+x)y &= \sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} \\ &\quad - \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1}. \end{aligned} \quad (12)$$

<sup>10</sup>Ferdinand Georg Frobenius (1849–1917) was (like Fuchs) a student and eventually a professor at the University of Berlin. He showed how to construct series solutions about regular singular points in 1874. His most distinguished work, however, was in algebra where he was one of the foremost early developers of group theory.

The last term in Eq. (12) can be written as  $\sum_{n=1}^{\infty} a_{n-1}x^{r+n}$ , so by combining the terms in Eq. (12) we obtain

$$2x^2y'' - xy' + (1+x)y = a_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty} \{[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}\}x^{r+n} = 0. \quad (13)$$

If Eq. (13) is to be satisfied for all  $x$ , the coefficient of each power of  $x$  in Eq. (13) must be zero. From the coefficient of  $x^r$  we obtain, since  $a_0 \neq 0$ ,

$$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (r-1)(2r-1) = 0. \quad (14)$$

Equation (14) is called the **indicial equation** for Eq. (8). Note that it is exactly the polynomial equation we would obtain for the Euler equation (9) associated with Eq. (8). The roots of the indicial equation are

$$r_1 = 1, \quad r_2 = 1/2. \quad (15)$$

These values of  $r$  are called the **exponents at the singularity** for the regular singular point  $x = 0$ . They determine the qualitative behavior of the solution (7) in the neighborhood of the singular point.

Now we return to Eq. (13) and set the coefficient of  $x^{r+n}$  equal to zero. This gives the relation

$$[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0 \quad (16)$$

or

$$\begin{aligned} a_n &= -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1} \\ &= -\frac{a_{n-1}}{[(r+n) - 1][2(r+n) - 1]}, \quad n \geq 1. \end{aligned} \quad (17)$$

For each root  $r_1$  and  $r_2$  of the indicial equation we use the recurrence relation (17) to determine a set of coefficients  $a_1, a_2, \dots$ . For  $r = r_1 = 1$ , Eq. (17) becomes

$$a_n = -\frac{a_{n-1}}{(2n+1)n}, \quad n \geq 1.$$

Thus

$$\begin{aligned} a_1 &= -\frac{a_0}{3 \cdot 1}, \\ a_2 &= -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)}, \end{aligned}$$

and

$$a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}.$$

In general we have

$$a_n = \frac{(-1)^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!} a_0, \quad n \geq 1. \quad (18)$$



Hence, if we omit the constant multiplier  $a_0$ , one solution of Eq. (8) is

$$y_1(x) = x \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!} \right], \quad x > 0. \quad (19)$$

To determine the radius of convergence of the series in Eq. (19) we use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{(2n+3)(n+1)} = 0$$

for all  $x$ . Thus the series converges for all  $x$ .

Corresponding to the second root  $r = r_2 = \frac{1}{2}$ , we proceed similarly. From Eq. (17) we have

$$a_n = -\frac{a_{n-1}}{2n(n-\frac{1}{2})} = -\frac{a_{n-1}}{n(2n-1)}, \quad n \geq 1.$$

Hence

$$\begin{aligned} a_1 &= -\frac{a_0}{1 \cdot 1}, \\ a_2 &= -\frac{a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)}, \\ a_3 &= -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)}, \end{aligned}$$

and in general

$$a_n = \frac{(-1)^n}{n![1 \cdot 3 \cdot 5 \cdots (2n-1)]} a_0, \quad n \geq 1. \quad (20)$$

Again omitting the constant multiplier  $a_0$ , we obtain the second solution

$$y_2(x) = x^{1/2} \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n![1 \cdot 3 \cdot 5 \cdots (2n-1)]} \right], \quad x > 0. \quad (21)$$

As before, we can show that the series in Eq. (21) converges for all  $x$ . Since the leading terms in the series solutions  $y_1$  and  $y_2$  are  $x$  and  $x^{1/2}$ , respectively, it follows that the solutions are linearly independent. Hence the general solution of Eq. (8) is

$$y = c_1 y_1(x) + c_2 y_2(x), \quad x > 0.$$

The preceding example illustrates that if  $x = 0$  is a regular singular point, then sometimes there are two solutions of the form (7) in the neighborhood of this point. Similarly, if there is a regular singular point at  $x = x_0$ , then there may be two solutions of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad (22)$$

that are valid near  $x = x_0$ . However, just as an Euler equation may not have two solutions of the form  $y = x^r$ , so a more general equation with a regular singular point may not have two solutions of the form (7) or (22). In particular, we show in the next section that if the roots  $r_1$  and  $r_2$  of the indicial equation are equal, or differ by an

integer, then the second solution normally has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (7) or (22); if  $r_1$  and  $r_2$  differ by an integer, this solution corresponds to the larger value of  $r$ . If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order or some other procedure can be invoked to determine the second solution in such cases. This is discussed in Sections 5.7 and 5.8.

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (7) or (22). Of course, these solutions are complex-valued functions of  $x$ . However, as for the Euler equation, it is possible to obtain real-valued solutions by taking the real and imaginary parts of the complex solutions.

Finally, we mention a practical point. If  $P$ ,  $Q$ , and  $R$  are polynomials, it is often much better to work directly with Eq. (1) than with Eq. (3). This avoids the necessity of expressing  $xQ(x)/P(x)$  and  $x^2R(x)/P(x)$  as power series. For example, it is more convenient to consider the equation

$$x(1+x)y'' + 2y' + xy = 0$$

than to write it in the form

$$x^2y'' + \frac{2x}{1+x}y' + \frac{x^2}{1+x}y = 0,$$

which would entail expanding  $2x/(1+x)$  and  $x^2/(1+x)$  in power series.

## PROBLEMS

In each of Problems 1 through 10 show that the given differential equation has a regular singular point at  $x = 0$ . Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ( $x > 0$ ) corresponding to the larger root. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

- |   |  |
|---|--|
| 1. $2xy'' + y' + xy = 0$                | 2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$ |
| 3. $xy'' + y = 0$                       | 4. $xy'' + y' - y = 0$                       |
| 5. $3x^2y'' + 2xy' + x^2y = 0$          | 6. $x^2y'' + xy' + (x - 2)y = 0$             |
| 7. $xy'' + (1 - x)y' - y = 0$           | 8. $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$        |
| 9. $x^2y'' - x(x + 3)y' + (x + 3)y = 0$ | 10. $x^2y'' + (x^2 + \frac{1}{4})y = 0$      |
11. The Legendre equation of order  $\alpha$  is

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

The solution of this equation near the ordinary point  $x = 0$  was discussed in Problems 22 and 23 of Section 5.3. In Example 5 of Section 5.4 it was shown that  $x = \pm 1$  are regular singular points. Determine the indicial equation and its roots for the point  $x = 1$ . Find a series solution in powers of  $x - 1$  for  $x - 1 > 0$ .

*Hint:* Write  $1 + x = 2 + (x - 1)$  and  $x = 1 + (x - 1)$ . Alternatively, make the change of variable  $x - 1 = t$  and determine a series solution in powers of  $t$ .

12. The Chebyshev equation is

$$(1 - x^2)y'' - xy' + \alpha^2y = 0,$$

where  $\alpha$  is a constant; see Problem 10 of Section 5.3.

- (a) Show that  $x = 1$  and  $x = -1$  are regular singular points, and find the exponents at each of these singularities.  
 (b) Find two linearly independent solutions about  $x = 1$ .

13. The Laguerre<sup>11</sup> differential equation is

$$xy'' + (1-x)y' + \lambda y = 0.$$

Show that  $x = 0$  is a regular singular point. Determine the indicial equation, its roots, the recurrence relation, and one solution ( $x > 0$ ). Show that if  $\lambda = m$ , a positive integer, this solution reduces to a polynomial. When properly normalized this polynomial is known as the Laguerre polynomial,  $L_m(x)$ .

