# ADVANCED ANALYSIS (201-2-5401)
# WINTER 2013/2014
# HOMEWORK ASSIGNMENT 1

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**Solve all exercises in the lecture notes. Hand in 2 of the exercises 1–4 and five of the exercises 5–15.**

Let $X$ be a locally compact Hausdorff apace. A continuous function $f$ on $X$ is said to *vanish at infinity* if for all $\epsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ for all $x \in X \setminus K$. In this exercise we will use $C_0(X;\mathbb{R})$ to denote the space of continuous real valued functions vanishing at infinity on $X$, and we will use $C_0(X)$ to denote the space of continuous complex valued functions vanishing at infinity.

*Exercise* 1. Let $A \subseteq C_0(X;\mathbb{R})$ be an algebra that separates points of $X$, in which there exists, for every $x \in X$, a function $f \in A$ such that $f(x) \neq 0$. Prove that $A$ is dense in $C_0(X;\mathbb{R})$. Deduce the complex valued version of this theorem.

*Exercise* 2. Prove directly that the (real) polynomials with zero constant coefficients are dense in $C_0((0,1];\mathbb{R})$.

*Exercise* 3. Let $T$ be a compact Hausdorff space, and let $W \subseteq T$ be a closed subset. Let $\mathcal{B}$ be an algebra of continuous functions on $T$ such that

(1) For all $f \in \mathcal{B}$, $f\big|_W = 0$.
(2) For every $t_1, t_2 \in T \setminus W$, there is an $f \in \mathcal{B}$ such that $f(t_1) \neq f(t_2)$.
(3) For every point $t$ in $T \setminus W$, there is an $f \in \mathcal{B}$ such that $f(t) \neq 0$.

Then the closure of the $*$-algebra generated by $\mathcal{B}$ in $C(T)$ (i.e., the algebra generated by all $f$ and $\overline{f}$, $f \in \mathcal{B}$) is equal to $\{f \in C(T) : f\big|_W = 0\}$.

*Exercise* 4. Make a series of drawings or figures to accompany the proof of the Stone-Weierstrass Theorem.

*Exercise* 5. For one of the following spaces $G$, decide (and prove) whether or not it is a Hilbert space.

(1) $G = PC[a,b]$.
(2) $G = \ell^2(S)$, where $S$ is an uncountable set.
(3) $G = L^2_a(\mathbb{D})$, which is defined to be the set of all analytic functions $f : \mathbb{D} \to \mathbb{C}$, such that $\int_{\mathbb{D}} |f(x+iy)|^2 dxdy < \infty$, with inner product

$$(f,g) = \int_{\mathbb{D}} f(x+iy)\overline{g(x+iy)}dxdy.$$

(4) $G = H^2(\mathbb{D})$, which is defined to be the set of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ with Taylor series $f(z) = \sum_{n=0}^\infty f_n z^n$, such that $\sum_{n=0}^\infty |f_n|^2 < \infty$, with inner product

$$(f,g) = \sum_{n=0}^\infty f_n \overline{g_n}.$$

*Exercise* 6. Regarding each of the following statements, decide whether (a) it is true (b) it is false (c) you don't know, but assuming the continuum hypothesis it is true (perhaps some are related to others).

(1) Let $G_1$ be a linear, dense, not closed subspace of $\ell^2(\mathbb{N})$. Let $G_2$ be a linear, dense, not closed subspace of $L^2[0,1]$. Then there exists a unitary map from $G_1$ to $G_2$, i.e., there is an isomorphism $U : G_1 \to G_2$ such that $(Ug, Uh) = (g, h)$ for all $g, h \in G_1$.
(2) Every *Hamel basis* (i.e., a basis in the sense of linear algebra) of $\ell^2(\mathbb{N})$ has cardinality $2^{\aleph_0}$.
(3) In an inner product space, every two complete orthonormal systems have the same cardinality.
(4) Two inner product spaces $G_1$ and $G_2$ are isomorphic if and only if for every complete orthonormal systems $E_1$ in $G_1$ and $E_2$ in $G_2$ have the same cardinality.
(5) In an inner product space, every closed o.n. system is complete.
(6) In an inner product space, every complete o.n. system is closed.

Reminder: recall that an o.n. system $\{e_i\}_{i\in I}$ in an inner product space $G$ is said to be closed if for all $x \in G$, $\|x\|^2 = \sum_{i\in I} |(x, e_i)|^2$.

