

## דבר 1

1.1. נניח שיש לנו שני וקטורים  $u, v$  במרחב  $V$ .  
נניח  $V$  הוא מרחב סקלרי מעל  $\mathbb{R}$ .  
נניח  $\langle u, v \rangle = \langle v, u \rangle$  לכל  $u, v \in V$ .

$$\frac{1}{4} \|u+v\|^2 - \frac{1}{4} \|u-v\|^2$$

$$= \frac{1}{4} \langle u+v, u+v \rangle - \frac{1}{4} \langle u-v, u-v \rangle$$

$$= \frac{1}{4} (\langle u, u \rangle + \langle v, u \rangle + \langle u, v \rangle + \langle v, v \rangle)$$

$$- \frac{1}{4} (\langle u, u \rangle - \langle v, u \rangle - \langle u, v \rangle + \langle v, v \rangle)$$

$$= \frac{1}{4} \cdot 4 \cdot \langle u, v \rangle = \langle u, v \rangle$$

כלומר

1.2. ~~נניח שיש לנו שני וקטורים  $u, v$  במרחב  $V$ .~~

$$\langle u, v \rangle = \frac{1}{4} \|u+v\|^2 - \frac{1}{4} \|u-v\|^2$$

נניח  $V$  הוא מרחב סקלרי מעל  $\mathbb{R}$ .

נניח  $\langle u, u \rangle = \|u\|^2$  לכל  $u \in V$ .

כלומר

$$\langle u, u \rangle = \frac{1}{4} \|u+u\|^2 - \frac{1}{4} \|u-u\|^2 = \frac{1}{4} \|2u\|^2 - \frac{1}{4} \|0\|^2$$

$$= \|u\|^2$$

$$u=0 \Leftrightarrow \|u\|=0 \Leftrightarrow \langle u, u \rangle = 0 \quad \text{or} \quad \langle u, u \rangle \geq 0 \quad \text{p.p.}$$

: p.p. נכונות     כן     = לכוון פל. p.p.

$$(1) \quad \langle u, v \rangle = \langle v, u \rangle$$

$$(2) \quad \langle \alpha u, v \rangle = \alpha \langle u, v \rangle$$

$$(3) \quad \langle u' + u'', v \rangle = \langle u', v \rangle + \langle u'', v \rangle$$

מכאן נגזר פ.ל.ו

: (1) R נכונות

$$\langle v, u \rangle = \frac{1}{4} \|u+v\|^2 - \frac{1}{4} \|v-u\|^2$$

$$= \frac{1}{4} \|u+v\|^2 - \frac{1}{4} \|u-v\|^2 = \langle u, v \rangle$$

: (3) R נכונות

~~$$\langle u' + u'', v \rangle = \frac{1}{4} \|u' + u'' + v\|^2 - \frac{1}{4} \|u' + u'' - v\|^2$$~~

$$\langle u', v \rangle + \langle u'', v \rangle = \frac{1}{4} \|u' + v\|^2 - \frac{1}{4} \|u' - v\|^2$$

$$+ \frac{1}{4} \|u'' + v\|^2 - \frac{1}{4} \|u'' - v\|^2$$

$$= \frac{1}{4} (\|u' + v\|^2 + \|u'' + v\|^2) - \frac{1}{4} (\|u' - v\|^2 + \|u'' - v\|^2)$$

אם נכונות פ.ל.ו

$$\frac{1}{2} (\|u' + u'' + 2v\|^2 + \|u' - u''\|^2) - \frac{1}{2} (\|u' + u'' - 2v\|^2 + \|u' - u''\|^2)$$

$$= \frac{1}{2} (\frac{1}{4} \|u' + u'' + 2v\|^2 - \frac{1}{4} \|u' + u'' - 2v\|^2)$$

$$= \frac{1}{2} \langle u' + u'', 2v \rangle$$

ע"ל ג' ע"פ עמ"ל  $u' = u, u'' = 0$  פ"ל של  $\mathcal{L}$

$\hookrightarrow$  ב"פ  $\langle 0, v \rangle = \frac{1}{4} \|0+v\|^2 - \frac{1}{4} \|0-v\|^2 = 0$  ז"ל פ"ל  $\mathcal{L}$

$$\langle u, v \rangle = \frac{1}{2} \langle u, 2v \rangle$$

$$\langle u, 2v \rangle = 2 \langle u, v \rangle \quad \text{ל"ל}$$

פ"ל  $\mathcal{L}$

$$\begin{aligned} \langle u' + u'', v \rangle &= \frac{1}{2} \langle u' + u'', 2v \rangle \\ &= \frac{1}{2} \cdot 2 \langle u' + u'', v \rangle \\ &= \langle u' + u'', v \rangle \end{aligned}$$