14. The Bessel equation of order zero is

$$x^2y'' + xy' + x^2y = 0.$$

Show that  $x = 0$  is a regular singular point; that the roots of the indicial equation are  $r_1 = r_2 = 0$ ; and that one solution for  $x > 0$  is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}.$$

Show that the series converges for all  $x$ . The function  $J_0$  is known as the Bessel function of the first kind of order zero.

15. Referring to Problem 14, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.

*Hint:* If  $y_2(x) = J_0(x)v(x)$ , then

$$y_2(x) = J_0(x) \int \frac{dx}{x[J_0(x)]^2}.$$

Find the first term in the series expansion of  $1/x[J_0(x)]^2$ .

16. The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

(a) Show that  $x = 0$  is a regular singular point; that the roots of the indicial equation are  $r_1 = 1$  and  $r_2 = -1$ ; and that one solution for  $x > 0$  is

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)! n! 2^{2n}}.$$

Show that the series converges for all  $x$ . The function  $J_1$  is known as the Bessel function of the first kind of order one.

(b) Show that it is impossible to determine a second solution of the form

$$x^{-1} \sum_{n=0}^{\infty} b_n x^n, \quad x > 0.$$

## 5.7 Series Solutions near a Regular Singular Point, Part II

Now let us consider the general problem of determining a solution of the equation

$$L[y] = x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \quad (1)$$

<sup>11</sup>Edmond Nicolas Laguerre (1834–1886), a French geometer and analyst, studied the polynomials named for him about 1879.

where

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad (2)$$

and both series converge in an interval  $|x| < \rho$  for some  $\rho > 0$ . The point  $x = 0$  is a regular singular point, and the corresponding Euler equation is

$$x^2 y'' + p_0 x y' + q_0 y = 0. \quad (3)$$

We seek a solution of Eq. (1) for  $x > 0$  and assume that it has the form

$$y = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad (4)$$

where  $a_0 \neq 0$ , and we have written  $y = \phi(r, x)$  to emphasize that  $\phi$  depends on  $r$  as well as  $x$ . It follows that

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}. \quad (5)$$

Then, substituting from Eqs. (2), (4), and (5) in Eq. (1) gives

$$\begin{aligned} & a_0 r(r-1)x^r + a_1(r+1)r x^{r+1} + \cdots + a_n(r+n)(r+n-1)x^{r+n} + \cdots \\ & + (p_0 + p_1 x + \cdots + p_n x^n + \cdots) \\ & \times [a_0 r x^r + a_1(r+1)x^{r+1} + \cdots + a_n(r+n)x^{r+n} + \cdots] \\ & + (q_0 + q_1 x + \cdots + q_n x^n + \cdots) \\ & \times (a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots) = 0. \end{aligned}$$

Multiplying the infinite series together and then collecting terms, we obtain

$$\begin{aligned} & a_0 F(r)x^r + [a_1 F(r+1) + a_0(p_1 r + q_1)]x^{r+1} \\ & + \{a_2 F(r+2) + a_0(p_2 r + q_2) + a_1[p_1(r+1) + q_1]\}x^{r+2} \\ & + \cdots + \{a_n F(r+n) + a_0(p_n r + q_n) + a_1[p_{n-1}(r+1) + q_{n-1}] \\ & + \cdots + a_{n-1}[p_1(r+n-1) + q_1]\}x^{r+n} + \cdots = 0, \end{aligned}$$

or in a more compact form,

$$\begin{aligned} L[\phi](r, x) = & a_0 F(r)x^r \\ & + \sum_{n=1}^{\infty} \left\{ F(r+n)a_n + \sum_{k=0}^{n-1} a_k [(r+k)p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0, \quad (6) \end{aligned}$$

where

$$F(r) = r(r-1) + p_0 r + q_0. \quad (7)$$

For Eq. (6) to be satisfied identically the coefficient of each power of  $x$  must be zero.

Since  $a_0 \neq 0$ , the term involving  $x^r$  yields the equation  $F(r) = 0$ . This equation is called the **indicial equation**; note that it is exactly the equation we would obtain in looking for solutions  $y = x^r$  of the Euler equation (3). Let us denote the roots of the indicial equation by  $r_1$  and  $r_2$  with  $r_1 \geq r_2$  if the roots are real. If the roots are complex, the designation of the roots is immaterial. Only for these values of  $r$  can we expect to find solutions of Eq. (1) of the form (4). The roots  $r_1$  and  $r_2$  are called the

**exponents at the singularity**; they determine the qualitative nature of the solution in the neighborhood of the singular point.

Setting the coefficient of  $x^{r+n}$  in Eq. (6) equal to zero gives the **recurrence relation**

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0, \quad n \geq 1. \quad (8)$$

Equation (8) shows that, in general,  $a_n$  depends on the value of  $r$  and all the preceding coefficients  $a_0, a_1, \dots, a_{n-1}$ . It also shows that we can successively compute  $a_1, a_2, \dots, a_n, \dots$  in terms of  $a_0$  and the coefficients in the series for  $xp(x)$  and  $x^2q(x)$  provided that  $F(r+1), F(r+2), \dots, F(r+n), \dots$  are not zero. The only values of  $r$  for which  $F(r) = 0$  are  $r = r_1$  and  $r = r_2$ ; since  $r_1 \geq r_2$ , it follows that  $r_1 + n$  is not equal to  $r_1$  or  $r_2$  for  $n \geq 1$ . Consequently,  $F(r_1 + n) \neq 0$  for  $n \geq 1$ . Hence we can always determine one solution of Eq. (1) in the form (4), namely,

$$y_1(x) = x^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_1)x^n \right], \quad x > 0. \quad (9)$$

Here we have introduced the notation  $a_n(r_1)$  to indicate that  $a_n$  has been determined from Eq. (8) with  $r = r_1$ . To specify the arbitrary constant in the solution we have taken  $a_0$  to be 1.

If  $r_2$  is not equal to  $r_1$ , and  $r_1 - r_2$  is not a positive integer, then  $r_2 + n$  is not equal to  $r_1$  for any value of  $n \geq 1$ ; hence  $F(r_2 + n) \neq 0$ , and we can also obtain a second solution

$$y_2(x) = x^{r_2} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_2)x^n \right], \quad x > 0. \quad (10)$$

Just as for the series solutions about ordinary points discussed in Section 5.3, the series in Eqs. (9) and (10) converge at least in the interval  $|x| < \rho$  where the series for both  $xp(x)$  and  $x^2q(x)$  converge. Within their radii of convergence, the power series  $1 + \sum_{n=1}^{\infty} a_n(r_1)x^n$  and  $1 + \sum_{n=1}^{\infty} a_n(r_2)x^n$  define functions that are analytic at  $x = 0$ . Thus the singular behavior, if there is any, of the solutions  $y_1$  and  $y_2$  is due to the factors  $x^{r_1}$  and  $x^{r_2}$  that multiply these two analytic functions. Next, to obtain real-valued solutions for  $x < 0$ , we can make the substitution  $x = -\xi$  with  $\xi > 0$ . As we might expect from our discussion of the Euler equation, it turns out that we need only replace  $x^{r_1}$  in Eq. (9) and  $x^{r_2}$  in Eq. (10) by  $|x|^{r_1}$  and  $|x|^{r_2}$ , respectively. Finally, note that if  $r_1$  and  $r_2$  are complex numbers, then they are necessarily complex conjugates and  $r_2 \neq r_1 + N$ . Thus, in this case we can always find two series solutions of the form (4); however, they are complex-valued functions of  $x$ . Real-valued solutions can be obtained by taking the real and imaginary parts of the complex-valued solutions. The exceptional cases in which  $r_1 = r_2$  or  $r_1 - r_2 = N$ , where  $N$  is a positive integer, require more discussion and will be considered later in this section.

