Overdetermined Systems on Lie Groups and Their Transfer Functions

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Abstract— **In this paper we define the notion of an overdetermined systems on a Lie group $\mathfrak{G}$ and its associated Lie algebra operator vessel. We define the transfer functions of such systems which turn out to be the joint characteristic functions of the associated vessel. We study the properties of the transfer functions via restriction to one-parameter subgroups of $\mathfrak{G}$. A formula connecting the values of the transfer function with the characteristic function of the infinitesimal generator of the group will be given and an example will be provided for $\mathfrak{G}$ the $ax+b$-group.**

I. Introduction

Commutative overdetermined systems and their transfer function were studied in great detail in [2], [6] and [9]. The non-commutative colligations and vessels were partly treated in [6] and in a very general setting in [5]. This paper is devoted to non-commutative overdetermined linear systems in the setting of Lie groups and Lie algebras. We work in the setting of Lie groups and algebras since we want to preserve invariance under group action, similarly to the commutative case. In the second section we will define the notions of such systems and their associated non-commutative operator vessels. Following [9] we develop a parallel non-commutative frequency domain theory for our overdetermined systems in the third section. We define the non-commutative joint and complete characteristic functions and the spaces on which they operate. The fourth section will consider the notion of one-parameter subgroups of the Lie group in question. We will show that restricting our system to the one-parameter subgroup a one dimensional operator colligation arises. In fact we show in a theorem in that chapter that the joint transfer function of our system is a direct integral of colligation transfer functions. In the fifth chapter we will give a concrete example of the $ax+b$ group and show how that theorem helps to compute the transfer function of a general non-commutative overdetermined system on this Lie group.

II. Lie Algebra Operator Vessels

This section will describe the Lie algebra vessels and their associated systems of differential equations on the Lie groups associated to the Lie algebra. We start with a few definitions. First, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{E}$ we will denote by $\mathcal{L}(\mathcal{H},\mathcal{E})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{E}$. For simplicity we will denote $\mathcal{L}(\mathcal{H},\mathcal{H})$ by $\mathcal{L}(\mathcal{H})$. Second, let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. A (quasi hermitian) $\mathfrak{g}$-operator vessel is a collection

$$\mathfrak{V} = (\mathcal{H},\mathcal{E},\rho,\Phi,\sigma,\gamma,\gamma_*)\,. \tag{1}$$

Here $\mathcal{H}$ and $\mathcal{E}$ are Hilbert spaces, $\Phi \in \mathcal{L}(\mathcal{H},\mathcal{E})$, and $\rho\colon \mathfrak{g} \to \mathcal{L}(\mathcal{H})$, $\sigma\colon \mathfrak{g} \to \mathcal{L}(\mathcal{E})$ and $\gamma,\gamma_*\colon \bigwedge^2(\mathfrak{g}) \to \mathcal{L}(\mathcal{E})$ are linear mappings such that the following hold:

- $\frac{1}{i}\rho$ is a representation of $\mathfrak{g}$, i.e., for all $a,b \in \mathfrak{g}$
$$\frac{1}{i}[\rho(a),\rho(b)] = \rho([a,b]). \tag{2}$$
- For all $a \in \mathfrak{g}$, $\sigma(a)^* = \sigma(a)$, and for all $a,b \in \mathfrak{g}$,
$$\begin{aligned}&\frac{1}{i}\left(\gamma(a\wedge b) - \gamma(a\wedge b)^*\right) = \\ &= \frac{1}{i}\left(\gamma_*(a\wedge b) - \gamma_*(a\wedge b)^*\right) = \\ &= \sigma([a,b]).\end{aligned} \tag{3}$$
- The colligation condition: for all $a \in \mathfrak{g}$
$$\frac{1}{i}\left(\rho(a) - \rho(a)^*\right) = \Phi^*\sigma(a)\Phi. \tag{4}$$
- The input vessel condition: for all $a,b \in \mathfrak{g}$
$$\sigma(a)\Phi\rho(b)^* - \sigma(b)\Phi\rho(a)^* = \gamma(a\wedge b)\Phi. \tag{5}$$
- The output vessel condition: for all $a,b \in \mathfrak{g}$
$$\sigma(a)\Phi\rho(b) - \sigma(b)\Phi\rho(a) = \gamma_*(a\wedge b)\Phi. \tag{6}$$
- The linkage condition: for all $a,b \in \mathfrak{g}$
$$\gamma_*(a\wedge b) - \gamma(a\wedge b) = i\left(\sigma(a)\Phi\Phi^*\sigma(b) - \sigma(b)\Phi\Phi^*\sigma(a)\right). \tag{7}$$

