Multivariable

OPERATOR THEORY

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A quick review of

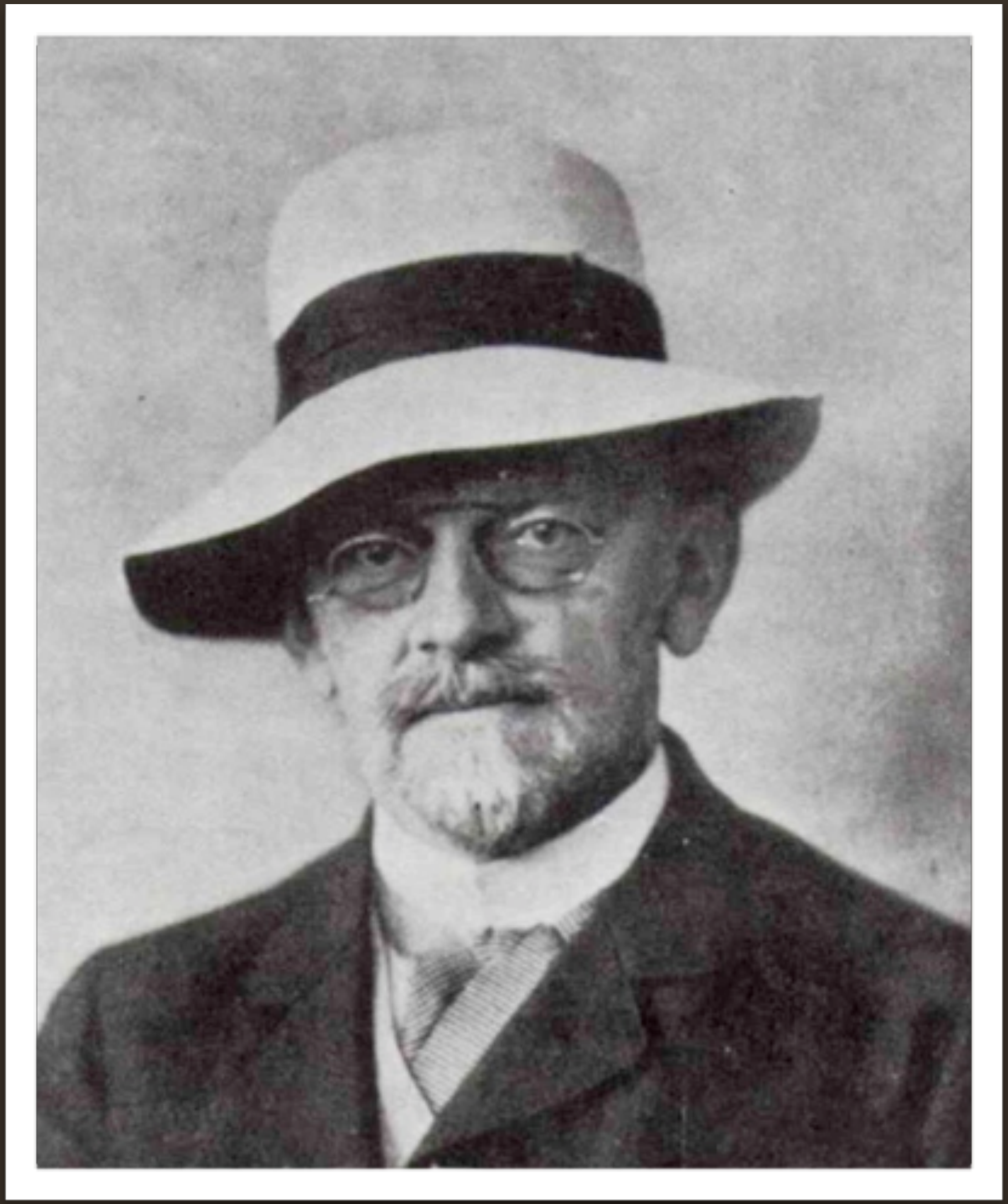
SOME BASIC FACTS

A Hilbert space is a complex vector space H with an inner product $\langle x, y \rangle$ and a norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Definition

An **operator** T on H is a linear mapping $T : H \to H$.

