Embeddability and Stresses of Graphs

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Abstract

Gluck [11] has proven that triangulated 2-spheres are generically 3-rigid. Equivalently, planar graphs are generically 3-stress free. We show that linklessly embeddable graphs are generically 4-stress free. Both of these results are corollaries of the following theorem: every K_{r+2}-minor free graph is generically r-stress free for $1 \leq r \leq 4$. (This assertion is false for $r \geq 6$.) We give an equivalent formulation of this theorem in the language of symmetric algebraic shifting and show that its analogue for exterior algebraic shifting also holds. Some further extensions are detailed.

1 Introduction

Gluck [11] has proven that triangulated 2-spheres are generically 3-rigid. His proof is based on two classical theorems, of Cauchy and of Steinitz. Cauchy's rigidity theorem [7] asserts that any bijection between the vertices of two (convex) 3-polytopes which induces a combinatorial isomorphism, and which induces an isometry of the facets, induces an isometry of the two polytopes. Gluck actually used Alexandrov's [1] extension of this theorem which relaxes the condition by replacing the boundaries of the 3-polytopes with arbitrary triangulations of them. Steinitz's theorem [26] asserts that any polyhedral 2-sphere is combinatorially isomorphic to the boundary complex of some 3-polytope. It is easy to see that a graph with n vertices and $3n-6$ edges is generically 3-rigid iff it is generically 3-stress free. Thus, Gluck's theorem can be stated as:

Theorem 1.1 *(Gluck) Planar graphs are generically 3-stress free.*

We show that also the following relation between embeddability and rigidity holds:

Theorem 1.2 *Linklessly embeddable graphs are generically 4-stress free.*

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Both of these theorems are corollaries of our main theorem:

Theorem 1.3 *For $2 \leq r \leq 6$, every K_r-minor free graph is generically $(r-2)$-stress free.*

The proof is by induction on the number of vertices, based on contracting edges possessing a certain property. We make an essential use of Mader's theorem [20] which gives an upper bound $(r-2)n - \binom{r-1}{2}$ on the number of edges in a K_r-minor free graph with n vertices, for $r \leq 7$. Indeed, Theorem 1.3 can be regarded as a strengthening of Mader's theorem, as being generically l-stress free implies having at most $ln - \binom{l+1}{2}$ edges, a fact which is clear from the equivalent formulation of Theorem 1.3 in terms of symmetric algebraic shifting, detailed below. This also shows that Theorem 1.3 fails for $r \geq 8$, as is demonstrated for $r = 8$ by $K_{2,2,2,2,2}$, and for $r > 8$ by repeatedly coning over the resulted graph for a smaller r (e.g. [25]). It would be interesting to find a proof of Theorem 1.3 that avoids using Mader's theorem (and derive Mader's theorem as a corollary).

Let Δ denote the algebraic shifting operator, for both symmetric and exterior versions. The symmetric case of the following result is equivalent to Theorem 1.3:

Theorem 1.4 *The following holds for symmetric and exterior shifting: for every $2 \leq r \leq 6$ and every graph G, if $\{r-1, r\} \in \Delta(G)$ then G has a K_r minor.*

As $\Delta(G)$ is shifted (i.e. if $\{a, b\} \in \Delta(G)$ and $a' \leq a, b' \leq b$ then $\{a', b'\} \in \Delta(G)$) it is k-colorable iff $\{k, k+1\} \notin \Delta(G)$; in this case a k-coloring f would be $f(i) = min\{i, k\}$. Hence, the following formulation á la Hadwiger

$$K_r \not< G \Rightarrow \chi(\Delta(G)) \leq r - 1$$

holds for $r \leq 6$ and is false for $r \geq 8$; the case $r = 7$ is still open. ($\chi(H)$ is the cromaric number of H and $H \not< G$ means that G is H-minor free.)

Problem 1.5 *Does Theorem 1.4 continue to hold when replacing "K_r minor" with "subdivision of K_r"?*

The answer is positive for $r = 2, 3, 4$ as in this case G has a K_r minor iff G has a subdivision of K_r ([10], Proposition 1.7.2). Mader proved that every graph on n vertices with more than $3n - 6$ edges contains a subdivision of K_5 [21]. A positive answer in the case $r = 5$ would strengthen this result.

Let $\mu(G)$ denote the Colin de Verdière's parameter of a graph G.