*Exercise* 7. Let $f_n$ be a sequence of continuous functions on $[a, b]$, and let $f, g \in C[a, b]$. Suppose that $f_n \to f$ in $L^2$ and that $f_n \to g$ uniformly. Prove that $f = g$. (**) What can be said if $f_n$ is known to converge to $g$ only pointwise, rather than uniformly?

*Exercise* 8. Let $f \in C^1[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. Prove that the Fourier series for $f$ converges uniformly to $f$. (Use only material from Infi 2 and things that you learned in this course).

*Exercise* 9. Describe an algorithm that, given $n < 100$, finds $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that the expression

$$\int_0^1 (e^{-x} - \sum_{i=0}^n a_i x^i)^2 dx + \int_0^1 (e^{-x} + \sum_{i=1}^n i a_i x^{i-1})^2 e^{-x^2} dx$$

is as small as possible.

*Exercise* 10. Evaluate the sum of the series $\sum_{n=1}^\infty \frac{1}{n^2}$ by considering the function $f(x) = x$ on the interval $[0, 1]$.

*Exercise* 11. Let $a \in \ell^2$, let $H$ be a Hilbert space, and let $\{e_n\}_{n=1}^\infty$ be an o.n. basis for $H$, and let $K = \{\sum x_n e_n : |x_i| \leq |a_i|\}$. Prove that $K$ is compact in $H$.

*Exercise* 12. Let $A \in M_n(\mathbb{C})$, and identify $A$ with the operator $T_A$ that it induces on $\mathbb{C}^n$. Prove that $\|A\|$ is equal to the square root of the greatest eigenvalue of $A^*A$.

*Exercise* 13. Let $P \in B(H)$, $P = P^2$. Such an operator is called a *projection.* Recall that an orthogonal projection was defined to be the best approximation mapping $P_M$ (here $M$ is a closed subspace) which maps every $x \in H$ to the unique $m \in M$ such that $\|x - m\|$ is minimal.
Prove that $P$ is an orthogonal projection if and only if $P^* = P$.
Prove that $P$ is an orthogonal projection if and only if $\|P\| \leq 1$.

*Exercise* 14. Let $H$ be a Hilbert space. Let $B : H \times H \to \mathbb{C}$ be a function that is a sesquilinear form, that is, it satisfies

$$B(ax + y, bz + w) = a\bar{b}B(x, z) + \bar{b}B(y, z) + aB(x, w) + B(y, w)$$

for all $x, y, z, w \in H$, and $a, b \in \mathbb{C}$. Assume further that $B$ is bounded, that is, there exists some $C$ such that

$$|B(x, y)| \leq C\|x\|\|y\|$$

for all $x, y \in H$. Prove that there exists a unique $A \in B(H)$ such that for all $x, y \in H$,

$$B(x, y) = (Ax, y).$$

The following exercise might be tedious but it is very useful.

*Exercise* 15. Let $\{H_i\}_{i\in I}$ be a collection of Hilbert spaces. Define a new space

$$H = \{x \in \Pi_{i\in I} H_i : \sum_{i\in I} \|x(i)\|^2 < \infty\}.$$

Endow $H$ with an inner product $(x, y) = \sum_{i\in I}\langle x(i), y(i)\rangle$. This space is called *the direct sum of the spaces* $H_i$ and is denoted $H = \oplus_{i\in I} H_i$.

(1) Prove that $H$ is indeed a Hilbert space.
(2) Suppose that $I = \{1, \ldots, n\}$¡ so that $H = \oplus_{i=1}^n H_i = H_1 \oplus \ldots \oplus H_n$. Then one can think of $h \in H$ as a column $h = (h_1, \ldots, h_n)^t$, where $h_i \in H_i$. Let $A$ be an $n \times n$ matrix such that the $ij$th entry is an operator $A_{ij}$ in $B(H_i, H_j)$. Show that multiplication by $A$ gives rise to an element $T_A$ in $B(H)$, by way of

$$T_A(h_1, \ldots, h_n)^t = (\sum_{j=1}^n A_{1j}h_j, \ldots, \sum_{j=1}^n A_{nj}h_j).$$

Show that every $T \in B(H)$ arises this way.
(3) Continuing with the same notation, show that the adjoint of $T_A$ is $T_{A^*}$ where the $A^*$ is the operator matrix that has $A_{ji}^*$ in the $ij$th place.
(4) Show that if $A$ and $B$ are operator matrices with the appropriate decomposition (figure out what that means), then $T_AT_B = T_{AB}$, where $AB$ is the matrix with $\sum_k A_{ik}B_{kj}$ in its $ij$th entry.