It is important to realize that  $r_1$  and  $r_2$ , the exponents at the singular point, are easy to find and that they determine the qualitative behavior of the solutions. To calculate  $r_1$  and  $r_2$  it is only necessary to solve the quadratic indicial equation

$$r(r-1) + p_0r + q_0 = 0, \quad (11)$$

whose coefficients are given by

$$p_0 = \lim_{x \rightarrow 0} xp(x), \quad q_0 = \lim_{x \rightarrow 0} x^2q(x). \quad (12)$$

Note that these are exactly the limits that must be evaluated in order to classify the singularity as a regular singular point; thus they have usually been determined at an earlier stage of the investigation.

Further, if  $x = 0$  is a regular singular point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad (13)$$

where the functions  $P$ ,  $Q$ , and  $R$  are polynomials, then  $xp(x) = xQ(x)/P(x)$  and  $x^2q(x) = x^2R(x)/P(x)$ . Thus

$$p_0 = \lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)}, \quad q_0 = \lim_{x \rightarrow 0} x^2 \frac{R(x)}{P(x)}. \quad (14)$$

Finally, the radii of convergence for the series in Eqs. (9) and (10) are at least equal to the distance from the origin to the nearest zero of  $P$  other than  $x = 0$  itself.

#### EXAMPLE

#### 1

Discuss the nature of the solutions of the equation

$$2x(1+x)y'' + (3+x)y' - xy = 0$$

near the singular points.

This equation is of the form (13) with  $P(x) = 2x(1+x)$ ,  $Q(x) = 3+x$ , and  $R(x) = -x$ . The points  $x = 0$  and  $x = -1$  are the only singular points. The point  $x = 0$  is a regular singular point, since

$$\begin{aligned} \lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} &= \lim_{x \rightarrow 0} x \frac{3+x}{2x(1+x)} = \frac{3}{2}, \\ \lim_{x \rightarrow 0} x^2 \frac{R(x)}{P(x)} &= \lim_{x \rightarrow 0} x^2 \frac{-x}{2x(1+x)} = 0. \end{aligned}$$

Further, from Eq. (14),  $p_0 = \frac{3}{2}$  and  $q_0 = 0$ . Thus the indicial equation is  $r(r-1) + \frac{3}{2}r = 0$ , and the roots are  $r_1 = 0$ ,  $r_2 = -\frac{1}{2}$ . Since these roots are not equal and do not differ by an integer, there are two linearly independent solutions of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)x^n \quad \text{and} \quad y_2(x) = |x|^{-1/2} \left[ 1 + \sum_{n=1}^{\infty} a_n(-\tfrac{1}{2})x^n \right]$$

for  $0 < |x| < \rho$ . A lower bound for the radius of convergence of each series is 1, the distance from  $x = 0$  to  $x = -1$ , the other zero of  $P(x)$ . Note that the solution  $y_1$  is bounded as  $x \rightarrow 0$ , indeed is analytic there, and that the second solution  $y_2$  is unbounded as  $x \rightarrow 0$ .

The point  $x = -1$  is also a regular singular point, since

$$\begin{aligned} \lim_{x \rightarrow -1} (x+1) \frac{Q(x)}{P(x)} &= \lim_{x \rightarrow -1} \frac{(x+1)(3+x)}{2x(1+x)} = -1, \\ \lim_{x \rightarrow -1} (x+1)^2 \frac{R(x)}{P(x)} &= \lim_{x \rightarrow -1} \frac{(x+1)^2(-x)}{2x(1+x)} = 0. \end{aligned}$$

In this case  $p_0 = -1$ ,  $q_0 = 0$ , so the indicial equation is  $r(r - 1) - r = 0$ . The roots of the indicial equation are  $r_1 = 2$  and  $r_2 = 0$ . Corresponding to the larger root there is a solution of the form

$$y_1(x) = (x + 1)^2 \left[ 1 + \sum_{n=1}^{\infty} a_n(2)(x + 1)^n \right].$$

The series converges at least for  $|x + 1| < 1$  and  $y_1$  is an analytic function there. Since the two roots differ by a positive integer, there may or may not be a second solution of the form

$$y_2(x) = 1 + \sum_{n=1}^{\infty} a_n(0)(x + 1)^n.$$

We cannot say more without further analysis.

Observe that no complicated calculations were required to discover the information about the solutions presented in this example. All that was needed was to evaluate a few limits and solve two quadratic equations.

We now consider the cases in which the roots of the indicial equation are equal, or differ by a positive integer,  $r_1 - r_2 = N$ . As we have shown earlier, there is always one solution of the form (9) corresponding to the larger root  $r_1$  of the indicial equation. By analogy with the Euler equation, we might expect that if  $r_1 = r_2$ , then the second solution contains a logarithmic term. This may also be true if the roots differ by an integer.

**Equal Roots.** The method of finding the second solution is essentially the same as the one we used in finding the second solution of the Euler equation (see Section 5.5) when the roots of the indicial equation were equal. We consider  $r$  to be a continuous variable and determine  $a_n$  as a function of  $r$  by solving the recurrence relation (8). For this choice of  $a_n(r)$  for  $n \geq 1$ , Eq. (6) reduces to

$$L[\phi](r, x) = a_0 F(r) x^r = a_0 (r - r_1)^2 x^r, \quad (15)$$

since  $r_1$  is a repeated root of  $F(r)$ . Setting  $r = r_1$  in Eq. (15), we find that  $L[\phi](r_1, x) = 0$ ; hence, as we already know,  $y_1(x)$  given by Eq. (9) is one solution of Eq. (1). But more important, it also follows from Eq. (15), just as for the Euler equation, that

$$\begin{aligned} L \left[ \frac{\partial \phi}{\partial r} \right] (r_1, x) &= a_0 \frac{\partial}{\partial r} [x^r (r - r_1)^2] \Big|_{r=r_1} \\ &= a_0 [(r - r_1)^2 x^r \ln x + 2(r - r_1)x^r] \Big|_{r=r_1} = 0. \end{aligned} \quad (16)$$

Hence, a second solution of Eq. (1) is

$$\begin{aligned} y_2(x) &= \frac{\partial \phi(r, x)}{\partial r} \Big|_{r=r_1} = \frac{\partial}{\partial r} \left\{ x^r \left[ a_0 + \sum_{n=1}^{\infty} a_n(r) x^n \right] \right\} \Big|_{r=r_1} \\ &= (x^{r_1} \ln x) \left[ a_0 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right] + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n \\ &= y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n, \quad x > 0, \end{aligned} \quad (17)$$



where  $a'_n(r_1)$  denotes  $da_n/dr$  evaluated at  $r = r_1$ .

It may turn out that it is difficult to determine  $a_n(r)$  as a function of  $r$  from the recurrence relation (8) and then to differentiate the resulting expression with respect to  $r$ . An alternative is simply to assume that  $y$  has the form of Eq. (17), that is,

$$y = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n x^n, \quad x > 0, \quad (18)$$

where  $y_1(x)$  has already been found. The coefficients  $b_n$  are calculated, as usual, by substituting into the differential equation, collecting terms, and setting the coefficient of each power of  $x$  equal to zero. A third possibility is to use the method of reduction of order to find  $y_2(x)$  once  $y_1(x)$  is known.

**Roots Differing by an Integer.** For this case the derivation of the second solution is considerably more complicated and will not be given here. The form of this solution is stated in Eq. (24) in the following theorem. The coefficients  $c_n(r_2)$  in Eq. (24) are given by

$$c_n(r_2) = \frac{d}{dr} [(r - r_2)a_n(r)] \Big|_{r=r_2}, \quad n = 1, 2, \dots, \quad (19)$$

where  $a_n(r)$  is determined from the recurrence relation (8) with  $a_0 = 1$ . Further, the coefficient  $a$  in Eq. (24) is

$$a = \lim_{r \rightarrow r_2} (r - r_2)a_N(r). \quad (20)$$

If  $a_N(r_2)$  is finite, then  $a = 0$  and there is no logarithmic term in  $y_2$ . A full derivation of the formulas (19), (20) may be found in Coddington (Chapter 4).