There are two fundamental examples of Hilbert spaces, one **finite-dimensional** and the other **infinite-dimensional**.



The **Euclidean-Hilbert space** $\mathbb{C}^n$ is the space of vectors

$$x = (x_1, x_2, \ldots, x_n)$$

with inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}, \quad x, y \in \mathbb{C}^n$$

and orthonormal basis of coordinate vectors

$$e_1, e_2, \ldots, e_n.$$

The operators on $\mathbb{C}^n$ are matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where

$$a_{ij} = \langle A\mathrm{e}_j, \mathrm{e}_i \rangle.$$

The Hilbert space ℓ^2 is the space of infinite sequences

$$x = (x_1, x_2, x_3, \ldots)$$

satisfying

$$\|x\|^2 = \sum_{i=1}^{\infty} |x_i|^2 < \infty$$

with inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}, \quad x, y \in \ell^2$$

and orthonormal basis of coordinate vectors

$$e_1, e_2, e_3, \ldots$$

The operators on ℓ^2 are infinite matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$a_{ij} = \langle Ae_j, e_i \rangle.$$

To make things interesting, we place some kind of **external** structure on our Hilbert space.

The **Hardy space** H^2 is the set of analytic functions on the complex unit disk $\mathbb{D}$,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

obtained by completing the polynomials $\mathbb{C}[z]$ in the inner product with corresponding orthonormal basis

$$1, z, z^2, z^3, \dots$$

This external structure leads to the consideration of “natural” operators.

Example (Unilateral shift)

The **unilateral shift** M_z is defined on the Hardy space H^2 by

$$(M_z f)(z) = zf(z), \quad f \in H^2.$$

In other words, M_z is the coordinate multiplication operator.

MULTIVARIABLE OPERATOR THEORY

The study of more than one operator at a time

Let $(A_1, \ldots, A_n)$ be an n-tuple of commuting operators (i.e. $A_iA_j = A_jA_i$) operators.

Motivating Question #1

For which functions f does it make sense to define $f(A_1, \ldots, A_n)$?

Let $(A_1, \ldots, A_n)$ be an n-tuple of (possibly noncommuting) operators. Let $\mathcal{A} = \mathrm{Alg}(A)$ denote the algebra they generate, i.e.

$$\mathcal{A} = \mathrm{span}\{A_{i_1} \cdots A_{i_k} \mid i_1, \ldots, i_k \in \{1, \ldots, n\}\}.$$

Motivating Question #2

When does an operator B belong to $\mathcal{A}$? When can B be approximated (in an appropriate sense) by operators in $\mathcal{A}$?

The case of

A SINGLE VARIABLE

Definition

Let $A : H \to H$ be an operator.

1. A is said to be a **contraction** if $\|Ax\| \leq \|x\|, \quad \forall x \in H$,
2. A is said to be an **isometry** if $\|Ax\| = \|x\|, \quad \forall x \in H$,
3. A is said to be a **unitary** if it is a surjective isometry (i.e. $\mathrm{Ran}(A) = H$).

Note: In finite dimensions, every isometry is a unitary.

Theorem (Sz.-Nagy (1954))

Every contraction A can be "dilated" to an isometry V. In other words, there are operators B and C such that

$$V = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

for an appropriate choice of basis.

By scaling A (if necessary), we can always take V to be a multiple of M_z, the unilateral shift.

The advantage of Sz.-Nagy's dilation theorem is that isometries are extremely well understood.