(3) - 2 ל"ל ע"ל

ע"פ ע"ל (2) ל"ל ע"ל  $n \in \mathbb{N}$

$\hookrightarrow n \in \mathbb{N}$  ב"פ

$$\langle nu, v \rangle = n \langle u, v \rangle$$

ע"פ ע"ל  $n=1$  - ע"פ ע"ל  $n=0$

ע"פ ע"ל  $\langle nu, v \rangle = n \langle u, v \rangle$

$$\begin{aligned} \langle (n+1)u, v \rangle &= \langle nu + u, v \rangle = \langle nu, v \rangle + \langle u, v \rangle \\ &\stackrel{(3) \text{ ב"פ}}{=} n \langle u, v \rangle + \langle u, v \rangle = (n+1) \langle u, v \rangle \end{aligned}$$

פ"ל  $\langle 0, u \rangle = 0 \langle u, v \rangle$  ל"ל ע"ל  $\langle 0, v \rangle = 0$

$$\langle u, v \rangle + \langle -u, v \rangle = \langle u + (-u), v \rangle = \langle 0, v \rangle = 0$$

$$\Rightarrow \langle -u, v \rangle = -\langle u, v \rangle$$

$$n \in \mathbb{Z} \quad \text{BP} \quad \langle nu, v \rangle = n \langle u, v \rangle \quad \text{pM}$$

$$m \in \mathbb{N} \quad \text{BP, pM}$$

$$\langle mu, v \rangle = m \langle u, v \rangle$$

$\Rightarrow$

$$\langle m \cdot \frac{1}{m} u, v \rangle = m \langle \frac{1}{m} u, v \rangle$$

$$\Rightarrow \langle \frac{1}{m} u, v \rangle = \frac{1}{m} \langle u, v \rangle$$

$$r = \frac{m}{n} \in \mathbb{Q} \quad \text{BP pM}$$

$$\begin{aligned} \langle ru, v \rangle &= \langle \frac{n}{m} u, v \rangle = n \langle \frac{1}{m} u, v \rangle \\ &= \frac{n}{m} \langle u, v \rangle = r \langle u, v \rangle \end{aligned}$$

$\mathbb{R} \setminus \{r_n\}_{n=1}^{\infty}$   $\Rightarrow$   $\alpha \in \mathbb{R}$   $\Rightarrow$   $\{r_n\}$   $\rightarrow \alpha$   $\Rightarrow$   $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

$$\langle \alpha u, v \rangle = \frac{1}{4} \| \alpha u + v \|^2 - \frac{1}{4} \| \alpha u - v \|^2$$

$\Rightarrow$   $\lim_{n \rightarrow \infty} \langle r_n u, v \rangle = \langle \alpha u, v \rangle$

$$\begin{aligned} &\lim_{n \rightarrow \infty} \| r_n u \| \\ &= \lim_{n \rightarrow \infty} \| u \| \end{aligned}$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \| r_n u + v \|^2 - \frac{1}{4} \lim_{n \rightarrow \infty} \| r_n u - v \|^2$$

$$= \langle r_n u, v \rangle \lim_{n \rightarrow \infty} \langle r_n u, v \rangle$$

$$= \lim_{n \rightarrow \infty} r_n \langle u, v \rangle = \alpha \langle u, v \rangle$$

$\Rightarrow$   $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$   $\Rightarrow$   $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$



גרס

$$\langle f, f \rangle = \int_{-1}^1 |f(x)|^2 dx + \int_{-1}^1 |f'(x)|^2 dx \geq 0$$

$\int_{-1}^1 |f(x)|^2 dx = 0$     לכן  $f(x) = 0$      $\langle f, f \rangle = 0$      $f(x) = 0$

$f = 0$      $\forall x \in \mathbb{R}$      $f(x) = 0$     (לכן)  $f(x) = 0$      $\langle f, f \rangle = 0$

בסיס,  $\{1, x, x^2\}$     בסיס אורתוגונלי     $\langle f, g \rangle = 0$      $f, g \in \mathbb{R}_2[x]$