In practice the best way to determine whether  $a$  is zero in the second solution is simply to try to compute the  $a_n$  corresponding to the root  $r_2$  and to see whether it is possible to determine  $a_N(r_2)$ . If so, there is no further problem. If not, we must use the form (24) with  $a \neq 0$ .

When  $r_1 - r_2 = N$ , there are again three ways to find a second solution. First, we can calculate  $a$  and  $c_n(r_2)$  directly by substituting the expression (24) for  $y$  in Eq. (1). Second, we can calculate  $c_n(r_2)$  and  $a$  of Eq. (24) using the formulas (19) and (20). If this is the planned procedure, then in calculating the solution corresponding to  $r = r_1$  be sure to obtain the general formula for  $a_n(r)$  rather than just  $a_n(r_1)$ . The third alternative is to use the method of reduction of order.

### Theorem 5.7.1

Consider the differential equation (1),

$$x^2 y'' + x[xp(x)]y' + [x^2q(x)]y = 0,$$

where  $x = 0$  is a regular singular point. Then  $xp(x)$  and  $x^2q(x)$  are analytic at  $x = 0$  with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for  $|x| < \rho$ , where  $\rho > 0$  is the minimum of the radii of convergence of the power series for  $xp(x)$  and  $x^2q(x)$ . Let  $r_1$  and  $r_2$  be the roots of the indicial equation

$$F(r) = r(r-1) + p_0r + q_0 = 0,$$

with  $r_1 \geq r_2$  if  $r_1$  and  $r_2$  are real. Then in either of the intervals  $-\rho < x < 0$  or  $0 < x < \rho$ , there exists a solution of the form

$$y_1(x) = |x|^{r_1} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_1)x^n \right], \quad (21)$$

where the  $a_n(r_1)$  are given by the recurrence relation (8) with  $a_0 = 1$  and  $r = r_1$ .

If  $r_1 - r_2$  is not zero or a positive integer, then in either of the intervals  $-\rho < x < 0$  or  $0 < x < \rho$ , there exists a second linearly independent solution of the form

$$y_2(x) = |x|^{r_2} \left[ 1 + \sum_{n=1}^{\infty} a_n(r_2)x^n \right]. \quad (22)$$

The  $a_n(r_2)$  are also determined by the recurrence relation (8) with  $a_0 = 1$  and  $r = r_2$ . The power series in Eqs. (21) and (22) converge at least for  $|x| < \rho$ .

If  $r_1 = r_2$ , then the second solution is

$$y_2(x) = y_1(x) \ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} b_n(r_1)x^n. \quad (23)$$

If  $r_1 - r_2 = N$ , a positive integer, then

$$y_2(x) = ay_1(x) \ln|x| + |x|^{r_2} \left[ 1 + \sum_{n=1}^{\infty} c_n(r_2)x^n \right]. \quad (24)$$

The coefficients  $a_n(r_1)$ ,  $b_n(r_1)$ ,  $c_n(r_2)$ , and the constant  $a$  can be determined by substituting the form of the series solutions for  $y$  in Eq. (1). The constant  $a$  may turn out to be zero, in which case there is no logarithmic term in the solution (24). Each of the series in Eqs. (23) and (24) converges at least for  $|x| < \rho$  and defines a function that is analytic in some neighborhood of  $x = 0$ .

## PROBLEMS

In each of Problems 1 through 12 find all the regular singular points of the given differential equation. Determine the indicial equation and the exponents at the singularity for each regular singular point.

1.  $xy'' + 2xy' + 6e^x y = 0$
2.  $x^2y'' - x(2+x)y' + (2+x^2)y = 0$
3.  $x(x-1)y'' + 6x^2y' + 3y = 0$
4.  $y'' + 4xy' + 6y = 0$
5.  $x^2y'' + 3(\sin x)y' - 2y = 0$
6.  $2x(x+2)y'' + y' - xy = 0$
7.  $x^2y'' + \frac{1}{2}(x + \sin x)y' + y = 0$
8.  $(x+1)^2y'' + 3(x^2-1)y' + 3y = 0$
9.  $x^2(1-x)y'' - (1+x)y' + 2xy = 0$
10.  $(x-2)^2(x+2)y'' + 2xy' + 3(x-2)y = 0$
11.  $(4-x^2)y'' + 2xy' + 3y = 0$
12.  $x(x+3)^2y'' - 2(x+3)y' - xy = 0$

In each of Problems 13 through 17:

- (a) Show that  $x = 0$  is a regular singular point of the given differential equation.
- (b) Find the exponents at the singular point  $x = 0$ .

(c) Find the first three nonzero terms in each of two linearly independent solutions about  $x = 0$ .

13.  $xy'' + y' - y = 0$
14.  $xy'' + 2xy' + 6e^x y = 0$ ; see Problem 1
15.  $x(x-1)y'' + 6x^2y' + 3y = 0$ ; see Problem 3
16.  $xy'' + y = 0$
17.  $x^2y'' + (\sin x)y' - (\cos x)y = 0$
18. Show that

$$(\ln x)y'' + \frac{1}{2}y' + y = 0$$

has a regular singular point at  $x = 1$ . Determine the roots of the indicial equation at  $x = 1$ .

Determine the first three nonzero terms in the series  $\sum_{n=0}^{\infty} a_n (x-1)^{r+n}$  corresponding to the larger root. Take  $x-1 > 0$ . What would you expect the radius of convergence of the series to be?

19. In several problems in mathematical physics (for example, the Schrödinger equation for a hydrogen atom) it is necessary to study the differential equation

$$x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0, \quad (i)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants. This equation is known as the **hypergeometric equation**.

- (a) Show that  $x = 0$  is a regular singular point, and that the roots of the indicial equation are 0 and  $1 - \gamma$ .
- (b) Show that  $x = 1$  is a regular singular point, and that the roots of the indicial equation are 0 and  $\gamma - \alpha - \beta$ .
- (c) Assuming that  $1 - \gamma$  is not a positive integer, show that in the neighborhood of  $x = 0$  one solution of (i) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \dots$$

What would you expect the radius of convergence of this series to be?

- (d) Assuming that  $1 - \gamma$  is not an integer or zero, show that a second solution for  $0 < x < 1$  is

$$y_2(x) = x^{1-\gamma} \left[ 1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)1!}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \dots \right].$$

(e) Show that the point at infinity is a regular singular point, and that the roots of the indicial equation are  $\alpha$  and  $\beta$ . See Problem 21 of Section 5.4.

20. Consider the differential equation

$$x^3y'' + \alpha xy' + \beta y = 0,$$

where  $\alpha$  and  $\beta$  are real constants and  $\alpha \neq 0$ .

- (a) Show that  $x = 0$  is an irregular singular point.
- (b) By attempting to determine a solution of the form  $\sum_{n=0}^{\infty} a_n x^{r+n}$ , show that the indicial equation for  $r$  is linear, and consequently there is only one formal solution of the assumed form.
- (c) Show that if  $\beta/\alpha = -1, 0, 1, 2, \dots$ , then the formal series solution terminates and therefore is an actual solution. For other values of  $\beta/\alpha$  show that the formal series solution has a zero radius of convergence, and so does not represent an actual solution in any interval.

21. Consider the differential equation

$$y'' + \frac{\alpha}{x^s} y' + \frac{\beta}{x^t} y = 0, \quad (\text{i})$$

where  $\alpha \neq 0$  and  $\beta \neq 0$  are real numbers, and  $s$  and  $t$  are positive integers that for the moment are arbitrary.

- (a) Show that if  $s > 1$  or  $t > 2$ , then the point  $x = 0$  is an irregular singular point.  
 (b) Try to find a solution of Eq. (i) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad x > 0. \quad (\text{ii})$$

Show that if  $s = 2$  and  $t = 2$ , then there is only one possible value of  $r$  for which there is a formal solution of Eq. (i) of the form (ii).