Conjecture 1.6 *Let G be a graph and let k be a positive integer. If $\mu(G) \leq k$ then $\{k+1, k+2\} \notin \Delta(G)$.*

For $k = 1, 2, 3, 4$ the conjecture holds true. Colin de Verdière [8] showed that the family $\{G : \mu(G) \leq k\}$ is closed under taking minors for every k. Note that $\mu(K_r) = r-1$. By Theorem 1.4 the conjecture holds for $k \leq 4$. Another "evidence" is that clique sums do not violate the conjecture: Suppose that G_1 and G_2 satisfy the conjecture, $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique. Let $max\{\mu(G_1), \mu(G_2)\} = k$. By hypothesis, $\{k+1, k+2\} \notin \Delta(G_i)$ for $i = 1, 2$. By [22], Thm.1.2 $\{k+1, k+2\} \notin \Delta(G)$. Also $\mu(G) \geq k$ and $\Delta(G)$ is shifted, hence G satisfy the conjecture. (Van der Holst, Lovász and Schrijver [12] investigated the behavior of Colin de Verdière's parameter under taking clique sums.) Conjecture 1.6 implies

$$\mu(G) \leq k \Rightarrow e \leq kv - \binom{k+1}{2}$$

(where e and v are the numbers of edges and vertices in G, respectively) which is not known either.

This paper is organized as follows: Section 2 provides relevant background in rigidity theory of graphs, Section 3 deals with graph minors, in Section 4 we prove the results about stress freeness mentioned in the Introduction, Section 5 deals with algebraic shifting - both symmetric and exterior, and concludes with a proof of Theorem 1.4 and some extensions concerning embeddability into 2-manifolds.

2 Rigidity

The presentation here is based mainly on Kalai's [15]. Let $G = (V, E)$ be a graph. Let $d(a, b)$ denote Euclidian distance between points a and b in Euclidian space. A d-embedding $f : V \to \mathbb{R}^d$ is called *rigid* if there exists an $\varepsilon > 0$ such that if $g : V \to \mathbb{R}^d$ satisfies $d(f(v), g(v)) < \varepsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$. Loosely speaking, f is rigid if any perturbation of it which preserves the lengths of the edges actually preserves the distances between any pair of vertices. G is called *generically* $d-rigid$ if the set of its rigid d-embeddings is open and dense in the topological vector space of all of its d-embeddings. Given a d-embedding $f : V \to \mathbb{R}^d$, a *stress* w.r.t. f is a function $w : E \to \mathbb{R}$ s.t. for every vertex $v \in V$

$$\sum_{u:\{v,u\}\in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

G is called *generically d-stress free* if the set of its d-embeddings which has a unique stress ($w = 0$) is open and dense in the space of all of its d-embeddings.

Rigidity and stress freeness can be related as follows: Let $V = [n]$, and let $Rig(G, f)$ be the $dn \times |E|$ matrix associated with a d-embedding f of $V(G)$ defined as follows: for its column corresponding to $\{v < u\} \in E$

put the vector $f(v) - f(u)$ (resp. $f(u) - f(v)$) at the entries of the rows corresponding to v (resp. u) and zero otherwise. G is generically d-stress free if $Ker(Rig(G, f)) = 0$ for a generic f (i.e. for an open dense set of embeddings). G is generically d-rigid if $Im(Rig(G, f)) = Im(Rig(K_V, f)$ for a generic f, where K_V is the complete graph on $V = V(G)$. The dimensions of the kernel and image of $Rig(G, f)$ are independent of the generic f we choose; we call $R(G) = Rig(G, f)$ the *rigidity matrix* of G.

$Im(Rig(K_V, f))$ can be described by the following linear equations: $(v_1, .., v_d) \in \bigoplus_{i=1}^{d} \mathbb{R}^n$ belongs to $Im(Rig(K_V, f))$ iff

$$\forall 1 \leq i \neq j \leq d \quad < f_i, v_j >=< f_j, v_i > \tag{1}$$

$$\forall 1 \leq i \leq d \quad < e, v_i >= 0 \tag{2}$$

where e is the all ones vector and f_i is the vector of the ith coordinate of the $f(v)$'s, $v \in V$. From this description it is clear that $rank(Rig(K_V, f)) = dn - \binom{d+1}{2}$ (see Asimov and Roth [2] for more details).

We need the following theorem of Whiteley:

Theorem 2.1 *(Whiteley [27]) Let G' be obtained from a graph G by contracting an edge $\{u, v\}$.*

(a)If u, v have at least $d - 1$ common neighbors and G' is generically d-rigid, then G is generically d-rigid.