Alternatively, let $e_1,\dots,e_l$ be a basis of $\mathfrak{g}$ and let c_{kj}^m be the corresponding structure constants:

$$[e_k,e_j] = \sum_m c_{kj}^m e_m. \tag{8}$$

Setting $A_k = \rho(e_k)$, $\sigma_k = \sigma(e_k)$, $\gamma_{kj} = \gamma(e_k\wedge e_j)$, $\gamma_{*kj} = \gamma_*(e_k\wedge e_j)$, we obtain the various vessel conditions in

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the basis dependent form:

$$\frac{1}{i}[A_k, A_j] = \sum_m c_{kj}^m A_m, \tag{9}$$

$$\frac{1}{i}\left(\gamma_{jk} - \gamma_{jk}^*\right) = \frac{1}{i}\left(\gamma_{*jk} - \gamma_{*jk}^*\right) = \sum_m c_{jk}^m \sigma_m, \tag{10}$$

$$\frac{1}{i}\left(A_k - A_k^*\right) = \Phi^* \sigma_k \Phi, \tag{11}$$

$$\sigma_k \Phi A_j^* - \sigma_j \Phi A_k^* = \gamma_{kj} \Phi, \tag{12}$$

$$\sigma_k \Phi A_j - \sigma_j \Phi A_k = \gamma_{*kj} \Phi, \tag{13}$$

$$\gamma_{*kj} - \gamma_{kj} = i\left(\sigma_k \Phi \Phi^* \sigma_j - \sigma_j \Phi \Phi^* \sigma_k\right). \tag{14}$$

A Lie algebra vessel is called strict if:

- $\Phi \mathcal{H} = \mathcal{E}$
- $\bigcap_{X \in \mathfrak{g}} \ker \sigma(X) = 0$

A Lie algebra vessel (1) corresponds to a left invariant conservative overdetermined linear system on a Lie group $\mathfrak{G}$ with Lie algebra (isomorphic to) $\mathfrak{g}$:

$$\begin{aligned} iXx + \rho(X)x &= \Phi^* \sigma(X) u, \\ y &= u - i\Phi x, \end{aligned} \tag{15}$$

with energy conservation law

$$X\langle x, x\rangle = \langle \sigma(X)u, u\rangle - \langle \sigma(X)y, y\rangle, \tag{16}$$

and with compatibility conditions for the input and the output signals:

$$\sigma(Y)Xu - \sigma(X)Yu + i\gamma(X \wedge Y)u = 0, \tag{17}$$

$$\sigma(Y)Xy - \sigma(X)Yy + i\gamma_*(X \wedge Y)y = 0. \tag{18}$$

Here $x \colon \mathfrak{G} \to \mathcal{H}$ is the state, $u \colon \mathfrak{G} \to \mathcal{E}$ is the input, and $y \colon \mathfrak{G} \to \mathcal{E}$ is the output, and X and Y are left invariant vector fields on $\mathfrak{G}$ (identified with elements of $\mathfrak{g}$). In terms of the corresponding basis $X_1, \ldots, X_l$ for left invariant vector fields, the system equations become:

$$\begin{aligned} iX_k(x) + A_k x &= \Phi^* \sigma_k u, \quad k = 1, \ldots, l, \\ y &= u - i\Phi x, \end{aligned} \tag{19}$$

$$X_k\langle x, x\rangle = \langle \sigma_k u, u\rangle - \langle \sigma_k y, y\rangle, \quad k = 1, \ldots, l, \tag{20}$$

$$\sigma_k X_j u - \sigma_j X_k u + i\gamma_{jk} u = 0, \quad j, k = 1, \ldots, l, \tag{21}$$

$$\sigma_k X_j y - \sigma_j X_k y + i\gamma_{*jk} y = 0, \quad j, k = 1, \ldots, l. \tag{22}$$

III. Frequency Response Functions

Recall that frequency domain theory in [2], [6] and [9] begins with looking at particular system trajectories:

$$\begin{aligned} u(t) &= e^{i\lambda t} u_0 \\ x(t) &= e^{i\lambda t} x_0 \\ y(t) &= e^{i\lambda t} y_0 \end{aligned} \tag{23}$$

