ADVANCED ANALYSIS (201-2-5401)
WINTER 2012/2013
HOMEWORK ASSIGNMENT 5

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Hand in one out of every two consecutive exercises, except the last exercise, which is not to be handed in. Due date:16/1/2014

Exercise 1. Let A be a nonunital commutative Banach algebra. Let $sp(A)$ denote the space of all non-zero multiplicative linear functionals on A, a.k.a. the maximal ideal space of A. Let $\tilde{A}$ denote the unitalization of A as in last week's exercises. Denote by $A \ni a \mapsto \hat{a} \in C(sp(A))$ the Gelfand transform: $\hat{a}(\rho) = \rho(a)$, for all $\rho \in sp(A)$. Prove the following.

(1) For all $a \in A$, $\sigma_{\tilde{A}}(a) = \{\rho(a) : \rho \in sp(A)\} \cup \{0\}$.
(2) $sp(A)$ is a locally compact space.
(3) The Gelfand transform is a contractive homomorphism of A onto a subalgebra of $C_0(sp(A))$ which separates points. In fact, $\|\hat{A}\|_\infty = r(a)$.
(4) $\sigma_{\tilde{A}}(a) = \hat{a}(sp(A)) \cup \{0\}$.

Exercise 2. Let A be the non-unital algebra $L^1(\mathbb{R})$ (with convolution as multiplication). Compute $sp(A)$ and describe the Gelfand transform.

Exercise 3. Let (X, μ) be a σ–finite measure space and let $A = L^\infty(X, \mu)$. For $f \in A$, we denote the *essential range of* f to be

$$R(f) = \{\lambda \in \mathbb{C} : \forall \epsilon > 0 . \mu(\{x : |f(x) - \lambda| < \epsilon\}) > 0\}.$$

Prove:

(1) $\sigma_A(f) = R(f)$.
(2) $r(f) = \|f\|_\infty = \sup\{|t| : t \in R(f)\}$.

Exercise 4. Let $1 \in B \subseteq A$ be Banach algebras.

(1) Prove that for all $x \in B$,
$$\partial\sigma_B(x) \subseteq \partial\sigma_A(x).$$
(2) Give an example showing that the converse does not hold.
(3) Deduce that if $\sigma_B(x) \subset \mathbb{R}$ then $\sigma_A(x) = \sigma_B(x)$.

Exercise 5. Let A be a unital commutative Banach algebra and $a \in A$. Denote by B the Banach algebra generated by 1 and a.

(1) Define $\tau : sp(A) \to \sigma_A(a)$ by $\tau(\rho) = \rho(a)$. Then τ is a surjective continuous map.
(2) If $B = A$ then τ is a homeomorphism.
(3) Suppose that $b \in B$ and that $\overline{alg(1, b)} = B$. Then $\sigma_B(a)$ and $\sigma_B(b)$ are homeomorphic. Can anything be said about $\sigma_A(a)$ and $\sigma_A(b)$ in this case?

(4) For every polynomial p, it holds that $\widehat{p(a)} = p \circ \tau$. Deduce the *spectral mapping theorem*: $p(\sigma_A(a)) = \sigma_A(p(a))$ for every polynomial p.

Exercise 6. Suppose that a, b are commuting elements in a unital Banach algebra A. Prove that $r(ab) \leq r(a)r(b)$ and that $r(a+b) \leq r(a) + r(b)$.

Exercise 7. Let A be a unital Banach algebra and $a \in A$.

(1) Let U be an open set containing $\sigma(a)$, and let f_n be a sequence of holomorphic functions in U which converge uniformly to f on compact subsets. Then $f \in H(a)$ and $f_n(a) \to f(a)$ in norm.
(2) Let $g \in H(a)$. Put $b = g(a)$ and let $f \in H(b)$. Then $f \circ g \in H(a)$ and $(f \circ g)(a) = f(b)$. In other words, $f \circ g(a) = f(g(a))$.
(3) Suppose that $f(z) = \frac{1}{(z-\alpha)^n}$. Show directly that if $\alpha \notin \sigma(a)$ then $f(a) = (a - \alpha \cdot 1)^{-n}$. Deduce that $f(a)$ attains the "obvious" value when f is a rational function with poles off $\sigma(a)$.

Exercise 8. Let E be a Banach space and $T \in B(E)$. Suppose that $\sigma(T)$ is disconnected. Prove that T has a non–trivial invariant subspace, i.e., there is a closed subspace $F \subset E$, $0 \neq F \neq E$, such that $T(F) \subseteq F$.

Exercise 9. (1) Let X and Y be compact Hausdorff spaces, and let $\phi : C(X) \to C(Y)$ be an *algebraic* isomorphism. Prove that there exists a homeomorphism $h : Y \to X$ such that $\phi(f) = f \circ h$ for all $f \in C(X)$.
(2) Take a moment to appreciate this: every isomorphism is in fact an isometric *–isomorphism.
(3) What happens if ϕ is only assumed to be a homomorphism.
(4) (No need to hand in the details of this one) Deduce that $X \mapsto C(X)$ gives rise to a contravariant equivalence of categories between compact Hausdorff spaces and unital commutative C*–algebras. What is the inverse functor?

Exercise 10. Let $T \in B(H)^{-1}$. Show that $\phi : A \mapsto TAT^{-1}$ is an automorphism of the algebra $B(H)$ which is a bounded and has a bounded inverse. Prove that ϕ is a *–isomorphism if and only if T is a scalar multiple of a unitary.

Exercise 11. Work out the Gelfand theory for commutative, non–unital C*-algebras.