- (c) Show that if  $s = 1$  and  $t = 3$ , then there are no solutions of Eq. (i) of the form (ii).  
 (d) Show that the maximum values of  $s$  and  $t$  for which the indicial equation is quadratic in  $r$  [and hence we can hope to find two solutions of the form (ii)] are  $s = 1$  and  $t = 2$ . These are precisely the conditions that distinguish a “weak singularity,” or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution we should point out that while it is sometimes possible to obtain a formal series solution of the form (ii) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 20 for an example.

## 5.8 Bessel's Equation



In this section we consider three special cases of Bessel's<sup>12</sup> equation,

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0, \quad (\text{1})$$

where  $\nu$  is a constant, which illustrate the theory discussed in Section 5.7. It is easy to show that  $x = 0$  is a regular singular point. For simplicity we consider only the case  $x > 0$ .

**Bessel Equation of Order Zero.** This example illustrates the situation in which the roots of the indicial equation are equal. Setting  $\nu = 0$  in Eq. (1) gives

$$L[y] = x^2 y'' + x y' + x^2 y = 0. \quad (\text{2})$$

Substituting

$$y = \phi(r, x) = a_0 x^r + \sum_{n=1}^{\infty} a_n x^{r+n}, \quad (\text{3})$$

<sup>12</sup>Friedrich Wilhelm Bessel (1784–1846) embarked on a career in business as a youth, but soon became interested in astronomy and mathematics. He was appointed director of the observatory at Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him in 1824 to make the first systematic analysis of the solutions, known as Bessel functions, of Eq. (1). He is also famous for making the first accurate determination (1838) of the distance from the earth to a star.

we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty} a_n[(r+n)(r+n-1) + (r+n)]x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= a_0[r(r-1) + r]x^r + a_1[(r+1)r + (r+1)]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \{a_n[(r+n)(r+n-1) + (r+n)] + a_{n-2}\}x^{r+n} = 0. \end{aligned} \quad (4)$$

The roots of the indicial equation  $F(r) = r(r-1) + r = 0$  are  $r_1 = 0$  and  $r_2 = 0$ ; hence we have the case of equal roots. The recurrence relation is

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1) + (r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2}, \quad n \geq 2. \quad (5)$$

To determine  $y_1(x)$  we set  $r$  equal to 0. Then from Eq. (4) it follows that for the coefficient of  $x^{r+1}$  to be zero we must choose  $a_1 = 0$ . Hence from Eq. (5),  $a_3 = a_5 = a_7 = \dots = 0$ . Further,

$$a_n(0) = -a_{n-2}(0)/n^2, \quad n = 2, 4, 6, 8, \dots,$$

or letting  $n = 2m$ , we obtain

$$a_{2m}(0) = -a_{2m-2}(0)/(2m)^2, \quad m = 1, 2, 3, \dots$$

Thus

$$a_2(0) = -\frac{a_0}{2^2}, \quad a_4(0) = \frac{a_0}{2^4 2^2}, \quad a_6(0) = -\frac{a_0}{2^6 (3 \cdot 2)^2},$$

and, in general,

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m} (m!)^2}, \quad m = 1, 2, 3, \dots \quad (6)$$

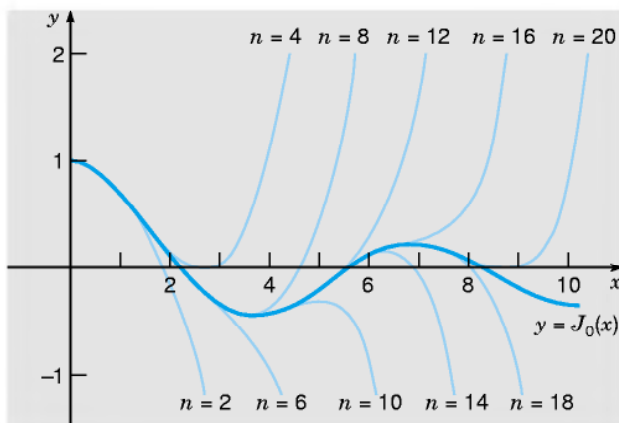
Hence

$$y_1(x) = a_0 \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \right], \quad x > 0. \quad (7)$$

The function in brackets is known as the Bessel function of the first kind of order zero and is denoted by  $J_0(x)$ . It follows from Theorem 5.7.1 that the series converges for all  $x$ , and that  $J_0$  is analytic at  $x = 0$ . Some of the important properties of  $J_0$  are discussed in the problems. Figure 5.8.1 shows the graphs of  $y = J_0(x)$  and some of the partial sums of the series (7).

To determine  $y_2(x)$  we will calculate  $a'_n(0)$ . The alternative procedure in which we simply substitute the form (23) of Section 5.7 in Eq. (2) and then determine the  $b_n$  is discussed in Problem 10. First we note from the coefficient of  $x^{r+1}$  in Eq. (4) that  $(r+1)^2 a_1(r) = 0$ . It follows that not only does  $a_1(0) = 0$  but also  $a'_1(0) = 0$ . It is easy to deduce from the recurrence relation (5) that  $a'_3(0) = a'_5(0) = \dots = a'_{2n+1}(0) = \dots = 0$ ; hence we need only compute  $a'_{2m}(0)$ ,  $m = 1, 2, 3, \dots$ . From Eq. (5) we have

$$a_{2m}(r) = -a_{2m-2}(r)/(r+2m)^2, \quad m = 1, 2, 3, \dots$$



**FIGURE 5.8.1** Polynomial approximations to  $J_0(x)$ . The value of  $n$  is the degree of the approximating polynomial.

By solving this recurrence relation we obtain

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2(r+4)^2 \cdots (r+2m-2)^2(r+2m)^2}, \quad m = 1, 2, 3, \dots \quad (8)$$

The computation of  $a'_{2m}(r)$  can be carried out most conveniently by noting that if

$$f(x) = (x - \alpha_1)^{\beta_1}(x - \alpha_2)^{\beta_2}(x - \alpha_3)^{\beta_3} \cdots (x - \alpha_n)^{\beta_n},$$

and if  $x$  is not equal to  $\alpha_1, \alpha_2, \dots, \alpha_n$ , then

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x - \alpha_1} + \frac{\beta_2}{x - \alpha_2} + \cdots + \frac{\beta_n}{x - \alpha_n}.$$

Applying this result to  $a_{2m}(r)$  from Eq. (8) we find that

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2 \left( \frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m} \right),$$

and, setting  $r$  equal to 0, we obtain

$$a'_{2m}(0) = -2 \left[ \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m} \right] a_{2m}(0).$$

Substituting for  $a_{2m}(0)$  from Eq. (6), and letting

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}, \quad (9)$$

we obtain, finally,

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m} (m!)^2}, \quad m = 1, 2, 3, \dots$$

The second solution of the Bessel equation of order zero is found by setting  $a_0 = 1$  and substituting for  $y_1(x)$  and  $a'_{2m}(0) = b_{2m}(0)$  in Eq. (23) of Section 5.7. We obtain

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m} (m!)^2} x^{2m}, \quad x > 0. \quad (10)$$

Instead of  $y_2$ , the second solution is usually taken to be a certain linear combination of  $J_0$  and  $y_2$ . It is known as the Bessel function of the second kind of order zero and is denoted by  $Y_0$ . Following Copson (Chapter 12), we define<sup>13</sup>

$$Y_0(x) = \frac{2}{\pi} [y_2(x) + (\gamma - \ln 2) J_0(x)]. \quad (11)$$

Here  $\gamma$  is a constant, known as the Euler–Mácheroni (1750–1800) constant; it is defined by the equation

$$\gamma = \lim_{n \rightarrow \infty} (H_n - \ln n) \cong 0.5772. \quad (12)$$

Substituting for  $y_2(x)$  in Eq. (11), we obtain

$$Y_0(x) = \frac{2}{\pi} \left[ \left( \gamma + \ln \frac{x}{2} \right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m} (m!)^2} x^{2m} \right], \quad x > 0. \quad (13)$$

The general solution of the Bessel equation of order zero for  $x > 0$  is

$$y = c_1 J_0(x) + c_2 Y_0(x).$$

Note that  $J_0(x) \rightarrow 1$  as  $x \rightarrow 0$  and that  $Y_0(x)$  has a logarithmic singularity at  $x = 0$ ; that is,  $Y_0(x)$  behaves as  $(2/\pi) \ln x$  when  $x \rightarrow 0$  through positive values. Thus if we are interested in solutions of Bessel's equation of order zero that are finite at the origin, which is often the case, we must discard  $Y_0$ . The graphs of the functions  $J_0$  and  $Y_0$  are shown in Figure 5.8.2.