Theorem (Lebesgue-von Neumann-Wold)

Every isometry V can be decomposed as

$$V = M_z^{(k)} \oplus U_a \oplus U_s,$$

where $M_z^{(k)}$ is a multiple of the unilateral shift, and U_a and U_s are unitaries with absolutely continuous and singular spectral measures respectively.

For a bounded analytic function f on the complex unit disk, we can define $f(M_z) = M_{f(z)}$, and we have

$$f(M_z) = \begin{pmatrix} f(A) & 0 \\ * & * \end{pmatrix}.$$

Corollary

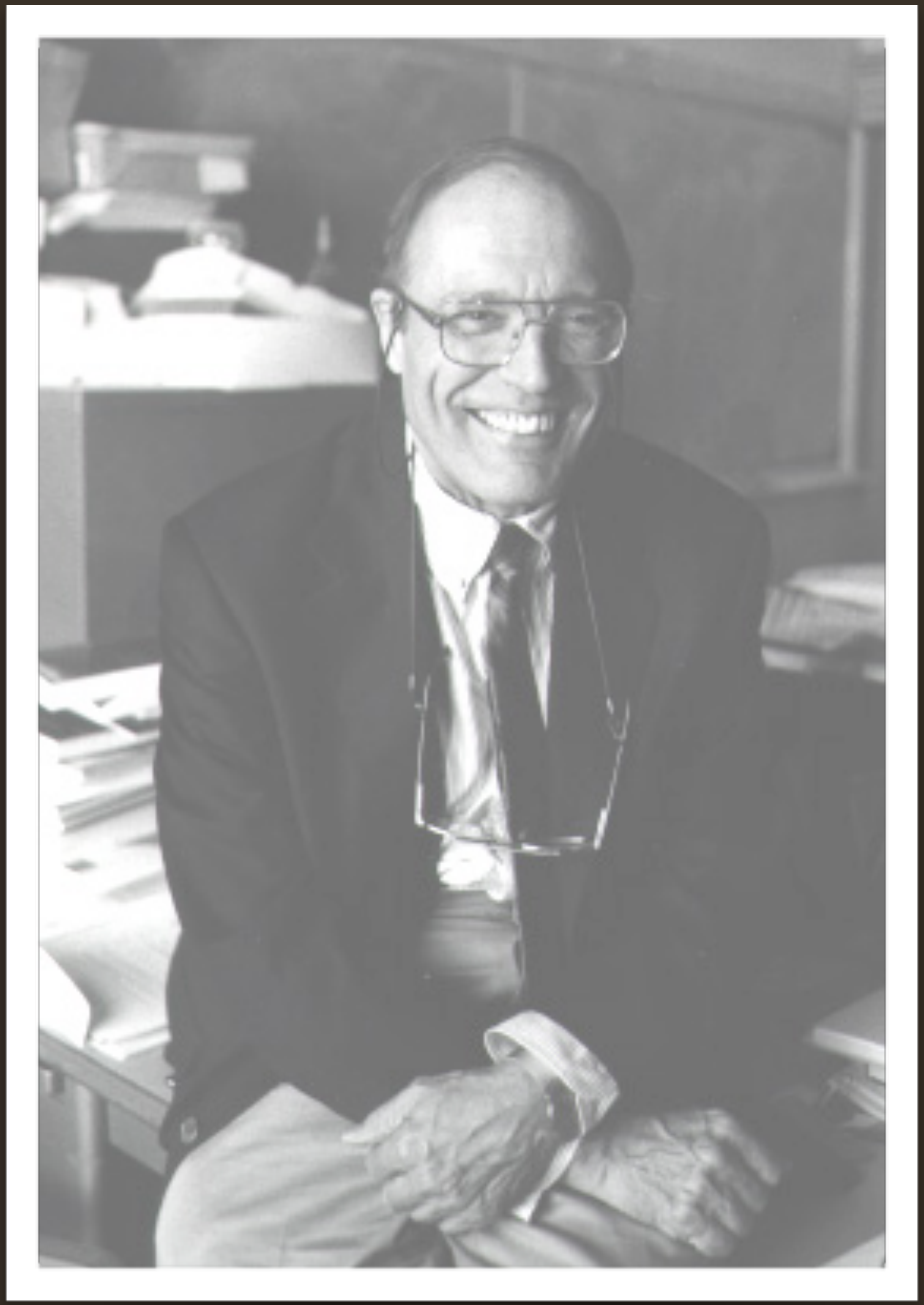
For a bounded analytic function f, we can define $f(A)$ by dilating A to M_z, then "compressing" to the top left corner of $f(M_z)$.