Here $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $t = (t_1, \ldots, t_l)$. Since in the classical case we deal with commutative Lie groups, their irreducible unitary representation are exactly $\pi_\lambda(t) = e^{i\lambda t}$. This leads us to consider the unitary dual $\widehat{\mathfrak{G}}$ of $\mathfrak{G}$ as the basis for frequency domain theory in our case. We consider a unitary irreducible representation $\pi \colon \mathfrak{G} \to \mathcal{L}(\mathcal{H}_\pi)$ and twist our system a bit. We look at $\mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E})$-valued and $\mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{H})$-valued system trajectories:

$$\begin{aligned} u(g) &= (\pi(g) \otimes I_{\mathcal{E}})\mathbf{u}_0 \\ x(g) &= (\pi(g) \otimes I_{\mathcal{H}})\mathbf{x}_0 \\ y(g) &= (\pi(g) \otimes I_{\mathcal{E}})\mathbf{y}_0 \end{aligned} \tag{24}$$

These functions are our generalized waves, where π is the frequency and $\mathbf{u}_0$, $\mathbf{x}_0$ and $\mathbf{y}_0$ are amplitudes. For $\mathbf{u}_0, \mathbf{y}_0 \in \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E})$ and $\mathbf{x}_0 \in \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{H})$. We twist the system equations appropriately:

$$\begin{aligned} iX\mathbf{x} + [I_{\mathcal{H}_\pi} \otimes \rho(X)]\mathbf{x} &= [I_{\mathcal{H}_\pi} \otimes \Phi^* \sigma(X)]\mathbf{u} \\ \mathbf{y} &= \mathbf{u} - i[I_{\mathcal{H}_\pi} \otimes \Phi]\mathbf{x} \end{aligned} \tag{25}$$

These are the system equations for operator valued trajectories. We will plug the generalized waves into them to obtain the frequency domain equations. In order to be able to do this we need to compute the action of the Lie algebra on the representation.

We proceed to define the action induced by π on the Lie algebra $\mathfrak{g}$. We follow [8, Ch. 2] and identify $\mathfrak{g}$ with $T_e\mathfrak{G}$. Every $X \in \mathfrak{g}$ is associated to a one-parameter subgroup of $\mathfrak{G}$, i.e., $\gamma_X(t) = \exp(tX)$ where $t \in \mathbb{R}$. Thus $V_{X,\pi}(t) = \pi(\gamma_X(t))$ is a one parameter subgroup of operators on $\mathcal{H}_\pi$. We define $\pi(X)$ to be the infinitesimal generator of $V_{X,\pi}$. Furthermore by [8, Eq. 2.3] $\pi(X)$ is a representation of the Lie algebra $\mathfrak{g}$. The image of this representation may contain unbounded operators as well.

Now by [8, Eq. 28], for every $u \in \mathcal{H}_\pi$ in the domain of definition of $\pi(X)$, $\pi(g)u$ is a smooth function on $\mathfrak{G}$ and we have:

$$\begin{aligned} \pi(X)\pi(g)u &= \lim_{h \to 0} \frac{\pi(\exp(hX))\pi(g)u - \pi(g)u}{h} = \\ &= \lim_{h \to 0} \frac{\pi(\exp(hX)g)u - \pi(g)u}{h} = \\ &= X\pi(g)u \end{aligned} \tag{26}$$

The last equality follows from the definition of $Xf(g)$, for a smooth function f on $\mathfrak{G}$, [8, Eq. 2.32].

Next we consider the input/output compatibility conditions (21) and (22) for the system equation and apply them to our trajectories (24), we get:

$$\left(\Lambda_k \otimes \sigma_j - \Lambda_j \otimes \sigma_k + I_{\mathcal{H}_\pi} \otimes \gamma_{kj}\right)\mathbf{u}_0 = 0 \tag{27}$$

$$\left(\Lambda_k \otimes \sigma_j - \Lambda_j \otimes \sigma_k + I_{\mathcal{H}_\pi} \otimes \gamma_{*kj}\right)\mathbf{y}_0 = 0 \tag{28}$$

