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ON POLYNOMIALS ORTHOGONAL TO ALL POWERS OF A CHEBYSHEV POLYNOMIAL ON A SEGMENT

BY

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ABSTRACT

In this paper we describe polynomials orthogonal to all powers of a Chebyshev polynomial on a segment.

1. Introduction

In the recent series of papers [1]–[5] by M. Briskin, J.-P. Francoise and Y. Yomdin the following "polynomial moment problem" arose as an infinitesimal version of the center problem for the Abel differential equation in the complex domain: *for a complex polynomial* $P(z)$ *and distinct* $a,b\in\mathbb{C}$ *to describe polynomials* $q(z)$ *such that*

$$\int_a^b P^i(z)q(z)\mathrm{d}z=0 \quad \textit{for all integers } i\geq 0. \tag{1}$$

The following "composition condition" imposed on $P(z)$ and $Q(z)=\int q(z)\mathrm{d}z$ is sufficient for polynomials $P(z),q(z)$ to satisfy (1): *there exist polynomials* $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(z)$ *such that*

$$P(z)=\tilde{P}(W(z)),\quad Q(z)=\tilde{Q}(W(z))\quad \textit{and}\quad W(a)=W(b). \tag{2}$$

Indeed, the sufficiency of condition (2) is a direct corollary of the Cauchy theorem, since after the change of variable $z\to W(z)$ the new way of integration

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is closed. It was suggested in the papers cited above ("the composition conjecture") that, under an additional assumption that $P(a) = P(b)$, condition (1) is actually equivalent to condition (2). This conjecture was verified in several special cases. In particular, when a, b are not critical points of $P(z)$ ([6]), when $P(z)$ is indecomposable ([8]), and in some other special cases ([1]–[5], [11], [9]). Nevertheless, in general the composition conjecture is not true.

A class of counterexamples to the composition conjecture was constructed in [7]. The simplest of them has the following form:

$$P(z) = T_6(z), \quad q(z) = T_3'(z) + T_2'(z), \quad a = -\sqrt{3}/2, \quad b = \sqrt{3}/2,$$

where $T_n(z) = \cos(n \arccos z)$ is the n-th Chebyshev polynomial. Indeed, since $T_2(\sqrt{3}/2) = T_2(-\sqrt{3}/2)$ it follows from the equality $T_6(z) = T_3(T_2(z))$ that (1) is satisfied for $P(z) = T_6(z)$ and $q_1(z) = T_2'(z)$. Similarly, from $T_6(z) = T_2(T_3(z))$ and $T_3(\sqrt{3}/2) = T_3(-\sqrt{3}/2)$ one concludes that (1) holds for $P(z) = T_6(z)$ and $q_2(z) = T_3'(z)$. Therefore, by linearity, condition (1) is satisfied also for $P(z) = T_6(z)$ and $q(z) = q_1(z) + q_2(z)$. Nevertheless, for $P(z) = T_6(z)$ and $Q(z) = T_3(z) + T_2(z)$ condition (2) does not hold.

More generally, it was shown in [7] that any polynomial "double decomposition" $A(B(z)) = C(D(z))$ such that $B(a) = B(b)$, $D(a) = D(b)$ supplies counterexamples to the composition conjecture whenever $\deg B(z)$, $\deg D(z)$ are coprime. Note that double decompositions with $\deg A(z) = \deg D(z)$, $\deg B(z) = \deg C(z)$ and $\deg B(z), \deg D(z)$ coprime are described explicitly by Ritt's theory of factorization of polynomials. They are equivalent either to decompositions with $A(z) = z^n R^m(z)$, $B(z) = z^m$, $C(z) = z^m$, $D(z) = z^n R(z^m)$ for a polynomial $R(z)$ and $(n, m) = 1$ or to decompositions with $A(z) = T_m(z)$, $B(z) = T_n(z)$, $C(z) = T_n(z)$, $D(z) = T_m(z)$ for Chebyshev polynomials $T_n(z)$, $T_m(z)$ and $(n, m) = 1$ (see [10], [12]).