It is interesting to note from Figure 5.8.2 that for  $x$  large both  $J_0(x)$  and  $Y_0(x)$  are oscillatory. Such a behavior might be anticipated from the original equation; indeed it

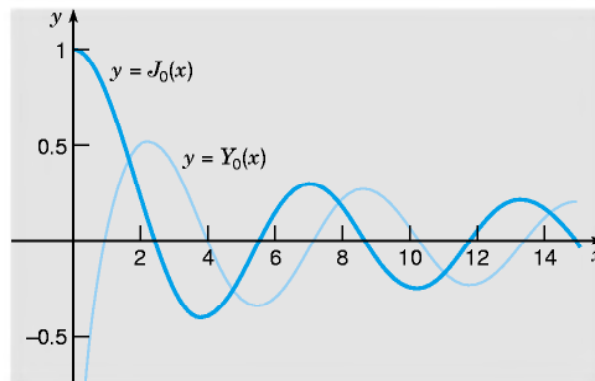


FIGURE 5.8.2 The Bessel functions of order zero.

<sup>13</sup>Other authors use other definitions for  $Y_0$ . The present choice for  $Y_0$  is also known as the Weber (1842–1913) function.



is true for the solutions of the Bessel equation of order  $\nu$ . If we divide Eq. (1) by  $x^2$ , we obtain

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

For  $x$  very large it is reasonable to suspect that the terms  $(1/x)y'$  and  $(\nu^2/x^2)y$  are small and hence can be neglected. If this is true, then the Bessel equation of order  $\nu$  can be approximated by

$$y'' + y = 0.$$

The solutions of this equation are  $\sin x$  and  $\cos x$ ; thus we might anticipate that the solutions of Bessel's equation for large  $x$  are similar to linear combinations of  $\sin x$  and  $\cos x$ . This is correct insofar as the Bessel functions are oscillatory; however, it is only partly correct. For  $x$  large the functions  $J_0$  and  $Y_0$  also decay as  $x$  increases; thus the equation  $y'' + y = 0$  does not provide an adequate approximation to the Bessel equation for large  $x$ , and a more delicate analysis is required. In fact, it is possible to show that

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right) \quad \text{as } x \rightarrow \infty, \quad (14)$$

and that

$$Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right) \quad \text{as } x \rightarrow \infty. \quad (15)$$

These asymptotic approximations, as  $x \rightarrow \infty$ , are actually very good. For example, Figure 5.8.3 shows that the asymptotic approximation (14) to  $J_0(x)$  is reasonably accurate for all  $x \geq 1$ . Thus to approximate  $J_0(x)$  over the entire range from zero to infinity, one can use two or three terms of the series (7) for  $x \leq 1$  and the asymptotic approximation (14) for  $x \geq 1$ .

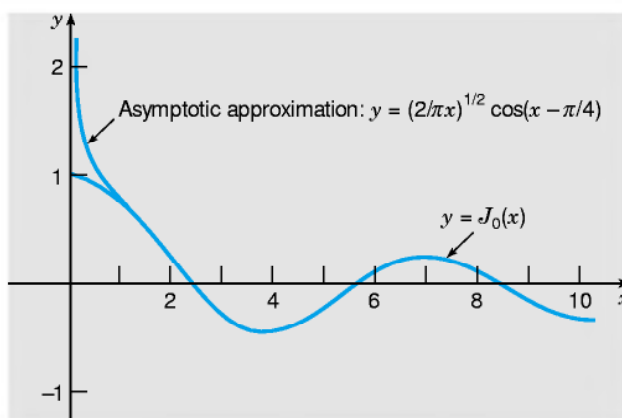


FIGURE 5.8.3 Asymptotic approximation to  $J_0(x)$ .

**Bessel Equation of Order One-Half.** This example illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting  $\nu = \frac{1}{2}$  in Eq. (1) gives

$$L[y] = x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0. \quad (16)$$

If we substitute the series (3) for  $y = \phi(r, x)$ , we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty} \left[ (r+n)(r+n-1) + (r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= \left(r^2 - \frac{1}{4}\right)a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1 x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \left\{ \left[(r+n)^2 - \frac{1}{4}\right]a_n + a_{n-2} \right\} x^{r+n} = 0. \end{aligned} \quad (17)$$

The roots of the indicial equation are  $r_1 = \frac{1}{2}$ ,  $r_2 = -\frac{1}{2}$ ; hence the roots differ by an integer. The recurrence relation is

$$\left[(r+n)^2 - \frac{1}{4}\right]a_n = -a_{n-2}, \quad n \geq 2. \quad (18)$$

Corresponding to the larger root  $r_1 = \frac{1}{2}$  we find from the coefficient of  $x^{r+1}$  in Eq. (17) that  $a_1 = 0$ . Hence, from Eq. (18),  $a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$ . Further, for  $r = \frac{1}{2}$ ,

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \quad n = 2, 4, 6, \dots,$$

or letting  $n = 2m$ , we obtain

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)}, \quad m = 1, 2, 3, \dots$$

By solving this recurrence relation we find that

$$a_2 = -\frac{a_0}{3!}, \quad a_4 = \frac{a_0}{5!}, \dots$$

and, in general,

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad m = 1, 2, 3, \dots$$

Hence, taking  $a_0 = 1$ , we obtain

$$y_1(x) = x^{1/2} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!} \right] = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}, \quad x > 0. \quad (19)$$

The power series in Eq. (19) is precisely the Taylor series for  $\sin x$ ; hence one solution of the Bessel equation of order one-half is  $x^{-1/2} \sin x$ . The Bessel function of the first kind of order one-half,  $J_{1/2}$ , is defined as  $(2/\pi)^{1/2} y_1$ . Thus

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0. \quad (20)$$

Corresponding to the root  $r_2 = -\frac{1}{2}$  it is possible that we may have difficulty in computing  $a_1$  since  $N = r_1 - r_2 = 1$ . However, from Eq. (17) for  $r = -\frac{1}{2}$  the coefficients of  $x^r$  and  $x^{r+1}$  are both zero regardless of the choice of  $a_0$  and  $a_1$ . Hence  $a_0$  and  $a_1$  can be chosen arbitrarily. From the recurrence relation (18) we obtain a set of even-numbered coefficients corresponding to  $a_0$  and a set of odd-numbered coefficients corresponding to  $a_1$ . Thus no logarithmic term is needed to obtain a second solution in this case. It is left as an exercise to show that, for  $r = -\frac{1}{2}$ ,

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \quad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \quad n = 1, 2, \dots$$

Hence

$$\begin{aligned} y_2(x) &= x^{-1/2} \left[ a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right] \\ &= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \quad x > 0. \end{aligned} \quad (21)$$

The constant  $a_1$  simply introduces a multiple of  $y_1(x)$ . The second linearly independent solution of the Bessel equation of order one-half is usually taken to be the solution for which  $a_0 = (2/\pi)^{1/2}$  and  $a_1 = 0$ . It is denoted by  $J_{-1/2}$ . Then

$$J_{-1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos x, \quad x > 0. \quad (22)$$

The general solution of Eq. (16) is  $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$ .

By comparing Eqs. (20) and (22) with Eqs. (14) and (15) we see that, except for a phase shift of  $\pi/4$ , the functions  $J_{-1/2}$  and  $J_{1/2}$  resemble  $J_0$  and  $Y_0$ , respectively, for large  $x$ . The graphs of  $J_{1/2}$  and  $J_{-1/2}$  are shown in Figure 5.8.4.