There are much deeper consequences of Sz.-Nagy's dilation theorem. For example:

Theorem (Brown-Chevreau-Pearcy (1988))

Every contractive operator with spectrum containing the complex unit circle has a nontrivial invariant subspace.

In the words of Rota, the unilateral shift is a **universal model** for a single contractive operator.



The case of
TWO VARIABLES

The **Hardy space** $H^2(\mathbb{D}^n)$, is the set of analytic functions on the complex polydisc $\mathbb{D}^n$,

$$f(z_1, \ldots, z_n) = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha z^\alpha,$$

obtained by completing the polynomials $\mathbb{C}[z_1, \ldots, z_n]$ in the inner product with corresponding orthonormal basis

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n.$$

We can consider the coordinate multiplication operators $M_{z_1}, \ldots, M_{z_n}$ on the Hardy space $H^2(\mathbb{D}^n)$ defined by

$$(M_{z_i}f)(z_1, \ldots, z_n) = z_i f(z_1, \ldots, z_n), \quad f \in H^2(\mathbb{D}^n).$$

Theorem (Ando (1963))

Every pair of commuting contractions (A_1, A_2) can be "dilated" to commuting isometries (V_1, V_2). In other words, there are operators B_1, B_2 and C_1, C_2 such that

$$V_1 = \begin{pmatrix} A_1 & 0 \\ B_1 & C_1 \end{pmatrix}, \; V_2 = \begin{pmatrix} A_2 & 0 \\ B_2 & C_2 \end{pmatrix},$$

for an appropriate choice of basis.

By scaling (A_1, A_2) (if necessary), we can always take (V_1, V_2) to be a multiple of (M_{z_1}, M_{z_2}), the coordinate multiplication operators on the Hardy space $H^2(\mathbb{D}^2)$.

The coordinate multiplication operators (M_{z_1}, M_{z_2}) on $H^2(\mathbb{D}^2)$ are a model for pairs of commuting contractions.

The case of MORE THAN TWO VARIABLES

Example (Parrott 1973)

There are commuting contractions (A_1, A_2, A_3) that cannot be dilated to commuting isometries. Hence, no analogue of the Sz.-Nagy-Ando dilation theorem holds for $n \geq 3$.

This phenomena is still not well understood.

The case for
NONCOMMUTATIVITY

The **noncommutative Hardy-Hilbert space** F_n^2 is the set of noncommutative “analytic” functions,

$$F(Z_1, \ldots, Z_n) = \sum_w a_w Z_w,$$

obtained by completing the noncommutative polynomials $\mathbb{C}\langle Z_1, \ldots, Z_n \rangle$ in the inner product with corresponding orthonormal basis

$$Z_w = Z_{w_1} \cdots Z_{w_k}, \quad w_1, \ldots, w_k \in \{1, \ldots, n\}.$$

We can consider the **unilateral** ***n*-shift**, i.e. the n-tuple of **left** coordinate multiplication operators $L_Z = (L_{Z_1}, \dots, L_{Z_n})$ on F_n^2 defined by

$$(L_{Z_i}F)(Z_1, \dots, Z_n) = Z_i F(Z_1, \dots, Z_n), \quad F \in F_n^2.$$

There are natural higher-dimensional analogues of familiar notions.