In this paper we give a solution of the polynomial moment problem (1) in the case when $P(z)$ is a Chebyshev polynomial $T_n(z)$. Denote by $V(T_n, a, b)$ the vector space over $\mathbb{C}$ consisting of complex polynomials $q(z)$ satisfying (1) for $P(z) = T_n(z)$. Note that any polynomial $T_m'(z)$ such that $T_d(a) = T_d(b)$ for $d = \mathrm{GCD}(n, m)$ is contained in $V(T_n, a, b)$ since $T_n(z) = T_{n/d}(T_d(z))$ and $T_m(z) = T_{m/d}(T_d(z))$.

Theorem 1: *For any $n \in \mathbb{N}$ and $a, b \in \mathbb{C}$, polynomials $T_m'(z)$ such that $T_d(a) = T_d(b)$ for $d = \mathrm{GCD}(n, m)$ form a basis of $V(T_n, a, b)$.*

For instance, it follows from the theorem that if a polynomial $q(z)$ is orthogonal to all powers of $T_6(z)$ on $[-\sqrt{3}/2, \sqrt{3}/2]$, then $\int q(z) \mathrm{d}z$ can be uniquely represented as a finite sum

$$\int q(z)\mathrm{d}z = \sum_k a_k T_{6k}(z) + \sum_k b_k T_{6k+2}(z) + \sum_k c_k T_{6k+3}(z) + \sum_k d_k T_{6k+4}(z)$$

for some $a_k, b_k, c_k, d_k \in \mathbb{C}$.

Theorem 1 implies the following corollary.

Corollary: *Non-zero polynomials orthogonal to all integer non-negative powers of $T_n(z)$ on $[a,b]$ exist if and only if $T_n(a) = T_n(b)$.*

Indeed, for $d|n$ condition $T_d(a) = T_d(b)$ implies that $T_n(a) = T_n(b)$ since $T_n(z) = T_{n/d}(T_d(z))$. On the other hand, if $T_n(a) = T_n(b)$ then for any $R(z) \in \mathbb{C}[z]$ the polynomial $R(T_n(z))T_n'(z)$ is contained in $V(T_n, a, b)$ by (2).

Furthermore, Theorem 1 implies that if $q(z) \in V(T_n, a, b)$ then $\int q(z)\mathrm{d}z$ can be represented as a *sum* of polynomials Q_j such that condition (2) holds for $P(z) = T_n(z)$, $Q(z) = Q_j(z)$. We show that actually the number of terms in such a representation can be reduced to two.

Theorem 2: *For any $q(z) \in V(T_n, a, b)$ there exist divisors d_1, d_2 of n such that $\int q(z)\mathrm{d}z = A(T_{d_1}(z)) + B(T_{d_2}(z))$ for some $A(z), B(z) \in \mathbb{C}[z]$ and the equalities $T_{d_1}(a) = T_{d_1}(b)$, $T_{d_2}(a) = T_{d_2}(b)$ hold.*

For instance, if a polynomial $q(z)$ is orthogonal to all powers of $T_6(z)$ on $[-\sqrt{3}/2, \sqrt{3}/2]$ then $\int q(z)\mathrm{d}z = A(T_3(z)) + B(T_2(z))$ for some $A(z), B(z) \in \mathbb{C}[z]$. Note that such a representation in general is not unique, in contrast to the one provided by Theorem 1.

2. Proofs

2.1 Reduction. First of all, we establish that Theorem 1 can be reduced to the following statement: *if $q(z) = Q'(z)$ is contained in $V(T_n, a, b)$, then*

$$T_d(a) = T_d(b) \quad \textit{for } d = \mathrm{GCD}(n, \deg Q). \tag{3}$$

In particular, $d > 1$.

Indeed, assuming that this statement is true the theorem can be deduced as follows. For $q(z) \in V(T_n, a, b)$, set $m_0 = \deg Q(z)$ and define $c_0 \in \mathbb{C}$ by the condition that the degree of $Q_1(z) = Q(z) - c_0 T_{m_0}(z)$ is strictly less than m_0. Since for $d_0 = \mathrm{GCD}(n, m_0)$ the equalities

$$T_n(z) = T_{n/d_0}(T_{d_0}(z)), \quad T_{m_0}(z) = T_{m_0/d_0}(T_{d_0}(z))$$

hold, it follows from $T_{d_0}(a) = T_{d_0}(b)$ that $T'_{m_0}(z) \in V(T_n, a, b)$. Therefore, by linearity, $Q'_1(z) \in V(T_n, a, b)$. If $\deg Q_1(z) = m_1$ then, similarly, for some $c_{m_1} \in \mathbb{C}$ we have $Q_1(z) = c_{m_1} T_{m_1}(z) + Q_2(z)$, where $Q'_2(z) \in V(T_n, a, b)$ and $\deg Q_2(z) < m_1$.

