Discrete Geometry [*]
Lecture 22

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Summary

Hypergraph Coloring.

1 Hypergraph Coloring

Let $H = (V, E)$ be a hypergraph. A *k-coloring* of the hypergraph H is a function $\varphi : V \to \{1, ..., k\} = [k]$. We call a k-coloring *proper* or *not monochromatic* if for every $S \in E$ such that $|S| \geq 2$ there is $x \neq y \in S$ such that $\varphi(x) \neq \varphi(y)$. A k-coloring φ is called *conflict-free* or *CF-coloring* if for every hyperedge $S \in E$ there is $x \in S$ such that $\varphi(x) \neq \varphi(y)$ for every $y \neq x \in S$. Namely, in every hyperedge there is at least one color that appears uniquely.

Motivation for studying conflict-free coloring arises in several applications. One such application is frequency assignment to cellular antennae. A cellular network consists of several antennae that are broadcasting in some frequencies. We wish that in any given point in the range of the antennae, there will be some antenna with some unique frequency, so we can receive the broadcast at that point without interference from other antennae. The range of an antenna can be modeled as a disc centered at the antenna. Let D denote the set of the discs that represent all the antenna, and let $H = (D, E)$ be

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[*] Lecture Notes for a course given by Dr. Shakhar Smorodinsky at the Ben-Gurion University, Israel.

a hypergraph where $E = \{D_p | p \in \mathbb{R}\}$ and $D_p = \{d \subset D | p \in d\}$. Hence, we are interested in a conflict-free coloring for the hypergraph H. Another motivation to study conflict-free coloring of hypergraphs is RFID-networks (radio frequency identification tags netowrks).

Let $H = (P, E)$ be a hypergraph where P is an n-point set in $\mathbb{R}^2$ and E is the family of all subsets $P \cap C \subset P$ where C is some disc. A natural question is what is the minimal number $f(n)$ such that every n-point set P can be conflict-free colored with respect to discs in at most $f(n)$ colors.

First lets examine a private case where all the points of P are on x-axis. It easy to see that $f(n) \leq \lfloor \log_2 n \rfloor + 1$. Indeed, we can assign an unique color to the median point p, and recursively assign colors to the two disjoint sets P_l, P_r where P_l is the set of all points to the left of p and p_r is the set of all points to the right of p. In the recursive procedure we use the same set of colors for P_l and P_r but we keep this colors disjoint from the color of p. In this way it is easy to see that the number of colors $f(n)$ used satisfy the inequality $f(n) \leq 1 + f(n/2)$ which imply the above inequality. On the other hand, we also have that $f(n) \geq \Omega(\log n)$. Indeed, let χ be any valid CF-coloring for P. There is at least one point that has a unique color in the disc containing all points of P. Denote this uniquely colored point by p_1. Either to the left or right of p_1, there are at least $\frac{n}{2}$ points. Denote this set of points by P_1. Since there is a disc containing only P_1 and not p_1, there is a point p_2 which is uniquely colored in P_1 and this color is different than the one assigned to p_1. Repeating this argument, it is easy to see that we will need at least $\log n$ distinct colors.

Theorem 1.1 (Even, Lotker, Ron, Smorodinsky)**.** *For a n-point set in* $\mathbb{R}^2$ *there is a CF-coloring (w.r.t discs) which uses at most* $\log_{4/3} n$ *colors.*

Before we show the proof we need a definition of a Delaunay graph.

Definition. *Let* $H = (V, E)$ *be some hypergraph. The* Delaunay graph *of* H *is the graph* $G(H) = (V, E^2)$ *such that* $E^2 = \{S \in E | |S| = 2\}$.

Proof. Let P be an n-point set. We provide an algorithm for finding a conflict free coloring for P with at most $\log_{4/3} n$ colors:

Algorithm 1: Algorithm for finding a conflict free coloring

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Input: $P \subset \mathbb{R}^2$ such that $|P| < \infty$
Output: A conflict free coloring $\varphi : P \rightarrow \{1, ..., \log_{\frac{4}{3}} n\}$
begin
    $i \longleftarrow 0$;
    while $P \neq \emptyset$ do
        find $P' \subseteq P$ (try to make $P'$ as large as possible) s.t. $P'$ forms
        an independent set in $G(P)$ (i.e. the Delaunay graph of $P$);
        $\forall x \in P'$ set $\varphi(x) = i$;
        $P \longleftarrow P \backslash P'$;
        $i++$;
    end
end
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First we count how many colors we use. Consider iteration i of the algorithm and let n_i be the number of remaining non-colored points after iteration $i-1$. We claim that at least $\frac{n_i}{4}$ points are colored by i in iteration i. This follows from the fact that the Delaunay graph in our case is a planar graph, so by the Four-Color theorem and the pigeon-hole principle it has an independent set of of size at least $\frac{n_i}{4}$. So, after each iteration we remain with $\frac{3}{4}$ of the points that we began with. Hence, the number of points in iteration i is $n \cdot (\frac{3}{4})^i$. After $\log_{\frac{4}{3}} n$ iterations we will remain with only one point, so the number of the iterations is at most $\log_{\frac{4}{3}} n + 1$ as asserted.
Next, we show that the coloring above is indeed a conflict free coloring. In fact, we will show the stronger property that for every possible disc d, the maximal indexed color in $C \cap P$ appears only once. Assume to the contradiction that there is disc d such that there are at least two points $p, q \in P$ with the maximal color i in d. The points p and q are independent in the Delaunay graph of the point set P_i of iteration i. Therefore, in the disc d, there is at least one more point $r \in d \cap P_i$. Since i is maximal in d then r must be colored i as well. By shrinking arguments, one can show that there is another disc d' such that: (i)$|d' \cap P_i| = 2$ and (ii) $d' \cap P_i \subset d \cap P_i$. This is a contradiction for the fact that the points in $d \cap P_i$ are independent in the Delaunay graph of P_i. □

The running time of the algorithm shown in the proof is $O(\sum_{i=1}^{\log_{\frac{4}{3}} n} n_i \log n_i) = O(\log n \sum_{i=1}^{\log_{\frac{4}{3}} n} n_i) = O(\log n \sum_{i=1}^{\log_{\frac{4}{3}} n} n(\frac{3}{4})^i) = O(n \log n \sum_{i=1}^{\log_{\frac{4}{3}} n} (\frac{3}{4})^i) = O(n \log n)$

where n_i is the number of points that remain after the i'th iteration.

The $O(\log n)$ bound provided above is tight as we know that there exists point set which require $\Omega(\log n)$ colors in any cf-coloring. In fact, Pach and Toth proved that one can not avoid using $\Omega(\log n)$ colors for any n-point set.

Theorem 1.2 (Pach, Toth)**.** *Any n-point set in $\mathbb{R}^2$ require at least $\Omega(\log n)$ colors in any cf-coloring w.r.t discs.*