ADVANCED ANALYSIS (201-2-5401)
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HOMEWORK ASSIGNMENT 2

ORR SHALIT

In all of the following X is normed space. Solve all exercises in the lecture notes. Hand in one out of every two consecutive exercises.

Exercise 1. The following are equivalent:

(1) X is Banach.
(2) For every $\{x_n\} \subset X$, if $\sum_{n=1}^{\infty} \|x_n\| < \infty$ then $\sum_{n=1}^{\infty} x_n$ converges in X.

Exercise 2. Let f be a nonzero linear functional on X. The following are equivalent.

(1) f is continuous.
(2) $N(f)$ is closed.
(3) $N(f)$ is not dense.

Exercise 3. True or false: A norm is induced by an inner product if and only if whenever $\|x\| = \|y\|$ and $a, b \in \mathbb{R}$, then $\|ax + by\| = \|bx + ay\|$ (you may assume, for simplicity, that the scalars are real).

Exercise 4. A *closed half space* in X is a set of the form $\{x \in X : \Re f(x) \leq c\}$ for some $c \in \mathbb{R}$ and some $f \in X^*$. True or false: a set C is a convex closed set if and only if it the intersection of closed half spaces.

Exercise 5. Let X be a normed space. Fix $\{x_i\}_{i \in I} \subset X$ and $\{a_i\}_{i \in I} \subset \mathbb{C}$. Show that there exists $f \in X^*$ such that $f(x_i) = a_i$ for all $i \in I$ if and only if there exists some constant $M \geq 0$ such that for every finite set of indices $i_1, \ldots, i_n \subseteq I$, and every sequence $\{b_k\}_{k=1}^n$, the inequality

$$|\sum_{k=1}^{n} b_k a_{i_k}| \leq M \| \sum_{k=1}^{n} b_k x_{i_k} \|$$

holds.

Exercise 6. Let $A, B \subseteq X$ be closed, and assume that at least one of them is compact. Prove that $A - B$ is closed. What happens if one does not assume that one of the sets is compact?

Exercise 7. Let X and Y be normed spaces. Define a new normed space

$$Z = X \oplus Y = \{(x, y) \in X \times Y\}$$

with the obvious vector space structure and the norm $\|(x, y)\|_Z = \|x\|_X + \|y\|_Y$.

(1) Prove that $Z^* = X^* \oplus Y^*$. What is the norm on Z^*? (The answer should be in terms of the norms of X^* and Y^*).
(2) What changes if the norm on Z is taken to be $\|(x, y)\|_Z = \sqrt{\|x\|^2 + \|y\|^2}$?

Exercise 8. True or false?

(1) If X and Y be isomorphic Banach spaces then X is reflexive if and only if Y is reflexive.
(2) Every finite dimensional normed space is reflexive.

Exercise 9. Let $X = \mathbb{R}^3$ with the ℓ^1 norm: $\|(x,y,z)\| = |x| + |y| + |z|$. Let $X_0 = \{(x,y,z) : z = 0 = x - 3y\}$. Let f_0 be the functional on X_0 given by $f_0(x,y,z) = x$. Find the general form of $f \in X^*$ that extends f_0 and satisfies $\|f\| = \|f_0\|$. What changes if the ℓ^1 norm is replaced by the Euclidean norm?

Exercise 10. Suppose that $H = \mathbb{C}^n$ with the standard inner product. Find the norm on $B(H)^*$, when $B(H)$ is given the operator norm.

Exercise 11. (Existence of a mean) Let $\ell^\infty_{\mathbb{R}}(\mathbb{Z})$ be the space of bounded real sequences:

$$\ell^\infty_{\mathbb{R}}(\mathbb{Z}) = \{a = (a_n)_{n=-\infty}^{\infty} : \sup_n |a_n| < \infty\}.$$

Let T be the translation operator given by $Ta = b = (b_n)_{n=-\infty}^{\infty}$, where $b_n = a_{n+1}$. Prove that there exists a linear functional $f \in \ell^\infty_{\mathbb{R}}$ of norm 1 that satisfies

(1) $f(Ta) = f(a)$ for all a (translation invariance).
(2) $\inf_n a_n \leq f(a) \leq \sup_n a_n$ for all a.

Such a functional is sometimes called a *mean*. (Hint: separate the set $\{a \in \ell^\infty_{\mathbb{R}}(\mathbb{Z}) : \inf a_n > 0\}$ and $\{a - Ta : a \in \ell^\infty_{\mathbb{R}}(\mathbb{Z})\}$)). Deduce that $(\ell^\infty(\mathbb{N}))^* \neq \ell^1(\mathbb{N})$.

Exercise 12. Let X a real vector space with two norms $\|\cdot\|_i$, $i = 1, 2$. Let f be a linear functional such that

$$f(x) \leq \max\{\|x\|_1, \|x\|_2\} \quad , \quad x \in X.$$

Prove that there exists $t \in [0,1]$ such that for all $x \in X$,

$$f(x) \leq t\|x\|_1 + (1-t)\|x\|_2.$$

(Hint: consider the convex hull of S (i.e., the smallest closed set containing S, where $S = \{(\|x\|_1 - f(x), \|x\|_2 - f(x)) : x \in X\} \subset \mathbb{R}^2$).

Exercise 13. Let c be the subspace of ℓ^∞ of that consists of convergent sequences. Let c_0 be the subspace of c that consists of sequences that converge to 0.

(1) Prove that c_0^* is isometric to ℓ^1 (hint: find a mapping from ℓ^1 into c_0^* and prove that it is isometric and surjective).
(2) Prove that c^* is isometric to ℓ^1 (hint: find a different mapping from the case of c_0).
(3) (**) Does the above imply that c_0 is isometric to c?

Exercise 14. (1) Let $T : [0,1] \to [0,1]$ be a measure preserving (piecsewise continuous, if you wish) transformation. Suppose that for every $f \in L^2[0,1]$, the sums $\frac{1}{N+1}\sum_{n=0}^N f \circ T^n$ converges in the mean to $\int_0^1 f(t)dt$. Prove that T is ergodic.
(2) Prove that the transformation $T(x) = x + \alpha(mod 1)$ is ergodic if and only if α is irrational.