Continuing in the same way and observing that $m_{i+1} < m_i$ we eventually arrive at the representation

$$\int q(z)\mathrm{d}z = \sum_{i=0}^{k} c_i T_{m_i}(z), \quad c_i \in \mathbb{C},$$

such that $T_{d_i}(a) = T_{d_i}(b)$ for $d_i = \mathrm{GCD}(n, m_i)$. Since polynomials of different degrees are linearly independent over $\mathbb{C}$, we conclude that the polynomials $T'_m(z)$ such that $T_d(a) = T_d(b)$ for $d = \mathrm{GCD}(n, m)$ form a basis of the vector space $V(T_n, a, b)$.

2.2 Proof of Theorem 1 for non-singular a, b. By 2.1 it is enough to show that condition (1) with $P(z) = T_n(z)$, $q(z) = Q'(z)$ implies condition (3). On the other hand, it is known (see [6] or [9]) that for any polynomial $P(z)$ such that a, b are not critical points of $P(z)$, conditions (1) and (2) are equivalent. Therefore, it is enough to prove that (2) with $P(z) = T_n(z)$ implies (3).

Suppose now that (2) holds and set $w = \deg W(z)$. Since by Engstrom's theorem (see, e.g., [12], Th. 5) for any double decomposition $A(B(z)) = C(D(z))$ we have

$$[\mathbb{C}(B, D) : \mathbb{C}(D)] = \deg D / \mathrm{GCD}(\deg B, \deg D),$$

it follows from the equality

$$T_n(z) = \tilde{P}(W(z)) = T_{n/w}(T_w(z))$$

that $\mathbb{C}(W) = \mathbb{C}(T_w)$. Therefore, since $W(z), T_w(z)$ are polynomials, there exists a linear function $\sigma(z)$ such that $W(z) = \sigma(T_w(z))$ and, hence, $W(a) = W(b)$ yields $T_w(a) = T_w(b)$. Since w is a divisor of $d = \mathrm{GCD}(n, \deg Q)$ the decomposition $T_d(z) = T_{d/w}(T_w(z))$ holds and, therefore, $T_w(a) = T_w(b)$ implies $T_d(a) = T_d(b)$.

2.3 Necessary condition for $P(z), q(z)$ to satisfy (1). To investigate the case when at least one of the points a, b is a critical point of $T_n(z)$, we will use a condition, obtained for the case when $P(a) = P(b)$ in [8] and in a general case in [9], which is necessary for polynomials $P(z)$, $q(z)$ to satisfy (1). To formulate this condition let us introduce the following notation. Say that a domain $U \subset \mathbb{C}$ is admissible with respect to the polynomial $P(z)$ if U is simply connected and contains no critical values of $P(z)$. By the monodromy theorem, in such a domain there exist $n = \deg P(z)$ single-valued branches of $P^{-1}(z)$. Let $U_{P(a)}$ (resp. $U_{P(b)}$) be an admissible domain such that its boundary contains the point $P(a)$ (resp. $P(b)$). Denote by $p_{u_1}^{-1}(z), p_{u_2}^{-1}(z), \ldots, p_{u_{d_a}}^{-1}(z)$ (resp. $p_{v_1}^{-1}(z)$, $p_{v_2}^{-1}(z), \ldots, p_{v_{d_b}}^{-1}(z)$) the branches of $P^{-1}(z)$ defined in $U_{P(a)}$ (resp. $U_{P(b)}$) which map points close to $P(a)$ (resp. $P(b)$) to points close to a (resp. b). In particular, the number d_a (resp. d_b) equals the multiplicity of the point a (resp. b) with respect to $P(z)$.