$\{1, x, x^2\}$     בסיס אורתוגונלי     $\langle f, g \rangle = 0$      $f, g \in \mathbb{R}_2[x]$

$$\langle 1, 1 \rangle = \int_{-1}^1 dx = 2 \Rightarrow p_1 = \frac{1}{\sqrt{2}}$$

$$p'_2 = x - \langle x, p_1 \rangle p_1$$

$$= x - \left( \int_{-1}^1 x \cdot \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}} = x$$

~~$\frac{1}{\sqrt{2}}$~~      $\frac{1}{\sqrt{2}}$      $\frac{1}{\sqrt{2}}$      $\frac{1}{\sqrt{2}}$   
 אורתוגונלי  
 (לכן)  $\langle f, g \rangle = 0$

$$\langle x, x \rangle = \int_{-1}^1 x^2 dx + \int_{-1}^1 dx = \frac{x^3}{3} \Big|_{-1}^1 + 2 = \frac{8}{3}$$

$$p_2 = \frac{p'_2}{\|p'_2\|} = \frac{x}{\sqrt{\frac{8}{3}}} = \sqrt{\frac{3}{8}} x$$

$$p'_3 = x^2 - \langle x^2, p_1 \rangle p_1 - \langle x^2, p_2 \rangle p_2$$

$$= x^2 - \underbrace{\left( \int_{-1}^1 x^2 \cdot \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}}}_{\frac{1}{2} \frac{x^3}{3} \Big|_{-1}^1 = \frac{1}{3}} - \underbrace{\left( \int_{-1}^1 x^2 \cdot \sqrt{\frac{3}{8}} x \right) \sqrt{\frac{3}{8}} x}_{\substack{\text{alle Potenzen} \\ \text{gerade Null}}}$$

$$= x^2 - \frac{1}{3}$$

$$\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle = \int_{-1}^1 (x^2 - \frac{1}{3})^2 dx + \int_{-1}^1 (2x)^2 dx$$

$$= \int_{-1}^1 \left( x^4 - \frac{2}{3}x^2 + \frac{1}{9} \right) dx + \int_{-1}^1 4x^2 dx$$

$$= \frac{x^5}{5} \Big|_{-1}^1 + \frac{10}{3} \frac{x^3}{3} \Big|_{-1}^1 + \frac{1}{9} \cdot 2$$

$$= \frac{2}{5} + \frac{20}{9} + \frac{2}{9} = \frac{18}{45} + \frac{110}{45} = \frac{128}{45}$$

$$p_3 = \frac{p_3'}{\|p_3'\|} = \sqrt{\frac{45}{128}} \left( x^2 - \frac{1}{3} \right)$$