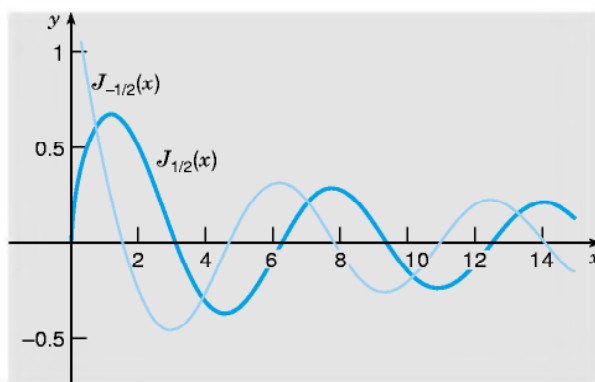


FIGURE 5.8.4 The Bessel functions  $J_{1/2}$  and  $J_{-1/2}$ .

**Bessel Equation of Order One.** This example illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting  $\nu = 1$  in Eq. (1) gives

$$L[y] = x^2 y'' + xy' + (x^2 - 1)y = 0. \quad (23)$$

If we substitute the series (3) for  $y = \phi(r, x)$  and collect terms as in the preceding cases, we obtain

$$\begin{aligned} L[\phi](r, x) &= a_0(r^2 - 1)x^r + a_1[(r + 1)^2 - 1]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \{[(r + n)^2 - 1]a_n + a_{n-2}\}x^{r+n} = 0. \end{aligned} \quad (24)$$

The roots of the indicial equation are  $r_1 = 1$  and  $r_2 = -1$ . The recurrence relation is

$$[(r + n)^2 - 1]a_n(r) = -a_{n-2}(r), \quad n \geq 2. \quad (25)$$

Corresponding to the larger root  $r = 1$  the recurrence relation becomes

$$a_n = -\frac{a_{n-2}}{(n+2)n}, \quad n = 2, 3, 4, \dots$$

We also find from the coefficient of  $x^{r+1}$  in Eq. (24) that  $a_1 = 0$ ; hence from the recurrence relation  $a_3 = a_5 = \dots = 0$ . For even values of  $n$ , let  $n = 2m$ ; then

$$a_{2m} = -\frac{a_{2m-2}}{(2m+2)(2m)} = -\frac{a_{2m-2}}{2^2(m+1)m}, \quad m = 1, 2, 3, \dots$$

By solving this recurrence relation we obtain

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m+1)!m!}, \quad m = 1, 2, 3, \dots \quad (26)$$

The Bessel function of the first kind of order one, denoted by  $J_1$ , is obtained by choosing  $a_0 = 1/2$ . Hence

$$J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}. \quad (27)$$

The series converges absolutely for all  $x$ , so the function  $J_1$  is analytic everywhere.

In determining a second solution of Bessel's equation of order one, we illustrate the method of direct substitution. The calculation of the general term in Eq. (28) below is rather complicated, but the first few coefficients can be found fairly easily. According to Theorem 5.7.1 we assume that

$$y_2(x) = aJ_1(x) \ln x + x^{-1} \left[ 1 + \sum_{n=1}^{\infty} c_n x^n \right], \quad x > 0. \quad (28)$$

Computing  $y_2'(x)$ ,  $y_2''(x)$ , substituting in Eq. (23), and making use of the fact that  $J_1$  is a solution of Eq. (23) give

$$2axJ_1'(x) + \sum_{n=0}^{\infty} [(n-1)(n-2)c_n + (n-1)c_n - c_n]x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0, \quad (29)$$

where  $c_0 = 1$ . Substituting for  $J_1(x)$  from Eq. (27), shifting the indices of summation in the two series, and carrying out several steps of algebra give

$$\begin{aligned} & -c_1 + [0 \cdot c_2 + c_0]x + \sum_{n=2}^{\infty} [(n^2 - 1)c_{n+1} + c_{n-1}]x^n \\ & = -a \left[ x + \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)x^{2m+1}}{2^{2m}(m+1)!m!} \right]. \end{aligned} \quad (30)$$

From Eq. (30) we observe first that  $c_1 = 0$ , and  $a = -c_0 = -1$ . Further, since there are only odd powers of  $x$  on the right, the coefficient of each even power of  $x$  on the left must be zero. Thus, since  $c_1 = 0$ , we have  $c_3 = c_5 = \cdots = 0$ . Corresponding to the odd powers of  $x$  we obtain the recurrence relation [let  $n = 2m + 1$  in the series on the left side of Eq. (30)]

$$[(2m+1)^2 - 1]c_{2m+2} + c_{2m} = \frac{(-1)^m (2m+1)}{2^{2m}(m+1)!m!}, \quad m = 1, 2, 3, \dots \quad (31)$$

When we set  $m = 1$  in Eq. (31), we obtain

$$(3^2 - 1)c_4 + c_2 = (-1)3/(2^2 \cdot 2!).$$

Notice that  $c_2$  can be selected *arbitrarily*, and then this equation determines  $c_4$ . Also notice that in the equation for the coefficient of  $x$ ,  $c_2$  appeared multiplied by 0, and that equation was used to determine  $a$ . That  $c_2$  is arbitrary is not surprising, since  $c_2$  is the coefficient of  $x$  in the expression  $x^{-1}[1 + \sum_{n=1}^{\infty} c_n x^n]$ . Consequently,  $c_2$  simply generates a multiple of  $J_1$ , and  $y_2$  is only determined up to an additive multiple of  $J_1$ . In accord with the usual practice we choose  $c_2 = 1/2^2$ . Then we obtain

$$\begin{aligned} c_4 &= \frac{-1}{2^4 \cdot 2} \left[ \frac{3}{2} + 1 \right] = \frac{-1}{2^4 2!} \left[ \left( 1 + \frac{1}{2} \right) + 1 \right] \\ &= \frac{(-1)}{2^4 \cdot 2!} (H_2 + H_1). \end{aligned}$$

It is possible to show that the solution of the recurrence relation (31) is

$$c_{2m} = \frac{(-1)^{m+1} (H_m + H_{m-1})}{2^{2m} m!(m-1)!}, \quad m = 1, 2, \dots$$

with the understanding that  $H_0 = 0$ . Thus

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x} \left[ 1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} m!(m-1)!} x^{2m} \right], \quad x > 0. \quad (32)$$

The calculation of  $y_2(x)$  using the alternative procedure [see Eqs. (19) and (20) of Section 5.7] in which we determine the  $c_n(r_2)$  is slightly easier. In particular the latter procedure yields the general formula for  $c_{2m}$  without the necessity of solving a recurrence relation of the form (31) (see Problem 11). In this regard the reader may also wish to compare the calculations of the second solution of Bessel's equation of order zero in the text and in Problem 10.

The second solution of Eq. (23), the Bessel function of the second kind of order one,  $Y_1$ , is usually taken to be a certain linear combination of  $J_1$  and  $y_2$ . Following Copson (Chapter 12),  $Y_1$  is defined as

$$Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)J_1(x)], \quad (33)$$

where  $\gamma$  is defined in Eq. (12). The general solution of Eq. (23) for  $x > 0$  is

$$y = c_1 J_1(x) + c_2 Y_1(x).$$

Notice that while  $J_1$  is analytic at  $x = 0$ , the second solution  $Y_1$  becomes unbounded in the same manner as  $1/x$  as  $x \rightarrow 0$ . The graphs of  $J_1$  and  $Y_1$  are shown in Figure 5.8.5.

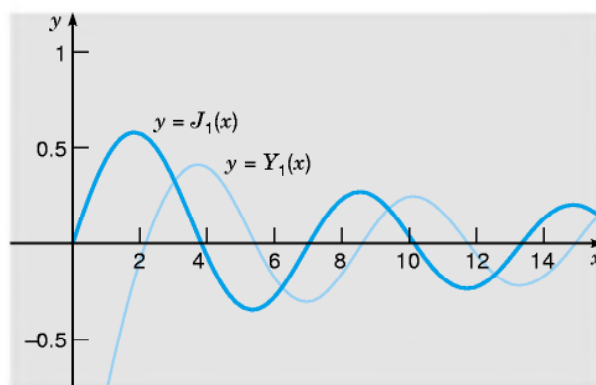


FIGURE 5.8.5 The Bessel functions  $J_1$  and  $Y_1$ .