Here we have set $\Lambda_p = \frac{1}{i}\pi(X_p)$. Hence similarly to the commutative case we are led to define the twin operator-valued functions on $\widehat{\mathfrak{G}}$:

$$\begin{aligned} U_{kj}(\pi) &= \Lambda_k \otimes \sigma_j - \Lambda_j \otimes \sigma_k + I_{\mathcal{H}_\pi} \otimes \gamma_{kj} \\ V_{kj}(\pi) &= \Lambda_k \otimes \sigma_j - \Lambda_j \otimes \sigma_k + I_{\mathcal{H}_\pi} \otimes \gamma_{*kj} \end{aligned} \tag{29}$$

In the basis-independent form we get the following two compatibility operators:

$$\begin{aligned} &U(\pi, X, Y) = \pi(X) \otimes \sigma(Y) - \pi(Y) \otimes \sigma(X) + \\ &+ iI_{\mathcal{H}_\pi} \otimes \gamma(X \wedge Y) \\ &V(\pi, X, Y) = \pi(X) \otimes \sigma(Y) - \pi(Y) \otimes \sigma(X) + \\ &+ iI_{\mathcal{H}_\pi} \otimes \gamma_*(X \wedge Y) \end{aligned} \tag{30}$$

Therefore one is led to define the following spaces:

$$\boldsymbol{\mathcal{E}}(\pi) = \{\mathbf{u} \in \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E}) | U_{kj}(\pi)\mathbf{u} = 0, \forall j, k\} \tag{31}$$

$$\boldsymbol{\mathcal{E}}_*(\pi) = \{\mathbf{y} \in \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E}) | V_{kj}(\pi)\mathbf{y} = 0, \forall j, k\} \tag{32}$$

In the basis independent form one has the following equivalent definitions:

$$\boldsymbol{\mathcal{E}}(\pi) = \{\mathbf{u} \in \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E}) | U(\pi, X, Y)\mathbf{u} = 0\} \tag{33}$$

$$\boldsymbol{\mathcal{E}}_*(\pi) = \{\mathbf{y} \in \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E}) | V(\pi, X, Y)\mathbf{y} = 0\} \tag{34}$$

Now we plug (24) into the first equation in (25) and obtain:

$$(i\pi(X) \otimes I_{\mathcal{H}} + I_{\mathcal{H}_\pi} \otimes \rho(X))\, \mathbf{x}_0 = (I_{\mathcal{H}_\pi} \otimes \Phi^* \sigma(X))\, \mathbf{u}_0 \tag{35}$$

Assume that we can choose $X \in \mathfrak{g}$ such that $i\pi(X) \otimes I_{\mathcal{H}} + I_{\mathcal{H}_\pi} \otimes \rho(X)$ is invertible. Then we have a solution to the above equation in the following form:

$$\mathbf{x}_0 = (i\pi(X) \otimes I_{\mathcal{H}} + I_{\mathcal{H}_\pi} \otimes \rho(X))^{-1} (I_{\mathcal{H}_\pi} \otimes \Phi^* \sigma(X))\, \mathbf{u}_0 \tag{36}$$

Plugging (24) and then the above result into the second equation of (25) we get:

$$\begin{aligned} &\mathbf{y}_0 = (I_{\mathcal{H}_\pi} \otimes I_{\mathcal{E}} - \\ &\quad - i\,(I_{\mathcal{H}_\pi} \otimes \Phi)\,(i\pi(X) \otimes I_{\mathcal{H}} + I_{\mathcal{H}_\pi} \otimes \rho(X))^{-1} \times \\ &\quad \times (I_{\mathcal{H}_\pi} \otimes \Phi^* \sigma(X)))\mathbf{u}_0 \end{aligned} \tag{37}$$

This computation leads us to define the joint transfer function (frequency response function) of the vessel:

$$\begin{aligned} &S(\pi) = (I_{\mathcal{H}_\pi} \otimes I_{\mathcal{E}} - \\ &\qquad - i\,(I_{\mathcal{H}_\pi} \otimes \Phi)\,(i\pi(X) \otimes I_{\mathcal{H}} + I_{\mathcal{H}_\pi} \otimes \rho(X))^{-1} \times \\ &\qquad \times (I_{\mathcal{H}_\pi} \otimes \Phi^* \sigma(X)))|_{\boldsymbol{\mathcal{E}}(\pi)} \end{aligned} \tag{38}$$

The domain of S is a subset of $\widehat{\mathfrak{G}}$ (where the resolvent exists), its values are operators from $\boldsymbol{\mathcal{E}}(\pi)$. For many cases of Lie algebras S in the definition above is independent of the choice of X. In other words $S(\pi)$ acts on operators from $\mathcal{H}_\pi$ to $\mathcal{H}_\pi \otimes \mathcal{E}$, namely $\mathbf{u}$ via composition to obtain $\mathbf{y}$. Furthermore by the definition above the range of $S(\pi)$ in fact lies in $\boldsymbol{\mathcal{E}}(\pi)$. Since the input functions must satisfy the input-compatibility conditions, to be eligible for our computations. Recalling the commutative case, one would expect that the range of $S(\pi)$ will lie in the "‘output bundle’", $\boldsymbol{\mathcal{E}}_*(\pi)$ and in fact it is so in our case. There exist an analog of the commutative intertwining relation in our case.