In the above notation a necessary condition for $P(z), q(z)$ to satisfy (1) has the following form: *if polynomials $P(z)$, $q(z) = Q'(z)$ satisfy (1) and $P(a) = P(b) = z_0$, then in any admissible domain U_{z_0} the equality*

$$\frac{1}{d_a}\sum_{s=1}^{d_a} Q(p_{u_s}^{-1}(z)) = \frac{1}{d_b}\sum_{s=1}^{d_b} Q(p_{v_s}^{-1}(z)) \tag{4}$$

holds. Furthermore, if $P(a) \neq P(b)$ then for any admissible domains $U_{P(a)}$, $U_{P(a)}$ we have

$$\frac{1}{d_a}\sum_{s=1}^{d_a} Q(p_{u_s}^{-1}(z)) = 0 \ \text{in } U_{P(a)}, \quad \frac{1}{d_b}\sum_{s=1}^{d_b} Q(p_{v_s}^{-1}(z)) = 0 \ \text{in } U_{P(b)}. \tag{4'}$$

Here $Q(z) = \int q(z)\mathrm{d}z$ is chosen in such a way that $Q(a) = Q(b) = 0$.

More precisely, conditions (4), (4′) hold whenever the function

$$H(t) = \int_a^b \frac{Q(z)P'(z)dz}{t - P(z)}$$

is algebraic near infinity; this is a corollary of general properties of the Cauchy type integrals of algebraic functions (see [9], section 3). On the other hand, using the integration by parts we have:

$$\frac{dH(t)}{dt} = -\int_a^b \frac{Q(z)P'(z)dz}{(t-P(z))^2} = \frac{Q(a)}{t-P(a)} - \frac{Q(b)}{t-P(b)} + \tilde{H}(t),$$

where

$$\tilde{H}(t) = \int_a^b \frac{q(z)dz}{t-P(z)}.$$

Hence, since condition (1) is equivalent to the requirement that $\tilde{H}(t) \equiv 0$ near infinity, it follows from $Q(a) = Q(b) = 0$ that $H(t)$ is algebraic. Therefore, conditions (4), (4′) hold.

2.4 Monodromy of $T_n(z)$. To make conditions (4), (4′) useful we must examine the monodromy group of $T_n(z)$. It follows from $T_n(\cos\phi) = \cos(n\phi)$, $n \geq 1$, that finite critical values of polynomial $T_n(z)$ are ± 1 and that preimages of the points ± 1 are points $\cos(\pi j/n)$, $j = 0, 1, \ldots, n$. To visualize the monodromy group of $T_n(z)$ consider the preimage $P^{-1}[-1,1]$ of the segment $[-1,1]$ under the map $P(z)\colon \mathbb{C} \to \mathbb{C}$. It is convenient to consider $P^{-1}[-1,1]$ as a bicolored graph λ embedded into the Riemann sphere. By definition, white (resp. black) vertices of λ are preimages of the point 1 (resp. -1) and edges of λ are preimages of the interval $(-1,1)$. Since the multiplicity of each critical point of $T_n(z)$ equals 2, the graph λ is a "chain-tree" and, as a point set in $\mathbb{C}$, coincides with the segment $[-1,1]$ (see Figure 1). In particular, non-critical points $-1, 1$ are vertices of valence 1; the vertex 1 is white while the vertex -1 is white or black depending on the parity of n.