(b)If u, v have at most $d - 1$ common neighbors and G' is generically d-stress free, then G is generically d-stress free.

In Section 5 we will prove an analogous statement in the language of exterior shifting. Theorem 2.1 gives an alternative proof of Gluck's theorem (Whiteley [27]): starting with a triangulated 2-sphere, repeatedly contract edges with exactly 2 common neighbors until a tetrahedron is reached (it is not difficult to show that this is always possible). By Theorem 2.1(a) it is enough to show that the tetrahedron is generically 3-rigid, as is well known (Asimov and Roth [2]).

For later use, we need the following result about stress-freeness of a union of graphs.

Theorem 2.2 *(Asimov and Roth [3]) Let $G_i = (V_i, E_i)$ be k-stress free graphs, $i = 1, 2$ s.t. $G_1 \cap G_2$ is k-rigid. Then $G_1 \cup G_2$ is k-stress free.*

3 Minors

All graphs we consider are simple, i.e. with no loops and no multiple edges. Let $e = \{v, u\}$ be an edge in a graph G. By *conrtacting* e we mean identifying the vertices v and u and deleting the loop and one copy of each double edge created by this identification, to obtain a new (simple) graph. A graph

H is called a *minor* of a graph G, denoted $H \prec G$, if by repeated contraction of edges we can obtain H from a subgraph of G. In the sequel we shall make an essential use of the following Theorem of Mader:

Theorem 3.1 *(Mader [20]) For $3 \leq r \leq 7$, if a graph G on n vertices has no K_r minor then it has at most $(r-2)n - \binom{r-1}{2}$ edges.*

Proposition 3.2 *For $3 \leq r \leq 5$: If G has an edge and each edge belongs to at least $r-2$ triangles, then G has a K_r minor.*

Proof: For $r = 3$ G actually contains K_3 as a subgraph. Let G have n vetrices and e edges. Assume (by contradiction) that $K_r \not\prec G$. W.l.o.g. G is connected.

For $r = 4$, by Theorem 3.1 $e \leq 2n - 3$ hence there is a vertex $u \in G$ with degree $d(u) \leq 3$. Denote by $N(u)$ the induced subgraph on the neighbors of u. For every $v \in N(u)$, the edge uv belongs to at least two triangles, hence $N(u)$ is a triangle, and together with u we obtain a K_4 as a subgraph of G, a contradiction.

For $r = 5$, by Theorem 3.1 $e \leq 3n - 6$ hence there is a vertex $u \in G$ with degree $d(u) \leq 5$. Also $d(u) \geq 4$ (as we may assume that u is not an isolated vertex). If $d(u) = 4$ then the induced subgraph on $\{u\} \cup N(u)$ is K_5, a contradiction. Otherwise, $d(u) = 5$. Every $v \in N(u)$ has degree at least 3 in $N(u)$, hence $e(N(u)) \geq \lceil 3 \cdot 5/2 \rceil = 8$. But $K_4 \not\prec N(u)$, hence $e(N(u)) \leq 2 \cdot 5 - 3 = 7$, a contradiction.$\blacksquare$

Proposition 3.3 *If G has an edge and each edge belongs to at least 4 triangles, then either G has a K_6 minor, or G is a clique sum over K_r for some $r \leq 4$ (i.e. $G = G_1 \cup G_2, G_1 \cap G_2 = K_r$, $G_i \neq K_r$, $i = 1, 2$).*

Proof: We proceed as in the proof of Proposition 3.2: Assume that $K_6 \not\prec G$. W.l.o.g. G is connected. By Theorem 3.1 $e \leq 4n - 10$ hence there is a vertex $u \in G$ with degree $d(u) \leq 7$, also $d(u) \geq 5$. If $d(u) = 5$ then $N(u) = K_5$, a contradiction. Actually, since $K_5 \not\prec N(u)$ and $N(u)$ has at most 7 vertices each of them of degree at least 4, Wagner's structure theorem for K_5-minor free graphs ([10], Theorem 8.3.4) asserts that $N(u)$ is planar.