:(in 6/7/8/9/10/11/12/13/14/15/16/17/18/19/20/21/22/23/24/25/26/27/28/29/30/31/32/33/34/35/36/37/38/39/40/41/42/43/44/45/46/47/48/49/50/51/52/53/54/55/56/57/58/59/60/61/62/63/64/65/66/67/68/69/70/71/72/73/74/75/76/77/78/79/80/81/82/83/84/85/86/87/88/89/90/91/92/93/94/95/96/97/98/99/100/101/102/103/104/105/106/107/108/109/110/111/112/113/114/115/116/117/118/119/120/121/122/123/124/125/126/127/128/129/130/131/132/133/134/135/136/137/138/139/140/141/142/143/144/145/146/147/148/149/150/151/152/153/154/155/156/157/158/159/160/161/162/163/164/165/166/167/168/169/170/171/172/173/174/175/176/177/178/179/180/181/182/183/184/185/186/187/188/189/190/191/192/193/194/195/196/197/198/199/200/201/202/203/204/205/206/207/208/209/210/211/212/213/214/215/216/217/218/219/220/221/222/223/224/225/226/227/228/229/230/231/232/233/234/235/236/237/238/239/240/241/242/243/244/245/246/247/248/249/250/251/252/253/254/255/256/257/258/259/260/261/262/263/264/265/266/267/268/269/270/271/272/273/274/275/276/277/278/279/280/281/282/283/284/285/286/287/288/289/290/291/292/293/294/295/296/297/298/299/300/301/302/303/304/305/306/307/308/309/310/311/312/313/314/315/316/317/318/319/320/321/322/323/324/325/326/327/328/329/330/331/332/333/334/335/336/337/338/339/340/341/342/343/344/345/346/347/348/349/350/351/352/353/354/355/356/357/358/359/360/361/362/363/364/365/366/367/368/369/370/371/372/373/374/375/376/377/378/379/380/381/382/383/384/385/386/387/388/389/390/391/392/393/394/395/396/397/398/399/400/401/402/403/404/405/406/407/408/409/410/411/412/413/414/415/416/417/418/419/420/421/422/423/424/425/426/427/428/429/430/431/432/433/434/435/436/437/438/439/440/441/442/443/444/445/446/447/448/449/450/451/452/453/454/455/456/457/458/459/460/461/462/463/464/465/466/467/468/469/470/471/472/473/474/475/476/477/478/479/480/481/482/483/484/485/486/487/488/489/490/491/492/493/494/495/496/497/498/499/500/501/502/503/504/505/506/507/508/509/510/511/512/513/514/515/516/517/518/519/520/521/522/523/524/525/526/527/528/529/530/531/532/533/534/535/536/537/538/539/540/541/542/543/544/545/546/547/548/549/550/551/552/553/554/555/556/557/558/559/560/561/562/563/564/565/566/567/568/569/570/571/572/573/574/575/576/577/578/579/580/581/582/583/584/585/586/587/588/589/590/591/592/593/594/595/596/597/598/599/600/601/602/603/604/605/606/607/608/609/610/611/612/613/614/615/616/617/618/619/620/621/622/623/624/625/626/627/628/629/630/631/632/633/634/635/636/637/638/639/640/641/642/643/644/645/646/647/648/649/650/651/652/653/654/655/656/657/658/659/660/661/662/663/664/665/666/667/668/669/670/671/672/673/674/675/676/677/678/679/680/681/682/683/684/685/686/687/688/689/690/691/692/693/694/695/696/697/698/699/700/701/702/703/704/705/706/707/708/709/710/711/712/713/714/715/716/717/718/719/720/721/722/723/724/725/726/727/728/729/730/731/732/733/734/735/736/737/738/739/740/741/742/743/744/745/746/747/748/749/750/751/752/753/754/755/756/757/758/759/760/761/762/763/764/765/766/767/768/769/770/771/772/773/774/775/776/777/778/779/780/781/782/783/784/785/786/787/788/789/790/791/792/793/794/795/796/797/798/799/800/801/802/803/804/805/806/807/808/809/810/811/812/813/814/815/816/817/818/819/820/821/822/823/824/825/826/827/828/829/830/831/832/833/834/835/836/837/838/839/840/841/842/843/844/845/846/847/848/849/850/851/852/853/854/855/856/857/858/859/860/861/862/863/864/865/866/867/868/869/870/871/872/873/874/875/876/877/878/879/880/881/882/883/884/885/886/887/888/889/890/891/892/893/894/895/896/897/898/899/900/901/902/903/904/905/906/907/908/909/910/911/912/913/914/915/916/917/918/919/920/921/922/923/924/925/926/927/928/929/930/931/932/933/934/935/936/937/938/939/940/941/942/943/944/945/946/947/948/949/950/951/952/953/954/955/956/957/958/959/960/961/962/963/964/965/966/967/968/969/970/971/972/973/974/975/976/977/978/979/980/981/982/983/984/985/986/987/988/989/990/991/992/993/994/995/996/997/998/999/1000)

$$\langle x^3, p_1 \rangle p_1 + \langle x^3, p_2 \rangle p_2 + \langle x^3, p_3 \rangle p_3$$

$$= \left( \int_{-1}^1 x^3 \cdot \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}} + \left( \int_{-1}^1 x^3 \sqrt{\frac{3}{8}} x dx + \int_{-1}^1 3x^2 \cdot \sqrt{\frac{3}{8}} dx \right)$$

$$+ \left( \int_{-1}^1 x^3 \frac{\sqrt{45}}{128} \left( x^2 - \frac{1}{3} \right) dx + \int_{-1}^1 3x^2 \frac{\sqrt{45}}{128} (2x) dx \right) \cdot \sqrt{\frac{3}{8}} x$$

$$\cdot \sqrt{\frac{45}{128}} \left(x^2 - \frac{1}{3}\right)$$

$$= \frac{3}{8} \left( \frac{x^5}{5} \Big|_{-1}^1 + 3 \frac{x^3}{3} \Big|_{-1}^1 \right) x$$

$$+ \frac{45}{128} f$$

$$= \frac{3}{8} \left( \frac{2}{5} + 2 \right) x = \frac{9}{10} x$$

$$\frac{3}{8} \cdot \frac{12}{5} = \frac{36}{40} = \frac{9}{10}$$

$\alpha = 0, \beta = \frac{9}{10}, \gamma = 0$       סדר הווקטור הנורמל

נניח  $A > 0$  בפרט נניח  $A = 3, 1$   
 $\mathcal{B} = C[-A, A]$

$$\langle f, g \rangle = \int_{-A}^A f(x) \overline{g(x)} e^{-x^2} dx \quad (\$)$$