Definition

Let $A = (A_1, \ldots, A_n)$ be an n-tuple of operators on a Hilbert space H. Consider A as a "row" operator $A : H^n \to H$.

1. A is said to be a contraction if $\|Ax\| \leq \|x\|, \quad \forall x \in H^n$,
2. A is said to be an isometry if $\|Ax\| = \|x\|, \quad \forall x \in H^n$,
3. A is said to be a unitary if it is a surjective isometry (i.e. $\mathrm{Ran}(A) = H$).

Theorem (Bunce-Frazho-Popescu (1982-1989))

Every contractive n-tuple of operators $A = (A_1, \ldots, A_n)$ can be dilated to an isometric n-tuple $V = (V_1, \ldots, V_n)$. In other words, for each i, there are operators B_i and C_i such that

$$V_i = \begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix},$$

for an appropriate choice of basis.

By scaling A (if necessary), we can always take V to be a multiple of L_Z, the n-tuple of left coordinate multiplication operators on the noncommutative Hardy space F_n^2.

An analogue of the Lebesgue-von Neumann-Wold decomposition holds for isometric tuples.

Theorem (K (2011))

Every isometric n-tuple $V = (V_1, \ldots, V_n)$ can be decomposed as

$$V = L_Z^{(k)} \oplus U_a \oplus U_s \oplus U_d,$$

where $L_Z^{(k)}$ is a multiple of the coordinate multiplication operators on F_n^2, U_a and U_s are absolutely continuous and singular unitary n-tuples respectively, and U_d arises as the dilation of some contractive n-tuple.

It turns out there are good noncommutative analogues of measure and function-theoretic notions like absolute continuity and singularity.

For example, an isometric n-tuple $V = (V_1, \ldots, V_n)$ is singular if and only if the algebra it generates is a von Neumann algebra.

Corollary (K (2011))

Let $\mathcal{V}$ denote the (weakly closed) algebra generated by an isometric n-tuple $V = (V_1, \ldots, V_n)$. An arbitrary operator T belongs to $\mathcal{V}$ if and only if every subspace invariant for $\mathcal{V}$ is also invariant for T.

We say that the n-tuple $L_Z = (L_{Z_1}, \ldots, L_{Z_n})$ is **reflexive**.

The unilateral n-shift M_Z is a **universal model** for an arbitrary contractive n-tuple of operators.

Back to the

COMMUTATIVE CASE

The **Drury-Arveson Hilbert space** H^2_n, is the set of analytic functions on the complex n-ball $\mathbb{B}_n$

$$f(z_1, \ldots, z_n) = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha z^\alpha$$

obtained by completing $\mathbb{C}[z_1, \ldots, z_n]$ in the inner product with corresponding orthonormal basis

$$\sqrt{\frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!}} z^\alpha, \quad \alpha \in \mathbb{N}_0^n.$$

The Drury-Arveson space has been rediscovered several times:

Algebraic geometers at the beginning of the 20th referred to the **apolar inner product**

Bombieri re-discovered the inner product the context of number theory, where people refer to the **Bombieri inner product**

Rota realized the inner product as a **Hopf algebra pairing** of the polynomials with the divided powers algebra

Arveson realized the Drury-Arveson space as the **symmetric subspace** of the non-commutative Hardy space

We can consider Arveson's n-**shift**, i.e. the n-tuple of coordinate multiplication operators $M_z = (M_{z_1}, \ldots, M_{z_n})$ on H_n^2 defined by

$$(M_{z_i} f)(z_1, \ldots, z_n) = z_i f(z_1, \ldots, z_n), \quad f \in H_n^2,$$

which arise from the left coordinate multiplication operators $L_Z = (L_{Z_1}, \ldots, L_{Z_n})$ on the noncommutative Hardy space F_n^2 as

$$L_{Z_i} = \begin{pmatrix} M_{z_i} & 0 \\ * & * \end{pmatrix}.$$

Theorem (Arveson (1998))

Every strictly contractive commuting n-tuple of operators $A = (A_1, \ldots, A_n)$ can be dilated to a multiple of the Arveson n-shift $M_z = (M_{z_1}, \ldots, M_{z_n})$, consisting of coordinate multiplication operators on H^2_n.