If $d(u) = 6$, then $12 = 3 \cdot 6 - 6 \geq e(N(u)) \geq 4 \cdot 6/2 = 12$ hence $N(u)$ is a triangulation of the 2-sphere S^2. If $d(u) = 7$, then $15 = 3 \cdot 7 - 6 \geq e(N(u)) \geq 4 \cdot 7/2 = 14$. We will show now that $N(u)$ cannot have 14 edges, hence it is a triangulation of S^2: Assume that $N(u)$ has 14 edges, so each of its vertices has degree 4, and $N(u)$ is a triangulation of S^2 minus an edge. Let us look on the unique square (in a planar embedding) and denote its vertices by A. Counting missing edges (there are 7 of them) shows that there is one missing edge between the vertices of $N(u) \setminus A = \{a, b, c\}$, say $\{b, c\}$. we now look at the neighborhood of a in a planar embedding (it is a

4-cycle): b, c must be opposite in this square as $\{b, c\}$ is missing. Hence for $v \in A \cap N(a)$ we get that v has degree 5, a contradiction.

Now we are left to deal with the case where $N(u)$ is a triangulation of S^2, and hence a maximal K_5-minor free graph. If G is the cone over $N(u)$ with apex u, then every edge in $N(u)$ belongs to at least 3 triangles in $N(u)$. By Proposition 3.2, $N(u)$ has a K_5 minor, a contradiction. Hence there exists a vertex $w \neq u$, $w \in G \setminus N(u)$. Denote by $[w]$ the set of all vertices in G connected to w by a path disjoint from $N(u)$. Denote by $N'(w)$ the induced graph on the vertices in $N(u)$ that are neighbors of some vertex in $[w]$. If $N'(w)$ is not a clique, there are two non-neighbors $x, y \in N'(w)$, and a path through vertices of $[w]$ connecting them. This path together with the cone over $N(u)$ with apex u form a subgraph of G with a K_6 minor, a contradiction.

Suppose $N'(w)$ is a clique (it has at most 4 vertices, as $N(u)$ is planar). Then G is a clique sum of two graphs that strictly contain $N'(w)$: Let G_1 be the induced graph on $[w] \cup N'(w)$ and let G_2 be the induced graph on $G \setminus [w]$. Then $G = G_1 \cup G_2$ and $G_1 \cap G_2 = N'(w)$. $\blacksquare$

Remark In view of Theorem 3.1 for the case $r = 7$, we may expect the following to be true:

Problem 3.4 *If G has an edge and each edge belongs to at least 5 triangles, then either G has a K_7 minor, or G is a clique sum over K_l for some $l \leq 6$.*

If true, it extends the assertion of Theorem 1.4 to the case $r = 7$. By now we can show only the weaker assertion

$$\{6, 7\} \in \Delta(G) \Rightarrow K_7^- \prec G,$$

using similar arguments to those used for proving Theorem 1.4 (K_7^- is K_7 minus an edge). However, if the assertion of Problem 3.4 holds for some r, it implies that Theorem 1.4 holds for this r, hence $e(G) = e(\Delta(G)) \leq (r-2)n - \binom{r-1}{2}$. But as mentioned in the Introduction, this is false for $r \geq 8$.

4 Proof of Theorems 1.1, 1.2 and 1.3

Proof of Theorem 1.3: For $r = 2$ the assertion of the theorem is trivial. Suppose $K_r \not\prec G$, and contract edges belonging to at most $r - 3$ triangles as long as it is possible. Denote the resulted graph by G'. Repeated application of Theorem 2.1 asserts that if G' is generically $(r-2)$-stress free, then so is G. In case G' has no edges, it is trivially $(r-2)$-stress free. Otherwise, G' has an edge, and each edge belongs to at least $r - 2$ triangles. For $2 < r < 6$, by Proposition 3.2 G' has a K_r minor, hence so has G, a contradiction. For $r = 6$, by Proposition 3.3 G' either has a K_6 minor which leads to a contradiction, or G' is a clique sum over K_r for some $r \leq 4$. In the later case, denote $G' = G_1 \cup G_2$, $G_1 \cap G_2 = K_r$. As the graph of a simplex is k-rigid for any k, by Theorem 2.2 it is enough to show that each G_i is generically $(r-2)$-stress free, which follows from induction hypothesis on the number of vertices. ■

Remark Note that we proved the case $r = 5$ without using Wagner's structure theorem for K_5-minor free graphs ([10], Theorem 8.3.4), but we used Theorem 3.1 of Mader. Alternatively, we can prove the case $r = 5$ avoiding Mader's theorem but using Wagner's theorem and the 'gluing lemma' Theorem 2.2. Using Wagner's structure theorem for $K_{3,3}$-minor free graphs ([10], ex.18 on p.185) and Theorem 2.2, we conclude that $K_{3,3}$-minor free graphs are generically 4-stress free.