If we use the basis notation, we can write down an expression for S in the basis dependent form:

$$\begin{aligned} S(\pi) = &(I_{\mathcal{H}_\pi} \otimes I_{\mathcal{E}} + i\,(I_{\mathcal{H}_\pi} \otimes \Phi) \times \\ &\times \left(\sum_{p=1}^{l} (\alpha_p \Lambda_p \otimes I_{\mathcal{H}} - \alpha_p I_{\mathcal{H}_\pi} \otimes A_p) \right)^{-1} \times \\ &\times (I_{\mathcal{H}_\pi} \otimes \Phi^*) \left(\sum_{q=1}^{l} \alpha_q I_{\mathcal{H}_\pi} \otimes \sigma_q \right))|_{\boldsymbol{\mathcal{E}}(\pi)} \end{aligned} \tag{39}$$

We now define the complete characteristic function of the vessel:

$$W(X, z) = I_{\mathcal{E}} - i\Phi\,(izI_{\mathcal{H}} + \rho(X))^{-1}\, \Phi^* \sigma(X) \tag{40}$$

Therefore $S(\pi) = W(X, \pi(X))|_{\boldsymbol{\mathcal{E}}(\pi)}$.

The complete characteristic function takes values in $\mathcal{L}(\mathcal{E})$. Hence $W(X, \pi(X))$ takes values in $\mathcal{L}(\mathcal{H}_\pi \otimes \mathcal{E})$. It acts on elements of $\mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\pi \otimes \mathcal{E})$ via composition. So the restriction above is to be understood as the restriction of this action.

In the basis-dependent representation, the complete characteristic function takes the following form:

$$\begin{aligned} W(\alpha_1, \ldots, \alpha_l, z) = &I_{\mathcal{E}} + \\ &+ i\Phi \left(zI_{\mathcal{H}} - \sum_{p=1}^{l} \alpha_p A_p \right)^{-1} \Phi^* (\sum_{q=1}^{l} \times \\ &\times \alpha_q \sigma_q) \end{aligned} \tag{41}$$

Hence we obtain that:

$$S(\pi) = W(\alpha_1, \ldots, \alpha_l, \alpha_1 \Lambda_1 + \ldots \alpha_l \Lambda_l)|_{\boldsymbol{\mathcal{E}}(\pi)} \tag{42}$$

The restriction is to be understood as above.

IV. One-Parameter Subgroups

Now we come to consider one parameter subgroups of our group $\mathfrak{G}$, which we assume is solvable. A one parameter subgroup is by definition a Lie group homomorphism $\phi\colon \mathbb{R} \to \mathfrak{G}$. We will restrict our system to those subgroups. To do this we consider all of our functions restricted to the image $\phi(\mathbb{R})$ with the Lie algebra $d\phi(\mathbb{R})$. Since the Lie algebra is generated by a single element $\frac{d}{dt}$ Hence we obtain that the image is generated by an element of the form $d\phi(\frac{d}{dt})$. We denote this element by X_ϕ. In fact the one parameter subgroup is given via:

$$\phi(t) = \exp(tX_\phi) \tag{43}$$

Where $exp\colon \mathfrak{g} \to \mathfrak{G}$ is the canonical exponential map. Restricting ρ and σ to this Lie algebra we denote $A_\phi = \rho(X_\phi)$ and $\sigma_\phi = \sigma(X_\phi)$. Since the algebra is one dimensional the restriction of γ and γ_* are in fact the zero functions. Therefore we obtain that the vessel conditions collapse into the following condition:

$$\frac{1}{i}(A_\phi - A_\phi^*) = \Phi^* \sigma_\phi \Phi \tag{44}$$

This condition is exactly the condition [9, 1-1] of a regular one dimensional operator vessel. Now for the system equations, plugging in the above definitions we obtain that the restricted system equations are:

$$\begin{aligned} &iX_\phi x \circ \phi(t) + A_\phi x \circ \phi(t) = \Phi^* \sigma_\phi u \circ \phi(t) \\ &y \circ \phi(t) = u \circ \phi(t) - i\Phi x \circ \phi(t) \end{aligned} \tag{45}$$

This system is exactly the restriction of our system to the one-parameter subgroup. In fact we can consider this system of equations as the usual operator colligation on $\mathbb{R}$, see for example [9, 1-6,7].