Basic Idea

1. Dilate $A = (A_1, \ldots, A_n)$ to the left coordinate multiplication operators $L_Z = (L_{Z_1}, \ldots, L_{Z_d})$ on the noncommutative Hardy space F^2_n.
2. "Compress" L_Z onto the coordinate multiplication operators $M_z = (M_{z_1}, \ldots, M_{z_n})$ on the Drury-Arveson space H^2_n.

Analogous to "modding out" by the commutator ideal of the tensor algebra to get the symmetric tensor algebra.

Operator-algebraic
GEOMETRY

Let I be an ideal of $\mathbb{C}[z_1, \ldots, z_n]$, considered as a subspace of the Drury-Arveson space H^2_n.

The closure of I is an invariant subspace for the coordinate multiplication operators $M_z = (M_{z_1}, \ldots, M_{z_n})$, so we can write

$$M_{z_i} = \begin{pmatrix} A_i & 0 \\ * & * \end{pmatrix},$$

where $A = (A_1, \ldots, A_n)$ is the n-tuple obtained by compressing M_z onto $I^\perp$.

The n-tuple A gives a "concrete" representation of the algebra

$$\mathbb{C}[z_1, \ldots, z_n]/I.$$

Example

Letting $I = \langle z_1^2 + z_2^2 - z_3^2 \rangle$ and compressing $M_{z_1}, M_{z_2}, M_{z_3}$ to the orthogonal complement $I^\perp$ gives operators A_1, A_2, A_3 satisfying

$$A_1^2 + A_2^2 = A_3^2.$$

By Arveson's dilation theorem, every contractive commuting n-tuple of operators arises in this way.

Consequence

The behavior of an arbitrary commuting n-tuple of operators is "determined" by this underlying geometry.

Let V be an algebraic variety in $\mathbb{C}^n$ with corresponding ideal

$$I(V) = \{p \in \mathbb{C}[z_1, \ldots, z_n] \mid p(z) = 0 \ \forall z \in V\},$$

and let $A = (A_1, \ldots, A_n)$ denote the n-tuple of operators obtained by compressing the coordinate multiplication operators $M_z = (M_{z_1}, \ldots, M_{z_n})$ to $I(V)^\perp$.

The self-adjoint algebra $\mathcal{A}_V$ generated by A can be thought of as a non-commutative coordinate ring for V.

Conjecture (Arveson 1999)