Theorems 1.1 and 1.2 now follow as easy corollaries:

Proof of Theorem 1.2: By (the easy part of) the theorem by Robertson, Seymour and Thomas characterizing linklessly embeddable graphs by a family of forbidden minors [24], a linklessly embeddable graph has no K_6 minor, hence by Theorem 1.3 it is generically 4-stress free. ■

Proof of Theorem 1.1: By (the easy part of) Kuratowski's criterion for planarity of graphs [17], a planar graph has no K_5 minor, hence by Theorem 1.3 it is generically 3-stress free. ■

5 Algebraic shifting

5.1 definition of algebraic shifting

Algebraic shifting is an operator which associates with each simplicial complex another simplicial complex which is combinatorially simpler. It was introduced by Kalai [13]. We follow the definitions and notation of [16]: Let K be a simplicial complex on a vertex set $[n]$. The i-th skeleton of K is $K_i = \{S \in K : |S| = i+1\}$. For each $1 \leq k \leq n$ let $<_L$ be the lexicographic order on $\binom{[n]}{k}$, i.e. $S <_L T \Leftrightarrow min\{a : a \in S \triangle T\} \in S$, and let $\triangleleft_P$ be the partial order defined by: Let $S = \{s_1 < \cdots < s_k\}, T = \{t_1 < \cdots < t_k\}$, $S \triangleleft_P T$ iff $s_i \leq t_i$ for every $1 \leq i \leq k$ (min and $\leq$ are taken with respect to the usual order on $\mathbb{N}$). K is called *shifted* if $S \triangleleft_P T \in K$ implies $S \in K$.

We now describe exterior shifting: Let V be an n-dimensional vector space over a field k of characteristic zero, with basis $\{e_1, \ldots, e_n\}$. Let $\bigwedge V$ be the graded exterior algebra over V. Denote $e_S = e_{s_1} \wedge \cdots \wedge e_{s_j}$ where $S = \{s_1 < \cdots < s_j\}$. Define the exterior algebra of K by the ring quotient

$$\bigwedge(K) = \bigwedge V/(e_S : S \notin K) = \bigwedge V/sp\{e_S : S \notin K\}.$$

Let $\{f_1, \ldots, f_n\}$ be a basis of V, generic over $\mathbb{Q}$ with respect to $\{e_1, \ldots, e_n\}$, which means that the entries of the corresponding transition matrix A are algebraically independent over $\mathbb{Q}$. Let $\tilde{f_S}$ be the image of $f_S \in \bigwedge V$ in $\bigwedge(K)$. We choose a basis for $\bigwedge(K)$ from these images in the greedy way, to construct the following collection of sets:

$$\Delta^e(K) = \bigcup_i \{S : \tilde{f}_S \notin sp\{\tilde{f}_{S'} : S' <_L S\}, |S| = i\}.$$

The construction is canonic (i.e. independent both of the numbering of the vertices of K and of the choice of the generic matrix A), and results in a shifted simplicial complex.

For symmetric shifting, let us look on the face ring (Stanley-Reisner ring) of K $k[K] = k[x_1,..,x_n]/I_K$ where I_K is the homogenous ideal generated by the monomials whose support is not in K (grading is by degree). Let $y_1,\ldots,y_n$ be generic linear combinations of $x_1,\ldots,x_n$. We choose a basis for each graded component of $k[K]$, up to degree $dim(K)+1$, from the canonic projection of the monomials in the y_i's, in the greedy way:

$$GIN(K) = \{m : \tilde{m} \notin sp\{\tilde{m'} : deg(m') = deg(m), m' <_L m\}\}$$

(where $\prod y_i^{a_i} <_L \prod y_i^{b_i}$ iff for $j = min\{i : a_i \neq b_i\}$ $a_j > b_j$). The combinatorial information in $GIN(K)$ is redundant: if $m \in GIN(K)$ of degree $i \leq dim(K)$ then $y_1 m,..,y_i m$ are also in $GIN(K)$. Thus, $GIN(K)$ can be reconstructed from its monomials of the form $m = y_{i_1} \cdot y_{i_2} \cdot .. \cdot y_{i_r}$ where $r \leq i_1 \leq i_2 \leq .. \leq i_r$, $r \leq dim(K)+1$. Denote this set by $gin(K)$, and define $S(m) = \{i_1 - r + 1, i_2 - r + 2, .., i_r\}$ for such m. The collection of sets

$$\Delta^s(K) = \cup\{S(m) : m \in gin(K)\}$$

carries the same combinatorial information as $GIN(K)$. It is a shifted simplicial complex. Again, the construction is canonic, in the same sense as for exterior shifting.