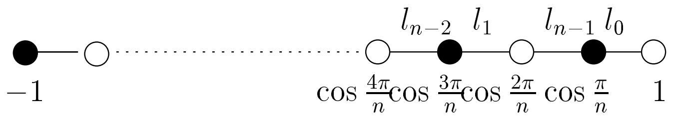

Figure 1

Let us fix an admissible with respect to $T_n(z)$ domain U such that U is unbounded and contains the interval $(-1,1)$. Any branch $T_{n,j}^{-1}(z)$, $0 \leq j \leq n-1$, of $T_n^{-1}(z)$ in U maps the interval $(-1,1)$ onto an edge of λ and we will label such an edge by the symbol l_j (an explicit numeration of the branches of $T_n^{-1}(z)$ will be defined later). Denote by $\pi_1 \in S_n$ (resp. $\pi_{-1}, \pi_\infty \in S_n$) the permutation defined by the condition that the analytic continuation of the functional element $\{U, T_{n,j}^{-1}(z)\}$, $0 \leq j \leq n-1$, along a clockwise oriented loop around 1 (resp. -1, ∞) is the functional element $\{U, T_{n,\pi_1(j)}^{-1}(z)\}$ (resp. $\{U, T_{n,\pi_{-1}(j)}^{-1}(z)\}$, $\{U, T_{n,\pi_\infty(j)}^{-1}(z)\}$). The tree λ represents the monodromy group of $T_n^{-1}(z)$ in the following sense: the edges of λ are identified with branches of $T_n^{-1}(z)$ and the permutation π_1 (resp. π_{-1}) is identified with the permutation arising under clockwise rotation of edges of λ around white (resp. black) vertices.* In order to fix a convenient numeration of branches of $T_n^{-1}(z)$ in U, consider an auxiliary domain $U_\infty = U \cap B$, where B is a disc with the center at the infinity such that

* Note that any polynomial with two finite critical values can be represented by an appropriate bicolored plane tree and vice versa; it is a very particular case of the Grothendieck correspondence between Belyi functions and graphs embedded into compact Riemann surfaces (see, e.g., [13]).

branches of $T_n^{-1}(z)$ can be represented in B by their Puiseux expansions at infinity. In more detail, if $z^{1/n}$ denotes a fixed branch of the algebraic function which is inverse to z^n in U_∞, then each branch of $T_n^{-1}(z)$ can be represented in U_∞ by the convergent series

$$\phi_j(z) = \sum_{k=-\infty}^{1} t_k \varepsilon_n^{jk} z^{k/n}, \quad t_k \in \mathbb{C}, \quad \varepsilon_n = \exp(2\pi i/n), \tag{5}$$

for certain j, $0 \le j \le n-1$.

Now we fix a numeration of branches of $T_n^{-1}(z)$ in U as follows: the branch $T_{n,j}^{-1}(z)$, $0 \le j \le n-1$, is the analytic continuation of $\phi_j(z)$ from U_∞ to U and the branch $z^{1/n}$ is defined by the condition that $T_{n,0}^{-1}(z)$ maps the interval $(-1,1)$ onto the interval $(\cos(\pi/n), 1)$. Since the result of the analytic continuation of the functional element $\{U_\infty, \varepsilon_n^j z^{1/n}\}$, $0 \le j \le n-1$, along a clockwise oriented loop around ∞ is the functional element $\{U_\infty, \varepsilon_n^{j+1} z^{1/n}\}$, such a choice of the numeration implies that $\pi_\infty = (012\ldots n-1)$. Furthermore, it follows from $\pi_\infty\pi_{-1}\pi_1 = 1$, taking into account the combinatorics of λ, that the numeration of edges of λ coincides with the one indicated on Figure 1 that is $\pi_{-1} = (0n-1)(1n-2)(2n-3)\ldots$ and $\pi_1 = (1n-1)(2n-2)(3n-3)\ldots$.

2.5 Proof of Theorem 1 for singular a, b. Again, it is enough to establish that (3) holds. Assume first that $T_n(a) = T_n(b)$. Let $Q'(z) \in V(T_n, a, b)$ with $\deg Q(z) = m$. Since at least one of points a, b is a critical point of $T_n(z)$, the number $z_0 = T_n(a) = T_n(b)$ equals ± 1. Suppose first that $z_0 = 1$. Then $a = \cos(2j_1\pi/n)$, $b = \cos(2j_2\pi/n)$ for certain j_1, j_2, $0 \le j_1, j_2 \le [n/2]$, and condition (4) has the following form:

$$Q(T_{n,j_1}^{-1}(z)) + Q(T_{n,n-j_1}^{-1}(z)) = Q(T_{n,j_2}^{-1}(z)) + Q(T_{n,n-j_2}^{-1}(z)), \tag{6}$$

where $T_{n,i}^{-1}(z)$ is represented in U_∞ by series (5). Since $t_1 \neq 0$, the comparison of the leading coefficients of the Puiseux expansions of the branches in (6) gives

$$\varepsilon_n^{j_1 m} + \varepsilon_n^{(n-j_1)m} = \varepsilon_n^{j_2 m} + \varepsilon_n^{(n-j_2)m}.$$