Having obtained this system, we look at it's characteristic function. The vessels characteristic function is given by ([9, 2-1]):

$$S(\lambda) = I_{\mathcal{E}} - i\Phi(A_\phi - \lambda I_{\mathcal{H}})^{-1}\Phi^*\sigma_\phi \tag{46}$$

On the other hand we can consider the restriction of any Lie group representation of $\mathfrak{G}$ to our one parameter subgroup. In general such a restriction will be highly reducible since the subgroup is abelian, and therefore by Schur's lemma admits only one-dimensional irreducible representations. Now by [4, Thm. 7.36] $\pi \circ \phi$ admits a decomposition as a direct integral of irreducible representations of $\mathbb{R}$:

$$\pi \circ \phi = \int_{\mathbb{R}}^{\oplus} \pi_\lambda d\nu(\lambda) \tag{47}$$

Here each π_λ is a multiplicative functional on $\mathbb{R}$ and the measure ν is a scalar measure corresponding to the direct integral decomposition. We now compute $\pi_\lambda(X_\phi)$. We can write $\pi_\lambda(h) = e^{i\lambda\phi^{-1}(h)}$. Hence if we differentiate π_λ then we will obtain:

$$d\pi_\lambda = i\lambda d\phi^{-1} \tag{48}$$

Hence $\pi_\lambda(X_\phi)s = d\pi_\lambda(X_\phi)s = i\lambda s$ for an arbitrary $s \in \mathbb{R}$.

Now since the formula of $S(\pi)$ is independent of the choice $X \in \mathfrak{g}$, we can write down:

$$\begin{aligned} S(\pi) = &I_{\mathcal{H}_\pi} \otimes I_{\mathcal{E}} - \\ &- i(I_{\mathcal{H}_\pi} \otimes \Phi)(i\pi(X_\phi \otimes I_{\mathcal{H}} + I_{\mathcal{H}_\pi} \otimes A_\phi)^{-1}) \times \\ &\times (I_{\mathcal{H}_\pi} \otimes \Phi^*\sigma_\phi) \end{aligned} \tag{49}$$

We plug in the result (47) and the formula for $d\pi_\lambda(X_\phi)$ and use [3, Prop. II.2.3] and [3, Prop. II.2.8] to obtain:

$$\begin{aligned} S(\pi) = &\int_{\mathbb{R}}^{\oplus} I_{\mathcal{E}} \otimes I_\lambda d\nu(\lambda) - \\ &- i(\int_{\mathbb{R}}^{\oplus} \Phi \otimes I_\lambda d\nu(\lambda))(\int_{\mathbb{R}}^{\oplus} A_\phi \otimes I_\lambda d\nu(\lambda))^{-1} \times \\ &\times (\int_{\mathbb{R}}^{\oplus} \Phi^*\sigma_\phi \otimes I_\lambda d\nu(\lambda) - \int_{\mathbb{R}}^{\oplus} \lambda I_{\mathcal{H}} \otimes I_\lambda d\nu(\lambda)) = \\ &= \int_{\mathbb{R}}^{\oplus} (I_{\mathcal{E}} - i\Phi(A_\phi - \lambda I_{\mathcal{H}})^{-1}\Phi^*\sigma_\phi) \otimes I_\lambda d\nu(\lambda) = \\ &= \int_{\mathbb{R}}^{\oplus} S(\lambda) \otimes I_\lambda d\nu(\lambda) \end{aligned} \tag{50}$$

In fact we have proved that:

Theorem 4.1: For every one parameter subgroup of $\mathfrak{G}$ we have that:

$$S(\pi) = \left(\int_{\mathbb{R}}^{\oplus} S(\lambda) \otimes I_\lambda d\nu(\lambda)\right)|_{\boldsymbol{\mathcal{E}}(\pi)} \tag{51}$$

or in other words:

$$S(\pi) = \left(\int_{-\infty}^{\infty} dE_(\lambda) \otimes S(\lambda)\right)|_{\boldsymbol{\mathcal{E}}(\pi)} \tag{52}$$

Where dE_λ is a $\mathcal{H}_\pi$-projection valued measure.