נניח  $\mathcal{B}$  בפרט נניח  $\mathcal{B}$  הוא

$$\langle g, f \rangle = \int_{-A}^A g(x) \overline{f(x)} e^{-x^2} dx = \overline{\int_{-A}^A f(x) \overline{g(x)} e^{-x^2} dx} = \overline{\langle f, g \rangle}$$

$$\langle \alpha f, g \rangle = \int_{-A}^A \alpha f(x) \overline{g(x)} e^{-x^2} dx = \alpha \langle f, g \rangle$$

$$\langle f_1 + f_2, g \rangle = \int_{-A}^A (f_1(x) + f_2(x)) \overline{g(x)} e^{-x^2} dx = \langle f_1, g \rangle + \langle f_2, g \rangle$$

$$\langle f, f \rangle = \int_{-A}^A |f(x)|^2 e^{-x^2} dx \geq 0$$

אם  $x \in \mathbb{R}$  ו- $f(x) \neq 0$  אז  $|f(x)|^2 e^{-x^2} > 0$  ולכן  $\langle f, f \rangle > 0$

אם  $f(x) = 0$  לכל  $x$  אז  $\langle f, f \rangle = 0$

$$f(x) = 0 \Rightarrow f = 0$$

התוצאה היא ש- $V$  היא תת-חלל סגור במרחב הריבועי

הריבועי  $L^2(\mathbb{R}, e^{-x^2} dx)$  הוא תת-חלל סגור במרחב הריבועי

כלומר  $\langle f, g \rangle = 0$  לכל  $f, g \in V$

אם  $f \in V$  אז  $\langle f, f \rangle = 0$  ולכן  $f = 0$

כלומר

כלומר  $V$  הוא תת-חלל סגור במרחב הריבועי

אם  $f \in V$  אז  $\langle f, f \rangle = 0$  ולכן  $f = 0$

כלומר  $V$  הוא תת-חלל סגור במרחב הריבועי

$$f \in V, \alpha \in \mathbb{C}$$

$$\int_{-\infty}^{\infty} |\alpha f(x)|^2 e^{-x^2} dx = |\alpha|^2 \int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx < +\infty$$

$$f_1, f_2 \in V$$

אם  $A > 0$  אז

$$\left( \int_{-A}^A |f_1(x) + f_2(x)|^2 dx \right)^{1/2} \leq \left( \int_{-A}^A |f_1(x)|^2 dx \right)^{1/2} + \left( \int_{-A}^A |f_2(x)|^2 dx \right)^{1/2}$$

כלומר  $V$  הוא תת-חלל סגור במרחב הריבועי

$$\leq \left( \int_{-\infty}^{\infty} |f_1(x)|^2 dx \right)^{1/2} + \left( \int_{-\infty}^{\infty} |f_2(x)|^2 dx \right)^{1/2}$$

היות שלילית ופונקציה  $f, g \in V$   $\int_{-A}^A |f(x) + g(x)|^2 e^{-x^2} dx$   $\leq$   $\int_{-A}^A |f(x)|^2 e^{-x^2} dx + \int_{-A}^A |g(x)|^2 e^{-x^2} dx$

היות כי המכפלה הפנימית שלוקחים היא  $\int_{-A}^A f(x) \overline{g(x)} e^{-x^2} dx$ .  $\int_{-A}^A |f(x)|^2 e^{-x^2} dx$   $\leq$   $\int_{-A}^A |f(x) + g(x)|^2 e^{-x^2} dx$

$$\int_{-A}^A |f(x)| |\overline{g(x)}| e^{-x^2} dx \leq \left( \int_{-A}^A |f(x)|^2 e^{-x^2} dx \right)^{1/2} \left( \int_{-A}^A |g(x)|^2 e^{-x^2} dx \right)^{1/2}$$

היות כי  $f, g \in V$   $\int_{-A}^A |f(x)| |\overline{g(x)}| e^{-x^2} dx$   $\leq$   $\left( \int_{-A}^A |f(x)|^2 e^{-x^2} dx \right)^{1/2} \left( \int_{-A}^A |g(x)|^2 e^{-x^2} dx \right)^{1/2}$