Therefore, the number $\varepsilon_n^{m/d}$, where $d = \mathrm{GCD}(n, m)$, is a root of the polynomial with integer coefficients

$$f(z) = z^{j_1 d} + z^{(n-j_1)d} - z^{j_2 d} - z^{(n-j_2)d}.$$

Since $\varepsilon_n^{m/d}$ is a primitive n-th root of unity and the n-th cyclotomic polynomial $\Phi_n(z)$ is irreducible over $\mathbb{Z}$, this fact implies that $\Phi_n(z)$ divides $f(z)$ in the ring

$\mathbb{Z}[z]$ and, therefore, that the primitive n-th root of unity ε_n also is a root of $f(z)$. Hence,

$$\varepsilon_n^{j_1 d}+\varepsilon_n^{-j_1 d}=\varepsilon_n^{j_2 d}+\varepsilon_n^{-j_2 d}.$$

Since

$$a=\cos(2j_1\pi/n)=\frac{1}{2}(\varepsilon_n^{j_1}+\varepsilon_n^{-j_1}),\quad b=\cos(2j_2\pi/n)=\frac{1}{2}(\varepsilon_n^{j_2}+\varepsilon_n^{-j_2}),$$

it follows now from

$$T_d\Big(\frac{1}{2}(z+\frac{1}{z})\Big)=\frac{1}{2}\Big(z^d+\frac{1}{z^d}\Big) \tag{7}$$

that $T_d(a)=T_d(b)$.

Similarly, if $z_0=-1$, assuming that

$$a=\cos((2j_1+1)\pi/n),\quad b=\cos((2j_2+1)\pi/n)$$

for certain j_1,j_2, $0\le j_1,j_2\le[(n-1)/2]$, we obtain the equality

$$T_{n,j_1}(z)+T_{n,n-j_1-1}(z)=T_{n,j_2}(z)+T_{n,n-j_2-1}(z),$$

which implies

$$\varepsilon_n^{j_1 m}+\varepsilon_n^{(n-j_1-1)m}=\varepsilon_n^{j_2 m}+\varepsilon_n^{(n-j_2-1)m}$$

and

$$\varepsilon_n^{j_1 d}+\varepsilon_n^{-(j_1+1)d}=\varepsilon_n^{j_2 d}+\varepsilon_n^{-(j_2+1)d}.$$

It yields that

$$\varepsilon_{2n}^{2j_1 d}+\varepsilon_{2n}^{-2(j_1+1)d}=\varepsilon_{2n}^{2j_2 d}+\varepsilon_{2n}^{-2(j_2+1)d},$$

where $\varepsilon_{2n}=\exp(2\pi i/2n)$, and, multiplying the last equality by ε_{2n}^d, we get

$$\varepsilon_{2n}^{(2j_1+1)d}+\varepsilon_{2n}^{-(2j_1+1)d}=\varepsilon_{2n}^{(2j_2+1)d}+\varepsilon_{2n}^{-(2j_2+1)d}.$$

Since

$$a=\frac{1}{2}(\varepsilon_{2n}^{2j_1+1}+\varepsilon_{2n}^{-(2j_1+1)}),\quad b=\frac{1}{2}(\varepsilon_{2n}^{2j_2+1}+\varepsilon_{2n}^{-(2j_2+1)}),$$

we conclude as above that $T_d(a)=T_d(b)$.

Let us prove now that $T_n(a)$ must be equal to $T_n(b)$. Indeed, equalities (4′) could hold only if $d_a>1$, $d_b>1$, that is only if both a,b are critical points of $P(z)$. Since $T_n(z)$ has only two critical values ±1, we see that if $T_n(a)\neq T_n(b)$ then either $T_n(a)=1$, $T_n(b)=-1$ or $T_n(a)=-1$, $T_n(b)=1$. Let, say, $T_n(a)=1$, $T_n(b)=-1$. Then $a=\cos(2j_1\pi/n)$, $b=\cos((2j_2+1)\pi/n)$ and (4′) imply

$$\varepsilon_n^{j_1 m}+\varepsilon_n^{(n-j_1)m}=0,\quad \varepsilon_n^{j_2 m}+\varepsilon_n^{(n-j_2-1)m}=0.$$

The analysis of these equalities similar to the above one leads to the equalities $T_d(a) = 0$, $T_d(b) = 0$. Since $T_n(z) = T_{n/d}(T_d(z))$ it contradicts $T_n(a) \neq T_n(b)$.

