Graph Theory: Homework 5, Due 23.1.2017

1. Given a set of lines in the plane with no three meeting at a common point, form a graph G whose vertices are the intersections of the lines with two vertices adjacent if they appear consecutive on one of the lines. Prove that $\chi(G) \leq 3$.

2. Let G be a graph such that every pair of odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

3. Without using the weak (nor the strong) perfect graph theorem, prove that the complement of a bipartite graph is perfect.

4. Prove that Brooks' Theorem is equivalent to the following statement: Every $k-1$ regular k-critical graph is a clique or an odd cycle.

5. Prove that for every graph G, $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$.

6. The Knesser graph $KG_{n,k}$ (for $1 \leq k \leq \frac{n}{2}$) is the graph whose vertices are all k-subsets (i.e., subsets of cardinality k) of an n element set and whose edges are all pairs of vertices that correspond to disjoint sets. Prove that $\chi(KG_{n,k}) \leq n-2k+2$.

7. Prove that the vertices of every connected graph on at least 8 vertices with maximum degree 6 can be colored with 3 colors so that no odd cycle is monochromatic.

8. Let G be a d-degenerate graph and let G' be the graph obtained from G by Mycielski's construction. Is it true that G' is $d+1$ degenerate?

9. (i) Let S be a finite family of closed axis-parallel squares of side length 1 in the plane. Assume also that any point in the plane belongs to at most 2 squares of S. Prove that the family S can be partitioned into four sub-families such that each sub-family consists of pairwise disjoint squares.
(ii) Let S be as above without the restriction on the side length. Thats is, S is a finite family of closed axis-parallel squares. Assume also that any point in the plane belongs to at most 2 squares of S. Prove that the family S can be partitioned into five sub-families such that each sub-family consists of pairwise disjoint squares.