For every projective algebraic variety, there is an exact sequence

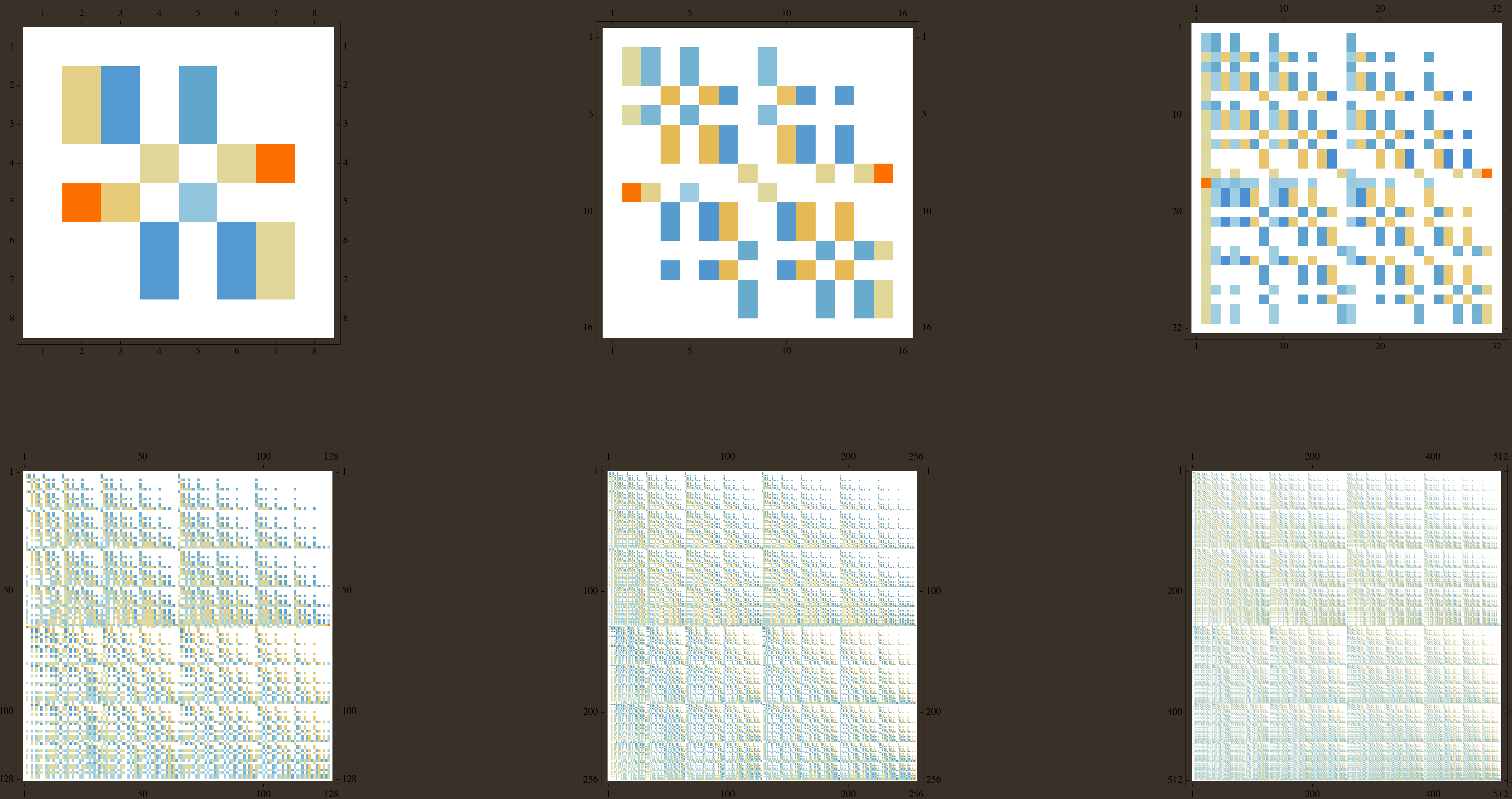
$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A}_V \longrightarrow C(V) \longrightarrow 0,$$

leading to connections between operator theory and algebraic geometry via index theory.

Conjecture is equivalent to the commutators

$$A_i^* A_j - A_j A_i^*, \quad 1 \leq i, j \leq n$$

being compact, i.e. finite rank plus an arbitrarily small remainder.



These commutators decompose as direct sums of finite matrices, and we can verify Arveson's conjecture experimentally.

Let I be an ideal of $\mathbb{C}[z_1, \ldots, z_n]$ generated by homogeneous polynomials.

Theorem (Arveson (2005))

If I is generated by monomials, then Arveson's conjecture holds.

Theorem (Guo-Wang (2008))

If I is singly generated, or if $n \leq 3$, then Arveson's conjecture holds.

Let I be an ideal of $\mathbb{C}[z_1, \ldots, z_n]$ generated by homogeneous polynomials.

Theorem (K (2012))

If there are "sufficiently nice" (for example, singly generated) ideals $I_1, \ldots, I_k$ such that

$$\bar{I} = \overline{I_1} + \ldots + \overline{I_k},$$

then Arveson's conjecture holds for I.