5.2 connection with rigidity and proof of Theorem 1.4

Let G be a graph. By the results of Lee [19], $\{d+1, d+2\} \notin \Delta^s(K)$ iff G is generically d-stress free, as both of these assertions are equivalent to a zero kernel of the rigidity matrix. We will describe now a similar statement for exterior shifting in more details; the exterior analogue of rigidity being Kalai's notion of hyperconnectivity [14].

We keep the notation from the previous subsection and follow the presentation in [14]. Fix $k = \mathbb{R}$. Let $(\bigwedge V)^* \cong \bigwedge(V^*)$ be the dual of $\bigwedge V$. Fixing the basis $e = \{e_1,\ldots,e_n\}$ induces an inner product on the degree j part of $\bigwedge V$, denoted $\wedge^j V$, for every j: $< f, g >= f^*(g)$ is a bilinear extension of $e_S^*(e_T) = \delta_{S,T}$, where $|S| = |T| = j$. Define a left interior product of g on f, where $g, f \in \wedge V$, denoted $g \lfloor f$, by the requirement:

$$< h, g\lfloor f >=< h \wedge g, f > \quad for\ all\ h \in \bigwedge V.$$

Thus, $g\lfloor f$ is a bilinear function, satisfying

$$e_T\lfloor e_S = \{^{\pm e_{S\setminus T}\ if\ T\subseteq S}_{0\ otherwise}$$

where the sign equals $(-1)^a$, where $a = |\{(s,t) \in S\times T : s \notin T, t < s\}|$.

This implies in particular that for g a wedge product of elements of degree 1, $g\lfloor$ is a boundary operation on $\bigwedge V$, and in particular on $\bigoplus_i M_i(K)$ where $M_i(K)$ is the subspace of $\bigwedge V$ spanned by $\{e_S : S \in K_i\}$. Consider the map

$$f(d,i,K) : M_i(K) \to \bigoplus_1^d M_{i-1}(K) \quad x \mapsto (f_1\lfloor x, ..., f_d\lfloor x).$$

The dimension of its kernel equals $|\{S \in \Delta^e K : |S| = i+1, S\cap[d] = \emptyset\}|$ (more details in [22]). Kalai [14] called a graph G *d-hyperconnected* if $Im(f(d,1,G)) = Im(f(d,1,K_{Ver(G)}))$, and *d-acyclic* if $Ker(f(d,1,G)) = 0$. With this terminology, G is d-acyclic iff $\{d+1, d+2\} \notin \Delta^e(K)$.

We shall prove now an exterior analogue of Theorem 2.1:

Proposition 5.1 *If G' is obtained from G by contracting an edge which belongs to at most $d-1$ triangles, and G' is d-acyclic, then so is G.*

Proof: Let $\{v,u\}$ be the edge we contract. Consider the $dn\times|E|$ matrix A of the map $f(d,1,G)$ w.r.t. the standard basis, where $f_i = \sum_{j=1}^n \alpha_{ij}e_j$, $n = |V|$: for its column corresponding to $\{v<u\} \in E$ put the vector $(\alpha_{1u}, .., \alpha_{du})^T$ (resp. $-(\alpha_{1v}, .., \alpha_{dv})^T$) at the entries of the rows corresponding to v (resp. u) and zero otherwise.

Now replace in A each α_{iv} with α_{iu} to obtain a new matrix $\hat{A}$. It is enough to show that the columns of $\hat{A}$ are independent: As the set of $dn\times|E|$ matrices with independent columns is open (in the Euclidian topology), by perturbing the α_{iu}'s in the places where $\hat{A}$ differs from A, we may obtain new generic α_{iv}'s forming a matrix with independent columns. But for every generic choice of f_i's, the map $f(d,1,G)$ has the same rank, hence we would conclude that the columns of A are independent as well.

