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A COUNTEREXAMPLE TO THE "COMPOSITION CONJECTURE"

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ABSTRACT. In this note we construct a class of counterexamples to the "composition conjecture" concerning an infinitesimal version of the center problem for the polynomial Abel equation in the complex domain.

In this note we treat the following "polynomial moment problem" proposed in [1], [2] as an infinitesimal version of the center problem for the polynomial Abel equation in the complex domain: *for complex polynomials $P(z), Q(z)$ to find conditions under which all moments*

$$m_i(P,Q,a,b) = \int_a^b P^i(z)\,\mathrm{d}\,Q(z) = \int_a^b P^i(z)Q'(z)\,\mathrm{d}\,z, \quad i \geq 1, \quad a,b \in \mathbb{C},$$

vanish under the assumption that $P(a) = P(b)$, $Q(a) = Q(b)$, $a \neq b$.

The "composition conjecture" suggested in [1] states that the following condition is necessary and sufficient: *there exist polynomials $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that*

$$(*) \qquad P(z) = \tilde{P}(W(z)), \qquad Q(z) = \tilde{Q}(W(z)), \qquad \deg W(z) > 1,$$

and $W(a) = W(b)$. Note that since $m_i(P,Q,a,b) = m_i(\tilde{P},\tilde{Q},W(a),W(b))$ and $W(a) = W(b)$, the composition condition is clearly sufficient and the problem is to decide whether this condition is necessary. Note also that since by Lüroth's theorem (see e.g. [6], p.13) each field k such that $\mathbb{C} \subset k \subset \mathbb{C}(z)$ and $k \neq \mathbb{C}$ is of the form $k = \mathbb{C}(R)$, $R \in \mathbb{C}(z)\backslash\mathbb{C}$, it is easy to see that the conditions $(*)$ hold if and only if the field $\mathbb{C}(P,Q)$ is a proper subfield of $\mathbb{C}(z)$. In its turn, since $[\mathbb{C}(z) : \mathbb{C}(P)] = \deg P$, the last condition is equivalent to the condition $[\mathbb{C}(P,Q) : \mathbb{C}(P)] < \deg P$.

Due to its connection with the center problem for the Abel equation and with the classical Poincare center-focus problem for polynomial vector fields on the plane, the polynomial moment problem has been studied in the recent papers [1]–[5]. In particular, the truth of the composition conjecture was established under the additional assumption that a,b are not critical points of $P(z)$ (see [4]) and under some other additional assumptions (see [2], [3] and [5]). In this note we construct a class of counterexamples to the composition conjecture.

Claim 1. *Let $B(z)$ and $D(z)$ be polynomials such that $\mathbb{C}(B) \cap \mathbb{C}(D)$ contains a polynomial $P(z)$, $\deg P(z) > 1$, and let $Q(z) = B(z) + D(z)$. Suppose that there exist*

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$a, b \in \mathbb{C}$ which satisfy $B(a) = B(b)$, $D(a) = D(b)$, $a \neq b$. Then $m_i(P, Q, a, b) = 0$ for all $i \geq 1$.

Proof. Indeed, since $P(z) = A(B(z))$ for some polynomial $A(z)$ and $B(a) = B(b)$, we have

$$m_i(P, B, a, b) = m_i(A, z, B(a), B(b)) = 0$$

for all $i \geq 1$. Similarly, $m_i(P, D, a, b) = 0$ for all $i \geq 1$. Therefore, also $m_i(P, Q, a, b) = 0$ for all $i \geq 1$. □

Claim 2. *Let $B(z)$ and $D(z)$ be polynomials such that $\mathbb{C}(B) \cap \mathbb{C}(D)$ contains a polynomial $P(z)$, $\deg P(z) > 1$, and let $Q(z) = B(z) + D(z)$. Suppose that $\mathbb{C}(B, D) = \mathbb{C}(z)$. Then $\mathbb{C}(P, Q) = \mathbb{C}(z)$.*

Proof. Assume the converse, i.e. that the conditions $(*)$ hold for some polynomials $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(z)$. To be definite suppose that $\deg B(z) \leq \deg D(z)$; then $\deg W(z) | \deg D(z)$. As $P(z) = C(D(z))$ for some polynomial $C(z)$, $\mathbb{C}(D) \cap \mathbb{C}(W)$ contains a non-constant polynomial $P(z)$. By Engstrom's theorem (see [6], Theorem 5, p. 18) this fact implies that $[\mathbb{C}(W, D) : \mathbb{C}(D)] = \deg D / (\deg W, \deg D)$. Therefore, $(\deg W, \deg D) = \deg W > 1$ implies that $[\mathbb{C}(D, W) : \mathbb{C}(D)] < \deg D$. Hence, $D(z) = \tilde{D}(F(z)), W(z) = \tilde{W}(F(z))$ for some polynomials $\tilde{D}(z)$, $\tilde{W}(z)$, $F(z)$, $\deg F > 1$. As $D(z) = \tilde{D}(F(z))$, $Q(z) = \tilde{Q}(W(z)) = \tilde{Q}(\tilde{W}(F(z)))$ and $B(z) = Q(z) - D(z)$, we see that $\mathbb{C}(B, D) \subset \mathbb{C}(F)$. Since this contradicts the condition $\mathbb{C}(B, D) = \mathbb{C}(z)$, we conclude that $\mathbb{C}(P, Q) = \mathbb{C}(z)$. □

In order to get a counterexample $P(z), Q(z)$ to the composition conjecture it is enough to find polynomials $B(z), D(z)$ which satisfy the assumptions of both claims above. Let us describe such polynomials. Since $P \in \mathbb{C}(B) \cap \mathbb{C}(D)$, $\deg P > 1$, and $\mathbb{C}(B, D) = \mathbb{C}(z)$, Engstrom's theorem implies that $(\deg B, \deg D) = 1$. By the second Ritt theorem (see [6], p. 24) conditions $P \in \mathbb{C}(B) \cap \mathbb{C}(D)$, $\deg P > 1$, and $(\deg B, \deg D) = 1$ yield that up to a linear change of variable either $B(z) = z^m$, $D(z) = z^n R(z^m)$, where $R(z) \in \mathbb{C}[z]$ and $(n, m) = 1$ (then $P(z) \in \mathbb{C}[z^{nm} R^m(z^m)]$) or $B(z) = T_n(z)$, $D(z) = T_m(z)$ for Chebyshev polynomials $T_n(z)$, $T_m(z)$ and $(n, m) = 1$ (then $P \in \mathbb{C}[T_{nm}(z)]$). In the first case, due to $(n, m) = 1$, the conditions $B(a) = B(b)$, $D(a) = D(b)$, $a \neq b$, are equivalent to $a^m = b^m = \zeta$, $a \neq b$, where ζ is a root of $R(z)$. Therefore, whenever $R(z) \neq z^l$, $l \geq 1$, we get a counterexample setting $Q(z) = z^m + z^n R(z^m)$, $P(z) \in \mathbb{C}[z^{nm} R^m(z^m)]$. In the second case, set $a = \alpha + 1/\alpha$, $b = \beta + 1/\beta$, where α, β satisfy $\alpha^m = \beta^m$, $(\alpha\beta)^n = 1$, $\alpha \neq \beta, 1/\beta$. Then $a \neq b$ and it follows from $T_k(z + \frac{1}{z}) = z^k + \frac{1}{z^k}$, $k \geq 1$, that $T_n(a) = T_n(b)$, $T_m(a) = T_m(b)$. In this case counterexamples have a form: $Q(z) = T_n(z) + T_m(z)$, $P(z) \in \mathbb{C}[T_{nm}(z)]$.

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