ADVANCED ANALYSIS (201-2-5401)
WINTER 2013/201
HOMEWORK ASSIGNMENT 3

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In all of the following X is a Banach space. Solve all exercises in the lecture notes. Hand in one out of every two consecutive exercises, where exercise 2 must be included. Due date: 10/12/2013

Exercise 1. The closed unit ball of X is compact in the norm topology if and only if X is finite dimensional.

Exercise 2. Prove the following Hahn–Banach type theorem. *Let Y be a subspace of X^* that separates the points of X. Let C be a convex set in X which is closed in the $\sigma(X,Y)$–topology, and let $x \notin C$. Then there exists $f \in Y$ that strictly separates C and x, i.e., there is some $\epsilon > 0$ such that $Ref(y) \leq f(x) - \epsilon$ for all $y \in C$.* (Hint: use the Hahn–Banach extension theorem in its most primitive form, and **carefully** adapt arguments from similar theorems that we have proved in class).

Exercise 3. Miscellaneous loose ends from the lectures and more:

(1) True or false: if M and N are closed subspaces of X, then $M+N = \{m+n : m \in M, n \in N\}$ is closed.
(2) True or false: The infimum in the definition of the quotient norm: $\|\dot{x}\| = \inf_{m \in M} \|x - m\|$ is always attained.
(3) True or false: in the space ℓ^1 weak convergence implies norm convergence.
(4) True or false: every separable Banach space is isomorphic to a closed subspace of ℓ^1.

Exercise 4. Let $M \subset L^1(\mathbb{R})$ be the range of the linear map $d/dx : C_c^\infty(\mathbb{R}) \to L^1(\mathbb{R})$ defined by $d/dxf = \frac{df}{dx}$. Consider $C_c^\infty(\mathbb{R})$ as a subspace of L^p, where p is not yet specified.

(1) Find the range of values of p for which the map $(d/dx)^{-1} : M \to L^p$ is bounded (here M is considered with the 1 norm).
(2) Prove that for every p in the range which you have found, and every $F \in L^q$ (where $q = p/(p-1)$ is the conjugate exponent of p), there exists a $u \in L^\infty$ that weakly solves the equation
$$\frac{du}{dx} = F.$$
(3) Prove that, conversely, if for all $F \in L^q$ there exists a weak solution in L^∞ to the above equation, then $p = q/(q-1)$ must lie in the range which you found in the first part of the problem.

Exercise 5. Every separable Banach space is isometric to a quotient space of ℓ^1. (Hint: Suppose that X is separable and let $\{x_n\}$ be a dense sequence in X_1. Define a map from ℓ^1 onto X using this sequence, and show that it does what you want it to).

Exercise 6. Recall that c_0 is the closed subspace of ℓ^∞ consisting of sequences that converge to 0. In this exercise we will show that c_0 is a non–complementable subspace of ℓ^∞.

(1) Prove that if $X = M \oplus N$, then X/M is isomorphic to N.
(2) Prove that if $f \in X^*$, then f induces a functional $\dot{f}$ on X/M by way of $\dot{f}(\dot{n}) = f\big|_N(n)$, where n is the unique element in $\dot{n}$ from N.
(3) From here on, assume for contradiction that $\ell^\infty = c_0 \oplus N$. Denote the projection from ℓ^∞ to ℓ^∞/c_0 by π.
(4) Prove the following seemingly unrelated lemma: *There exists a family $\mathcal{F} \subseteq 2^{\mathbb{N}}$ (so $\mathcal{F}$ consists of subsets of $\mathbb{N}$) such that $|\mathcal{F}| = 2^{\aleph_0}$ and $A \cap B$ is finite for all $A, B \in \mathcal{F}$ such that $A \neq B$.*
(5) For every $A \subseteq \mathbb{N}$, denote the characteristic function of A by 1_A. Prove that if $A_1, \ldots, A_n$ are distinct elements in $\mathcal{F}$, then for all $c_1, \ldots, c_n \in \mathbb{C}$,
$$\left\| \pi\left(\sum_{i=1}^n c_i 1_{A_i}\right)\right\| \leq \max_{1\leq i\leq n} |c_i|.$$
(6) Fix $g \in (\ell^\infty)^*$. Prove that for every n, the number of elements $A \in \mathcal{F}$ for which $|g(\pi 1_A)| > 1/n$, is finite. Deduce that there are at most a countable number of As in $\mathcal{F}$ such that $g(\pi 1_A) \neq 0$.
(7) Let f_i be the functional on ℓ^∞ given by $f_i(a_1, a_2, \ldots) = a_i$. Prove that if for some $z \in \ell^\infty/c_0$ it holds that $\dot{f}_i(z) = 0$ for all i, then $z = 0$.
(8) Contradict the previous item by exhibiting a nonzero $z \in \ell^\infty/c_0$ such that $\dot{f}_i(z) = 0$ for all i.

Exercise 7. In contrast to the previous exercise, prove that X^* is always a complementable subspace of X^{***}. (Hint: consider the natural maps $i : X \to X^{**}$, $j : X^* \to X^{***}$, their adjoints, and the various compositions you can form).

Exercise 8. Recall that c is the closed subspace of ℓ^∞ consisting of all convergent sequences. Prove that c_0 and c are isomorphic, but not isometrically isomorphic.

Exercise 9. Give two proofs for the fact that c_0 is not a dual space (i.e., there is no X such that X^* is isometrically isomorphic to c_0). Proof 1: consider exercises 6 and 7. Proof 2: find the extreme points of $(c_0)_1$.

Exercise 10. Let V be an isometry on a Hilbert space H. Prove that V is an extreme point in the unit ball of $B(H)$.

Exercise 11. **Remark:** In this exercise we will denote by $C(\mathbb{T})$ the continuous functions on the unit circle (where $\mathbb{T}$ denotes the unit circle). This is the same space as $C_{per}([0,1])$ from the lectures, only that the latter notation is *never* used, and the former is used all the time.

For $f \in C(\mathbb{T})$, let us denote by
$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\, d\theta$$

the Fourier coefficients. Prove that given a complex valued sequence $\{\gamma_n\}_{n=-\infty}^{\infty}$, there exists a complex Borel measure μ on $\mathbb{T}$ such that

$$\gamma_n = \int_{\mathbb{T}} e^{-in\theta}\, d\mu(e^{i\theta})$$

for all n (i.e., γ_n are the Fourier coefficients of μ) iff for every $f \in C(\mathbb{T})$ there exists a $g \in C(\mathbb{T})$ with $\hat{g}(n) = \gamma_n \hat{f}(n)$ for all n. (Hint: one direction is a straightforward calculation; for the other direction, use the closed graph theorem to show that $f \mapsto g$ is a bounded operator on $C(\mathbb{T})$, and then apply the Riesz representation theorem to an appropriate bounded linear functional on $C(\mathbb{T})$.)

Exercise 12. Let X be a compact Hausdorff space.

(1) Let A be a closed subspace of $C(X)$, let μ be an extreme point of the closed unit ball of $A^{\perp} \subseteq M(X)$, and let $f \in C(X)$ be a real valued function so that

$$\int fg\, d\mu = 0$$

for all $g \in A$. Show that f is constant on $\operatorname{supp} \mu$.

(2) Show that μ is an extreme point of the closed unit ball of $M(X)$ iff $\mu = \lambda\delta_x$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and some $x \in X$ (here δ_x denotes the unit point mass measure concentrated at x: $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ if $x \notin E$).