Graph Theory: Homework 6, Due 13.01.2014

1. Prove that every simple 9-regular graph on 100 vertices contains a subgraph with maximum degree at most 5 and at least 225 edges.

2. Let G be a graph such that $\chi'(G) = 2$. Prove that:

   (a) $ch(G) = 2$

   (b) $ch'(G) = 2$ where $ch'(G) = ch(L(G))$

3. Given a finite set of lines in the plane with no three meeting at a common point, and a circle that contain all intersection points of the lines in its interior, form a graph G whose vertices are the intersections of the pair of lines and the intersections of the lines with the circle with two vertices adjacent if they appear consecutive on one of the lines or consecutive on the circle. Prove that $\chi(G) \leq 3$.

4. Let $G = (V, E)$ be an even cycle. Let $\{L_x\}_{x \in V \cup E}$ be a family of sets, each of cardinality 4. Prove that there is a function $f : V \cup E \to \bigcup_{x \in V \cup E} L_x$ such that:

   (a) $\forall x \in V \cup E f(x) \in L_x$

   (b) $f|_V$ is a proper vertex coloring of G and $f|_E$ is a proper edge coloring of G.

   (c) $\forall v \in V \; e \in E$ if $v \in e$ then $f(v) \neq f(e)$.

5. Prove that for every graph $G = (V, E)$ with n vertices $ch(G) < \chi(G) \ln n + 1$.

6. Prove that a plane graph is bipartite if and only if every face has even length.

7. prove that the Petersen graph is nonplanar.

8. Suppose G is the union of two simple planar graphs. Prove that $\chi(G) \leq 12$.

9. Show that for every m and n $4n \leq m \leq \binom{n}{2}$ there exists a graph G with n vertices and m edges such that $Cr(G) = O(\frac{m^3}{n^2})$.

10. A graph G is called *quasi-planar* if it can be drawn in the plane with no three pairwise crossing edges. Prove that a quasi-planar graph with n vertices has at most $O(n^{3/2})$ edges.