## PROBLEMS

In each of Problems 1 through 4 show that the given differential equation has a regular singular point at  $x = 0$ , and determine two linearly independent solutions for  $x > 0$ .

1.  $x^2 y'' + 2xy' + xy = 0$
2.  $x^2 y'' + 3xy' + (1+x)y = 0$
3.  $x^2 y'' + xy' + 2xy = 0$
4.  $x^2 y'' + 4xy' + (2+x)y = 0$
5. Find two linearly independent solutions of the Bessel equation of order  $\frac{3}{2}$ ,

$$x^2 y'' + xy' + (x^2 - \frac{9}{4})y = 0, \quad x > 0.$$

6. Show that the Bessel equation of order one-half,

$$x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0, \quad x > 0,$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable  $y = x^{-1/2}v(x)$ . From this conclude that  $y_1(x) = x^{-1/2} \cos x$  and  $y_2(x) = x^{-1/2} \sin x$  are solutions of the Bessel equation of order one-half.

7. Show directly that the series for  $J_0(x)$ , Eq. (7), converges absolutely for all  $x$ .
8. Show directly that the series for  $J_1(x)$ , Eq. (27), converges absolutely for all  $x$  and that  $J_0'(x) = -J_1(x)$ .

9. Consider the Bessel equation of order  $\nu$ ,

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0.$$

Take  $\nu$  real and greater than zero.

- (a) Show that  $x = 0$  is a regular singular point, and that the roots of the indicial equation are  $\nu$  and  $-\nu$ .

- (b) Corresponding to the larger root  $\nu$ , show that one solution is

$$y_1(x) = x^\nu \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)} \left(\frac{x}{2}\right)^{2m} \right].$$

- (c) If  $2\nu$  is not an integer, show that a second solution is

$$y_2(x) = x^{-\nu} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\cdots(m-1-\nu)(m-\nu)} \left(\frac{x}{2}\right)^{2m} \right].$$

Note that  $y_1(x) \rightarrow 0$  as  $x \rightarrow 0$ , and that  $y_2(x)$  is unbounded as  $x \rightarrow 0$ .

- (d) Verify by direct methods that the power series in the expressions for  $y_1(x)$  and  $y_2(x)$  converge absolutely for all  $x$ . Also verify that  $y_2$  is a solution provided only that  $\nu$  is not an integer.

10. In this section we showed that one solution of Bessel's equation of order zero,

$$L[y] = x^2 y'' + xy' + x^2 y = 0,$$

is  $J_0$ , where  $J_0(x)$  is given by Eq. (7) with  $a_0 = 1$ . According to Theorem 5.7.1 a second solution has the form ( $x > 0$ )

$$y_2(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} b_n x^n.$$

- (a) Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2x J_0'(x). \quad (i)$$

- (b) Substituting the series representation for  $J_0(x)$  in Eq. (i), show that

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty} (n^2 b_n + b_{n-2}) x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n} (n!)^2}. \quad (ii)$$

- (c) Note that only even powers of  $x$  appear on the right side of Eq. (ii). Show that  $b_1 = b_3 = b_5 = \cdots = 0$ ,  $b_2 = 1/2^2(1!)^2$ , and that

$$(2n)^2 b_{2n} + b_{2n-2} = -2(-1)^n (2n)/2^{2n} (n!)^2, \quad n = 2, 3, 4, \dots$$

Deduce that

$$b_4 = -\frac{1}{2^2 4^2} \left(1 + \frac{1}{2}\right) \quad \text{and} \quad b_6 = \frac{1}{2^2 4^2 6^2} \left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

The general solution of the recurrence relation is  $b_{2n} = (-1)^{n+1} H_n / 2^{2n} (n!)^2$ . Substituting for  $b_n$  in the expression for  $y_2(x)$  we obtain the solution given in Eq. (10).

11. Find a second solution of Bessel's equation of order one by computing the  $c_n(r_2)$  and  $a$  of Eq. (24) of Section 5.7 according to the formulas (19) and (20) of that section. Some guidelines along the way of this calculation are the following. First, use Eq. (24) of this



section to show that  $a_1(-1)$  and  $a_1'(-1)$  are 0. Then show that  $c_1(-1) = 0$  and, from the recurrence relation, that  $c_n(-1) = 0$  for  $n = 3, 5, \dots$ . Finally, use Eq. (25) to show that

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+1)(r+3)^2 \cdots (r+2m-1)^2(r+2m+1)},$$

for  $m = 1, 2, 3, \dots$ , and calculate

$$c_{2m}(-1) = (-1)^{m+1} (H_m + H_{m-1}) / 2^{2m} m!(m-1)!.$$

12. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show that a solution of

$$x^2 y'' + (\alpha^2 \beta^2 x^{2\beta} + \frac{1}{4} - \nu^2 \beta^2) y = 0, \quad x > 0$$

is given by  $y = x^{1/2} f(\alpha x^\beta)$  where  $f(\xi)$  is a solution of the Bessel equation of order  $\nu$ .

13. Using the result of Problem 12 show that the general solution of the Airy equation

$$y'' - xy = 0, \quad x > 0$$

is  $y = x^{1/2} [c_1 f_1(\frac{2}{3} i x^{3/2}) + c_2 f_2(\frac{2}{3} i x^{3/2})]$  where  $f_1(\xi)$  and  $f_2(\xi)$  are linearly independent solutions of the Bessel equation of order one-third.

14. It can be shown that  $J_0$  has infinitely many zeros for  $x > 0$ . In particular, the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 5.8.1). Let  $\lambda_j$ ,  $j = 1, 2, 3, \dots$ , denote the zeros of  $J_0$ ; it follows that

$$J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}$$

Verify that  $y = J_0(\lambda_j x)$  satisfies the differential equation

$$y'' + \frac{1}{x} y' + \lambda_j^2 y = 0, \quad x > 0.$$

Hence show that

$$\int_0^1 x J_0(\lambda_i x) J_0(\lambda_j x) dx = 0 \quad \text{if} \quad \lambda_i \neq \lambda_j.$$

This important property of  $J_0(\lambda_j x)$ , known as the orthogonality property, is useful in solving boundary value problems.

*Hint:* Write the differential equation for  $J_0(\lambda_i x)$ . Multiply it by  $x J_0(\lambda_j x)$  and subtract it from  $x J_0(\lambda_j x)$  times the differential equation for  $J_0(\lambda_i x)$ . Then integrate from 0 to 1.

## REFERENCES

- Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice Hall, 1961; New York: Dover, 1989).
- Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable* (Oxford: Oxford University, 1935).
- Proofs of Theorems 5.3.1 and 5.7.1 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, or Chapters 3 and 4 of:
- Rainville, E. D., *Intermediate Differential Equations* (2nd ed.) (New York: Macmillan, 1964).

Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 21 of Section 5.4. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of:

Coddington, E. A., and Levinson, N., *Theory of Ordinary Differential Equations* (New York: McGraw-Hill, 1955).

More complete discussions of the Bessel equation, the Legendre equation, and many of the other named equations can be found in advanced books on differential equations, methods of applied mathematics, and special functions. A text dealing with special functions such as the Legendre polynomials and the Bessel functions is:

Hochstadt, H., *Special Functions of Mathematical Physics* (New York: Holt, 1961).

An excellent compilation of formulas, graphs, and tables of Bessel functions, Legendre functions, and other special functions of mathematical physics may be found in:

Abramowitz, M., and Stegun, I. A. (eds.), *Handbook of Mathematical Functions* (New York: Dover, 1965); originally published by the National Bureau of Standards, Washington, DC, 1964.