Proof: The first equation was proved in the discussion above. Now we prove the second form. Note that in fact $\pi(\exp(tX_\phi))$ is a strongly continuous one-parameter group of unitary operators, by Stone's theorem (see for example [1, Eq. 73.1]) we have that, there exists an $\mathcal{H}_\pi$-projection valued measure on $\mathbb{R}$ such that:

$$\pi(\exp(tX_\phi)) = \int_{-\infty}^{\infty} e^{i\lambda t} dE_\lambda \tag{53}$$

Now furthermore by Stone's theorem there exists an infinitesimal self-adjoint generator A for this group. In other words:

$$\pi(\exp(tX_\phi)) = e^{itA} \tag{54}$$

Now by [10, Rem. 3.36] we have that:

$$\begin{aligned} \pi(X_\phi) = \frac{d}{dt}|_{t=0}(\pi(\exp(tX_\phi))) = \frac{d}{dt}|_{t=0} \int_{-\infty}^{\infty} e^{i\lambda t} dE_\lambda = \\ = \int_{-\infty}^{\infty} i\lambda dE_\lambda \end{aligned} \tag{55}$$

Now we plug it into the equation for $S(\pi)$ to obtain the desired result. ■

This proposition in fact gives a representation of the joint transfer function of a vessel in terms of the "diagonalization"' of the operator $\pi(X)$ for a fixed $X \in \mathfrak{g}$.

V. $ax + b$-GROUP EXAMPLE

The $ax+b$ group is one of the simplest Lie groups. It is the lowest dimensional non-commutative Lie group. The $ax + b$ group is the group of affine direction preserving transformations of the real line $\mathbb{R}$, i.e., the transformations:

$$(a, b) : \mathbb{R} \to \mathbb{R}x \mapsto ax + b, a > 0 \tag{56}$$

We denote the $ax + b$ group by G.

The unitary dual of the $ax + b$ group consists of a family of one-dimensional representation:

$$\pi_t^0(a, b) = a^{it}, t \in \mathbb{R} \tag{57}$$

and two infinite dimensional representations on $\mathcal{H}_+ = L^2(\mathbb{R}_+)$ and $\mathcal{H}_- = L^2(\mathbb{R}_-)$

$$\pi_+(a, b)f(t) = \sqrt{a}e^{2\pi ibt}f(at), f(t) \in L^2(\mathbb{R}_+, dt/t) \tag{58}$$

$$\pi_-(a, b)f(t) = \sqrt{a}e^{2\pi ibt}f(at), f(t) \in L^2(\mathbb{R}_-, dt/|t|) \tag{59}$$

The exact construction of the unitary dual can be found in Example 1 of [4, Sec. 6.7]. The same example states that in fact the Plancherel measure of the family of one-dimensional representation is 0. Hence the Plancherel measure on the unitary dual is in fact a counting measure on the two element set $\{\pi_+, \pi_-\}$.

We will restrict our attention to system-trajectories of the form $(\pi_+(a,b) \otimes I_{\mathcal{E}})\mathbf{u}_0$, where $\mathbf{u}_0 \in \mathrm{TC}(L^2(\mathbb{R}_+), L^2(\mathbb{R}_+) \otimes \mathcal{E})$. We can consider such a trajectory as an operator-valued function on G that for each $(a,b) \in G$ gives us an operator from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+) \otimes \mathcal{E}$. Since we are considering only trace-class operators, we recall that to every trace-class operator T on $L^2(\mathbb{R}_+)$, there exists a kernel function $K \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ such that:

$$[Tf](t) = \int_0^\infty K(t,s) f(s) \frac{ds}{s} \tag{60}$$

See for example [4, Thm. 7.16] or [7, Thm. IV.23]. We can thus identify $\mathbf{u}$ with an integral operator on $L^2(\mathbb{R}_+)$ with an $\mathcal{E}$-valued kernel. in other words:

$$\mathbf{u} f(t) = \int_0^\infty K_{\mathbf{u}}(t,s) f(s) \frac{ds}{s} \tag{61}$$

Here $K_{\mathbf{u}}(t,s) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{E})$. Now we compute the action of the Lie algebra of G on π_+. We recall that $X_1 = a\frac{\partial}{\partial a}$ and $X_2 = a\frac{\partial}{\partial b}$ are the basis for the Lie algebra of our group.

