Combinatorial Geometry: Exercise 1.
Due date: November 11, 2010.

1. Prove that for $X = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}^d$ the set $CH(X)$ equals the set of all convex combinations of X, that is, $\{\sum_{i=1}^{n} \lambda_i a_i \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \forall i\}$.

2. Let A be a $k \times d$ matrix and let $b \in \mathbb{R}^k$. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ be a function defined by $f(v) = Av + b$. Prove that if $C \subset \mathbb{R}^d$ is convex then $f(C)$ is convex.

3. Let $K \subset \mathbb{R}^d$ be a convex set and let $C_1, \ldots, C_n \subset \mathbb{R}^d$, $n \geq d + 1$, be convex sets such that the intersection of any $d + 1$ of them contains a translated copy of K. Prove that the intersection of all the sets C_i also contains a translated copy of K.

4. (i) Prove that if a family of convex sets $\{C_1, \ldots, C_n\}$ in $\mathbb{R}^2$ has the property that out of every four sets, some triple have a non-empty intersection then there is a point that stabs at least $\frac{n}{12}$ of the sets.

   (ii) Prove that there is a constant $c = c(d, p)$ $(p \geq d+1)$ such that if a finite family of n convex sets in $\mathbb{R}^d$ have the property that out of every p members, some $d + 1$ of them have a non-empty intersection then there is a point that stabs at least cn of the sets.

   (iii) Prove that there is a constant $c = c(d)$ such that a finite family $\mathcal{F} = \{C_1, \ldots, C_n\}$ of convex sets in $\mathbb{R}^d$ has the property that out of every $d+2$ members of $\mathcal{F}$, some $d+1$ of them have a non-empty intersection, then $\mathcal{F}$ can be partitioned into at most $c \log n$ intersecting sub-families. In other words, the members of $\mathcal{F}$ can be "stabbed" by at most $c \log n$ points.

5. An *axis-parallel* box in $\mathbb{R}^d$ is a set of the form $I_1 \times I_2 \times \cdots \times I_d$ where each I_j is a non-empty interval:

   (i) Let $\mathcal{B}$ be a finite family of axis-parallel boxes in $\mathbb{R}^d$ such that any two have a non-empty intersection. Prove that they all have a non-empty intersection.
   (ii) Let $\mathcal{R}$ be a finite collection of axis-parallel rectangles in $\mathbb{R}^2$ such that out of any three, there are some two that have a non-empty intersection. Prove that the rectangles in $\mathcal{R}$ can be stabbed by 3 points. Show that the number 3 is best possible here.

6. Let L be a set of n lines in $\mathbb{R}^d$ such that for any subset $L' \subset L$ of $d+1$ lines, there is a point p such that the distance from p to each of the lines in L' is at most 1. Prove that there is a point of distance at most 1 from all lines in L.