2.6 Lemma about values of Chebyshev polynomials. In this subsection we prove the following lemma. *Let $a, b \in \mathbb{C}$ and $p_1, p_2, p_3 \in \mathbb{N}$. Suppose that*

$$T_{p_1}(a) = T_{p_2}(b), \quad T_{p_2}(a) = T_{p_2}(b), \quad T_{p_3}(a) = T_{p_3}(b). \tag{8}$$

Set $l_1 = \mathrm{GCD}(p_1, p_2)$, $l_2 = \mathrm{GCD}(p_1, p_3)$, $l_3 = \mathrm{GCD}(p_2, p_3)$. Then $T_{l_i}(a) = T_{l_i}(b)$ at least for one i, $1 \leq i \leq 3$.

Choose $\alpha, \beta \in \mathbb{C}$ such that $\cos\alpha = a$, $\cos\beta = b$. Since $T_n(\cos\phi) = \cos(n\phi)$, equalities (8) imply that

$$p_1\alpha = \mu_1 p_1\beta + 2\pi g_1, \quad p_2\alpha = \mu_2 p_2\beta + 2\pi g_2, \quad p_3\alpha = \mu_3 p_3\beta + 2\pi g_3, \tag{10}$$

where $\mu_1, \mu_2, \mu_3 = \pm 1$ and $g_1, g_2, g_3 \in \mathbb{Z}$. Clearly, at least two numbers from the set $\{\mu_1, \mu_2, \mu_3\}$ are equal between themselves. To be definite suppose that $\mu_1 = \mu_2$. Choose $u, v \in \mathbb{Z}$ such that $up_1 + vp_2 = l_1$. Adding to the first equation in (10) multiplied by u the second one multiplied by v, we see that $l_1\alpha = \mu_1 l_1\beta + 2\pi g$, where $g \in \mathbb{Z}$. It implies that $\cos l_1\alpha = \cos l_1\beta$ and, therefore, $T_{l_1}(a) = T_{l_1}(b)$.

2.7 Proof of Theorem 2. Suppose $q(z) \in V(T_n, a, b)$. Then, by Theorem 1, $\int q(z)\mathrm{d}z$ can be represented as a sum

$$\int q(z)\mathrm{d}z = \sum_{i=0}^{k} c_i T_{m_i}(z), \quad c_i \in \mathbb{C},$$

where $T_{d_i}(a) = T_{d_i}(b)$ for $d_i = \mathrm{GCD}(n, m_i)$, $0 \leq i \leq k$. We will prove the corollary by induction on k. Since $T_{m_i}(z) = T_{m_i/d_i}(T_{d_i}(z))$ and $T_{d_i}(a) = T_{d_i}(b)$, the corollary is true for $k = 0, 1$. Suppose now that $k > 1$. By the inductive assumption there exist $r, s \in \mathbb{N}$ and $A(z), B(z) \in \mathbb{C}[z]$ such that

$$\sum_{i=0}^{k-1} c_i T_{m_i}(z) = A(T_r(z)) + B(T_s(z)), \quad T_r(a) = T_r(b), \quad T_s(a) = T_s(b).$$

Since $T_{m_k}(a) = T_{m_k}(b)$ it follows from lemma 2.6 that either $T_d(a) = T_d(b)$ for $d = \mathrm{GCD}(r, s)$ and

$$\int q(z)\mathrm{d}z = C(T_d(z)) + c_k T_{m_k}(z) \quad \text{with } C(z) = A(T_{r/d}(z)) + B(T_{s/d}(z)),$$

or $T_e(a)=T_e(b)$ for $e=\mathrm{GCD}(r,m_k)$ and

$$\int q(z)\mathrm{d}z = E(T_e(z))+B(T_s(z)) \quad \text{with } E(z)=A(T_{r/e}(z))+c_kT_{m_k/e}(z),$$

or $T_f(a)=T_f(b)$ for $f=\mathrm{GCD}(s,m_k)$ and

$$\int q(z)\mathrm{d}z = A(T_r(z))+F(T_f(z)) \quad \text{with } F(z)=B(T_{s/f}(z))+c_kT_{m_k/f}(z).$$

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