Using the above representation we compute that the input and output compatibility conditions translate to compatibility conditions for the kernels $K_{\mathbf{u}}$ at the input and $K_{\mathbf{y}}$ at the output:

$$\begin{aligned} &\frac{1}{2}\sigma_2 K_{\mathbf{u}}(t,s) + t\sigma_2 \frac{\partial}{\partial t} K_{\mathbf{u}}(t,s) - \\ &- 2\pi i t \sigma_1 K_{\mathbf{u}}(t,s) + i\gamma K_{\mathbf{u}}(t,s) = 0 \end{aligned} \tag{62}$$

$$\begin{aligned} &\frac{1}{2}\sigma_2 K_{\mathbf{y}}(t,s) + t\sigma_2 \frac{\partial}{\partial t} K_{\mathbf{y}}(t,s) - \\ &- 2\pi i t \sigma_1 K_{\mathbf{y}}(t,s) + i\gamma_* K_{\mathbf{y}}(t,s) = 0 \end{aligned} \tag{63}$$

Similarly the system equations translate to equations for the kernels of the respective operators:

$$\frac{i}{2} K_{\mathbf{x}}(t,s) + it\frac{\partial}{\partial t} K_{\mathbf{x}}(t,s) + A_1 K_{\mathbf{x}}(t,s) = \Phi^* \sigma_1 K_{\mathbf{u}}(t,s) \tag{64}$$

$$A_2 K_{\mathbf{x}}(t,s) - 2\pi t K_{\mathbf{x}}(t,s) = \Phi^* \sigma_2 K_{\mathbf{u}}(t,s) \tag{65}$$

$$K_{\mathbf{y}}(t,s) = K_{\mathbf{u}}(t,s) - i\Phi K_{\mathbf{x}}(t,s) \tag{66}$$

The above equations are true for every $s \in \mathbb{R}_+$. Now we restrict our attention to the one parameter subgroup given by $\mathfrak{H} = \{\exp(rX_2) | r \in \mathbb{R}\}$. We consider the family of unitary operators $\pi(\exp(tX_1))$. Since π_+ and π_- are both sub representations of π (a representation of G on $L^2(\mathbb{R} \setminus \{0\}, dt/|t|)$).

$$\pi(\exp(rX_2))g(s) = e^{2\pi i r s} g(s) \tag{67}$$

This operator admits a fairly simple spectral analysis:

$$\pi(\exp(rX_2)) = \int_{-\infty}^{\infty} e^{ir\lambda} dE_\lambda \tag{68}$$

Here E is an $L^2(\mathbb{R})$-projection valued measure. Hence in particular:

$$\pi(X_2) = \int_{-\infty}^{\infty} i\lambda dE_\lambda \tag{69}$$

Finally we obtain that:

$$S(\pi) = \int_{-\infty}^{\infty} (I_{\mathcal{E}} - i\Phi(A_2 - \lambda I_{\mathcal{H}})^{-1}\Phi^*\sigma_2) \otimes dE_\lambda \tag{70}$$

Now we note that in fact if we recall that $\frac{1}{i}\int_{-\infty}^{\infty} \lambda dE_\lambda$, we can use some spectral functional calculus to obtain that:

$$S(\pi) K_{\mathbf{u}}(t,s) = \left(I_{\mathcal{E}} - i\Phi(A_2 - 2\pi t I_{\mathcal{H}})^{-1}\Phi^*\sigma_2\right) K_{\mathbf{u}}(t,s) \tag{71}$$

This follows from the fact that $\pi(X_2)g(t) = 2\pi t g(t)$. The kernel $K_{\mathbf{y}}$ is obtained by applying the right hand side to $K_{\mathbf{u}}$.

VI. Conclusions and Future Works

In this short paper we have re-introduced the notion of non-commutative overdetermined system. We have shown that in fact the transfer function of such systems can be reconstructed from one-dimensional data, by restricting the system to a one-parameter subgroup. This technique can be employed to study systems not only on the $ax+b$-group, as in the example, but for groups interesting for physics, such as the Heisenberg groups. On the other hand fully developing the theory of Lie algebra operator vessels is an interesting topic of research in the purely mathematical context of operator theory.

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