Theorem (K (2012))

If I is generated by polynomials in disjoint variables, then Arveson's conjecture holds for I.

Let V be an algebraic variety in $\mathbb{C}^n$.

Theorem (K-Shalit (2012))

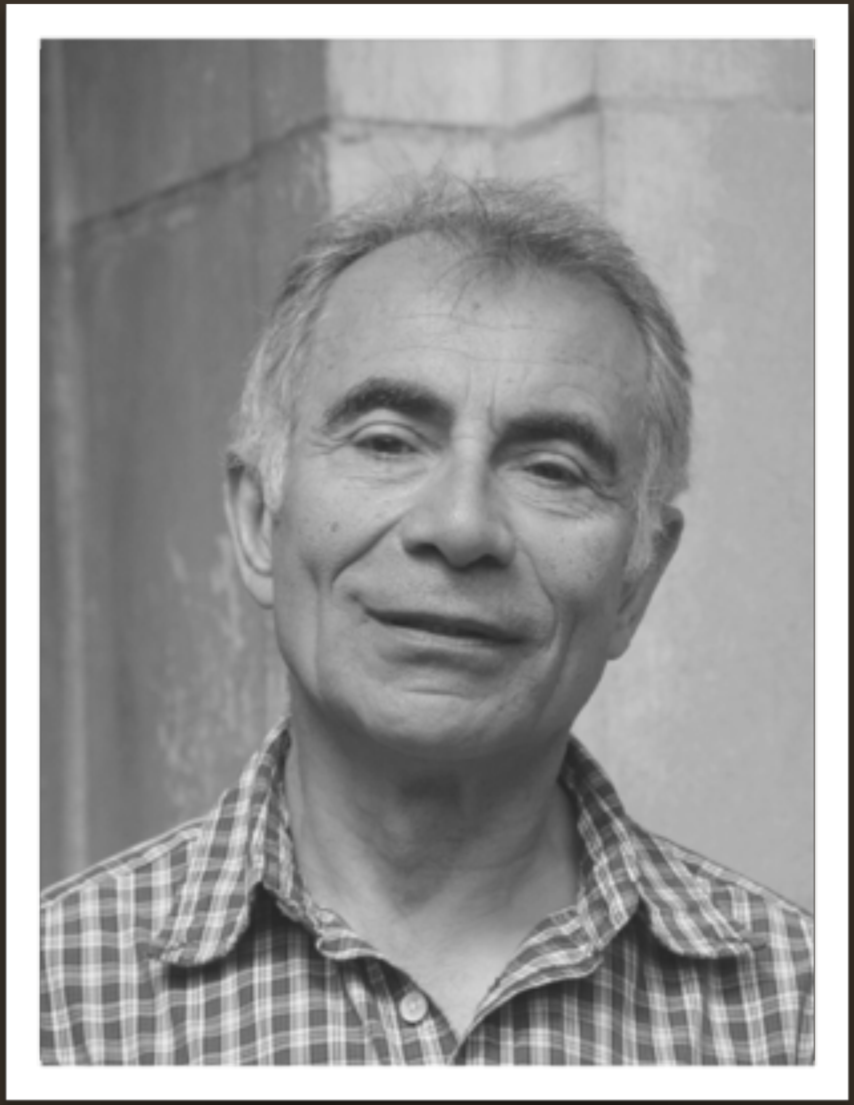
If Arveson's conjecture holds for V, then it holds for every variety with an isomorphic coordinate ring.

Theorem (K-Shalit (2012))

If there are "sufficiently nice" varieties (for example, subspaces) $V_1, \ldots, V_k$ such that

$$V = V_1 \cup \cdots \cup V_k \text{ or } V = V_1 \cap \cdots \cap V_k,$$

then Arveson's conjecture holds for V.

Thanks!