Suppose a linear combination of the columns of $\hat{A}$ equals zero. Let $\bar{A}$ be obtained from $\hat{A}$ by adding the rows of v to the corresponding rows of u, and deleting the rows of v. Thus, the a linear combination with the same coefficients of the columns of $\bar{A}$ also equals zero. $\bar{A}$ is obtained from the matrix of $f(d,1,G')$ by adding a zero column (for the edge $\{v,u\}$) and doubling the columns which corresponds to common neighbors of v and u in G. As $Ker(f(d,1,G')) = 0$, apart from the above mentioned columns the rest have coefficient zero, and pairs of columns we doubled have opposite sign. Let us look at the submatrix of $\hat{A}$ consisting of the 'doubled' columns with vertex v and the column of $\{v,u\}$, restricted to the rows of v: it has generic coefficients, d rows and at most d columns, hence its columns are independent. Thus, all coefficients in the above linear combination are zero. ■

We need the following exterior analogue of Theorem 2.2:

Theorem 5.2 *(Kalai [14]) Let $G_i = (V_i, E_i)$ be k-acyclic graphs, $i = 1, 2$ s.t. $G_1 \cap G_2$ is k-hyperconnected. Then $G_1 \cup G_2$ is k-acyclic.*

Proof of Theorem 1.4: As explained in subsection 5.2, Theorem 1.3 is equivalent to the symmetric case of Theorem 1.4. In the exterior case, the case $r = 2$ is trivial as shifting preserves the f-vector. Now we repeat the proof of Theorem 1.3 almost word by word, introducing the following modifications. Replace "Theorem 2.1" by "Proposition 5.1". Replace "stress free" by "acyclic", and "rigid" by "hyperconnected" everywhere. Replace "Theorem 2.2" by "Theorem 5.2". As G is $(r-2)$-acyclic iff $\{r-1, r\} \notin \Delta^e(G)$, the proof is completed. ■

5.3 embedding into 2-manifolds

Theorem 1.1 may be extended to other 2-manifolds as follows:

Theorem 5.3 *Let $M \neq S^2$ be a compact connected 2-manifold without boundary, and let G be a graph. Suppose that $\{r-1, r\} \in \Delta(G)$ and K_r can not be embedded in M. Then G can not be embedded in M.*

Proof: Let $g = g(M) > 0$ be the genus of M (e.g. the torus has genus 1, the projective plane has genus 1/2). Assume by contradiction that G embeds in M. By looking at the rigidity matrix we note that deleting from G a vertex of degree at most $r-2$ preserves the existence of $\{r-1, r\}$ in the shifted graph. Deletion preserves embeddability in M as well. Thus we may assume that G has minimal degree $\delta(G) \geq r-1$. By Euler formula $e \leq 3v - 6 + 6g$ (where e and v are the numbers of edges and vertices in G respectively). Also $e \geq (r-1)v/2$, hence $v \leq \frac{12g-12}{(r-1)-6}$. Thus $(r-1)^2 - 5(r-1) + (6-12g) \leq 0$ which implies $r \leq (7 + \sqrt{1+48g})/2$. But K_r can not be embedded in M, hence by Ringel and Youngs [23] proof of Heawood's map-coloring conjecture $r > (7 + \sqrt{1+48g})/2$, a contradiction.■

Remark For any compact connected 2-manifold without boundary of positive genus, M, embedded in $\mathbb{R}^3$, two linked simple closed curves on it exist. One may ask whether the graph of any triangulated such M is always not linkless. For the projective plane this is true. It follows from the fact that the two minimal triangulations of the projective plane (w.r.t. edge contraction), determined by Barnette [4], have a minor from the Petersen family, and hence are not linkless, by the result of Robertson, Seymour and Thomas [24]. Moreover, the graph of any polyhedral map of the projective plane is not linkless, as its 7 minimal polyhedral maps (w.r.t. edge contraction), determined by Barnette [5], have graphs equal to 6 of the members in Petersen family.

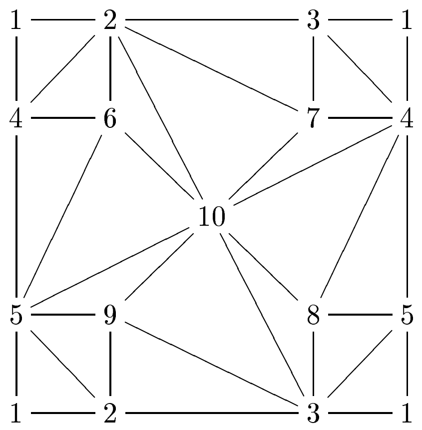

Figure 1: Linkless graph of a torus

Examining the 21 minimal triangulations of the torus, see Lavrenchenko [18], we note that 20 of them have a K_6 minor, and hence are not linkless, but the last one is linkless, see Figure 1 (one checks that it contains no minor from Petersen's family). Taking connected sums of this triangulation, we obtain linkless graphs triangulating any oriented surface of positive genus. By performing stellar operations we obtain linkless graphs with arbitrarily many vertices triangulating any oriented surface of positive genus.

