

Combinatorial Geometry: Exercise 2.

Due date: November 25, 2010.

1. Let $P \subset \mathbb{R}^2$ be a set of n points such that no four points of P are co-circular (i.e., there is no circle passing through four or more points). Prove that there is a partition $P = B \cup S \cup R$ such that: $|B|, |R| \leq 3/4n$ and $|S| \leq 2\sqrt{n}$ and such that for any disc d if $d \cap B \neq \emptyset$ and $d \cap R \neq \emptyset$ then $d \cap S \neq \emptyset$.
2. Consider the curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^{2d}$ where $\gamma(t) = (t, t^2, \dots, t^{2d})$. Let $P = \{p_1, p_2, \dots, p_{d+1}\} \subset \mathbb{R}^{2d}$ where $p_i = \gamma(t_i), i \in \{1, \dots, d+1\}$ and $t_1 < t_2 < \dots < t_{d+1}$ are some given $d+1$ reals.
 - (i) Prove that for any set $Q = \{q_1, q_2, \dots, q_{d+1}\} \subset \mathbb{R}^{2d}$ where $q_i = \gamma(s_i), i \in \{1, \dots, d+1\}$ and $s_1 < t_1 < s_2 < t_2 \dots < s_{d+1} < t_{d+1}$ $CH(Q) \cap CH(P) \neq \emptyset$. Hint: In how many points can a given hyperplane intersect γ ?
 - (ii) Put $k = d(2d+1) + 1$. Let $P = \{p_1, p_2, \dots, p_k\} \subset \mathbb{R}^{2d}$ where $p_i = \gamma(t_i), i \in \{1, \dots, k\}$ and $t_1 < t_2 < \dots < t_k$ are some given k reals. Prove that there is a point $q \in \mathbb{R}^{2d}$ such that for any set $Q = \{q_1, q_2, \dots, q_k\} \subset \mathbb{R}^{2d}$ where $q_i = \gamma(s_i), i \in \{1, \dots, k\}$ and $s_1 < t_1 < s_2 < t_2 \dots < s_k < t_k$ the set $CH(Q)$ must contain q .
3.
 - (i) Let P be a set of n points in \mathbb{R}^2 . Let $G = (P, E)$ be a simple graph on P with m edges. Assume that for each edge $e = (p, q) \in E$ we pick a disc D_e such that the circle C_e bounding the disc D_e contains exactly p and q (i.e., $C_e \cap P = \{p, q\}$). Denote the set of m chosen discs by \mathcal{D} . Show that if $m \geq 4n$, then there is a point in p that is contained in the interior of at least $\frac{m^2}{16n^2}$ discs of \mathcal{D} . Hint: Show that if $m \geq 3n$ then at least one of the discs contains a point from P in its interior.
 - (ii) Show that there is a constant $0 < \alpha \leq 1$ such that for any set P of n points in the plane, there exists a pair $p, q \in P$ such that any disc that contains p and q in its interior, must contain at least αn points of P in its interior.
4. Let P be a set of n points in the plane. Show that there are at most $O(n^{7/3})$ triangles of unit area whose vertices are from P . Hint: Use the Szemerédi-Trotter theorem.