היות כי  $f, g \in V$   $\int_{-A}^A |f(x)| |\overline{g(x)}| e^{-x^2} dx$   $\leq$   $\left( \int_{-A}^A |f(x)|^2 e^{-x^2} dx \right)^{1/2} \left( \int_{-A}^A |g(x)|^2 e^{-x^2} dx \right)^{1/2}$

3.2. היות כי  $V$  היא מרחב הילברט

נסמן  $x^n$   $\frac{x^n}{n}$   $\in V$   $\int_{-A}^A |x^n|^2 e^{-x^2} dx$

היות כי  $\lim_{n \rightarrow \infty} x^n e^{-|x|} = 0$

$$|x^n| < e^{M|x|} \Leftrightarrow |x^n e^{-|x|}| < 1$$

$$|x^{2n}| e^{-x^2} < e^{-x^2 + 2|x|}$$



$$= -\frac{1}{2} \left( \underbrace{x e^{-x^2}}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2} dx \right)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$p_2 = \frac{p_1'}{\|p_1'\|} = \frac{\sqrt{2}}{\sqrt{\pi}} x$$

$$p_3' = x^2 - \langle x^2, p_1 \rangle p_1 - \langle x^2, p_2 \rangle p_2$$

$$= x^2 - \left( \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{\pi}} e^{-x^2} dx \right) \frac{1}{\sqrt{\pi}} - \left( \int_{-\infty}^{\infty} x^2 \cdot \frac{\sqrt{2}}{\sqrt{\pi}} x dx \right) \frac{\sqrt{2}}{\sqrt{\pi}} x$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2}$$

$$= \frac{1}{2}$$

$$= x^2 - \frac{1}{2}$$

$$\langle x^2 - \frac{1}{2}, x^2 - \frac{1}{2} \rangle = \int_{-\infty}^{\infty} (x^2 - \frac{1}{2})^2 e^{-x^2} dx$$

$$= \int_{-\infty}^{\infty} (x^4 - x^2 + \frac{1}{4}) e^{-x^2} dx$$

$$\int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \int_{-\infty}^{\infty} x^3 \cdot -\frac{1}{2} d(e^{-x^2})$$

$$= -\frac{1}{2} \left( \underbrace{x^3 e^{-x^2}}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2} 3x^2 dx \right)$$

$$= \frac{3}{2} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{3}{2} \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\begin{aligned}
 \langle x^2 - \frac{1}{2}, x^2 - \frac{1}{2} \rangle &= \int_{-\infty}^{\infty} x^4 e^{-x^2} dx - \int_{-\infty}^{\infty} x^2 e^{-x^2} dx \\
 &\quad + \frac{1}{4} \int_{-\infty}^{\infty} e^{-x^2} dx \\
 &= \frac{3\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{4} \\
 &= \frac{2\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2}
 \end{aligned}$$

$$p_3 = \frac{p_3'}{\|p_3'\|} = \frac{\sqrt{2}}{\sqrt{\pi}} \left( x^2 - \frac{1}{2} \right)$$

∴  $\langle x^3, p_1 \rangle p_1 + \langle x^3, p_2 \rangle p_2 + \langle x^3, p_3 \rangle p_3$

$$\begin{aligned}
 &\langle x^3, p_1 \rangle p_1 + \langle x^3, p_2 \rangle p_2 + \langle x^3, p_3 \rangle p_3 \\
 &= \left( \int_{-\infty}^{\infty} x^3 \cdot \frac{1}{\sqrt{\pi}} e^{-x^2} dx \right) \frac{1}{\sqrt{\pi}} \\
 &\quad + \left( \int_{-\infty}^{\infty} x^3 \frac{\sqrt{2}}{\sqrt{\pi}} x e^{-x^2} dx \right) \frac{\sqrt{2}}{\sqrt{\pi}} x \\
 &\quad + \left( \int_{-\infty}^{\infty} x^3 \frac{\sqrt{2}}{\sqrt{\pi}} \left( x^2 - \frac{1}{2} \right) e^{-x^2} dx \right) \frac{\sqrt{2}}{\sqrt{\pi}} \left( x^2 - \frac{1}{2} \right) \\
 &= \frac{2}{\sqrt{\pi}} \left( \int_{-\infty}^{\infty} x^4 e^{-x^2} dx \right) x = \frac{2}{\sqrt{\pi}} \frac{3\sqrt{\pi}}{4} x \\
 &= \frac{3}{2} x
 \end{aligned}$$