Problem 5.4 *Is the graph of a triangulated non orientable* 2*-manifold always not linkless?*

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References

[1] A.D. Alexandrov, Convex Polyhedra. Moscow 1950 (Russian); German transl., Konvexe Polyeder. Berlin: Akademie-Verlag 1958. *Trans. Amer. Math. Soc.*, **245** (1978), 279-289.

[2] L. Asimov and B. Roth, The rigidity of graphs, *Trans. Amer. Math. Soc.*, **245** (1978), 279-289.

[3] L. Asimov and B. Roth, The rigidity of graphs: part II, *J. Math. Anal. Appl.*, **68** (1979), 171-190.

[4] D. Barnette, Generating the triangulations of the projective plane, *J. Combi. Th. Ser. B.*, **33**(3) (1982), 222-230.

[5] D. Barnette, Generating the projective plane polyhedral maps, *J. Combi. Th. Ser. B.*, **51** (1991), 277-291.

[6] A. Björner and G. Kalai, An extended Euler-Poincaré theorem, *Acta Math.*, **161** (1988), 279-303.

[7] L. Cauchy, Sur les polygones et les polyèdres, *Second Memoire, I. Ecole Polytechnique*, **245** (1813) (=Oeuvres complètes d'Augustin Cauchy, 2nd sér., Tome 1 (1905), pp. 26-38.

[8] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, *J. Combi. Th. Ser. B*, **50** (1990), 11-21.

[9] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, in Graph Structure Theory (N. Robertson and P. Seymour eds.), Contemp. Math., A.M.S., (1993), pp. 137-147.

[10] R. Diestel, *Graph Theory*, Second edition, Springer-Verlag, N.Y., (2000).

[11] H. Gluck, Almost all simply connected closed surfaces are rigid, in Geometric topology. Lecture Notes in Math., Vol.438, pp.225-239 (1975).

[12] H. Van der Holst, L. Lovász and A. schrijver, On the invariace of Colin de Verdière's graph parameter under clique sums, *Lin. Alg. Appl.*, **226-228** (1995), 509-517.

[13] G. Kalai, A characterization of f-vectors of families of convex sets in $\mathbb{R}^d$, Part 1: Necessity of Eckhoff's conditions, *Israel j.Math.*, **48**, (1984), 175-195.

[14] G. Kalai, Hyperconnectivity of graphs, *Graphs and Combi.*, **1**, (1985), 65-79.

[15] G. Kalai, Rigidity and the lower bound theorem, *Inven. Math.*, **88**, (1987), 125-151.

[16] G. Kalai, Algebraic Shifting, *Advanced Studies in Pure Math.*, **33** (2002), 121-163.

[17] K. Kuratowski, Sur le probléme des courbes gauches en topologie, *Fund. Math.*, **15**, (1930), 271-283.

[18] S.A. Lavrenchenko, Irreducible triangulations of the torus, *Ukrain. Geom. Sb.*, **30**, (1987), 52-62.

[19] K. W. Lee, Generalized stress and motion, in Polytopes: Abstract, Convex and Computational (T. Briztriczky et al., eds.), pp.249-271 (1995).

[20] W. Mader, Homomorphiesätze für Graphen, *Math. Ann.* **178**, (1968), 154-168.

[21] W. Mader, $3n-5$ edges do force a subdivision of K_5, *Combinatorica* Vol.**18**, **4**, (1998), 569-595.

[22] E. Nevo, Algebraic shifting and basic constructions on simplicial complexes, submitted.

[23] G. Ringel and J.W.T. Youngs, Solution of the Heawood map-coloring problem, *Proc. Nat. Acad. Sci. USA*, **60**, (1968), 438-445.

[24] N. Robertson, P.D. Seymour and R. Thomas, Linkless embeddings of graphs in 3-space, *Bull. Amer. Math. Soc.*, **28**, (1993), 84-89.

[25] Z. Song, The extremal function for K_8^- minors, submitted, www.math.gatech.edu/ song/K8.pdf.

[26] E. Steinitz and H. Rademacher, Volgesungen über die Theorie der Polyeder. Berlin-Göttingen Springer 1934.

[27] W. Whitely, Vertex splitting in isostatic frameworks, *Struc. Top.*, **16** (1989), 23-30.