EXERCISES ON DISTRIBUTIONS

Problem 1. Let g be a C^∞ function on $(-\infty, +\infty)$, and let ϕ be a distribution.

(1) Show that $\int_{-\infty} g(x)\phi(x)\, u(x)\, dx = \int_{-\infty}^{\infty} \phi(x)\, g(x)u(x)\, dx$, where u is a test function, defines a distribution. (In other words, we can define a product of a C^∞ function and a distribution; notice that a product of two distributions is in general not defined.)

(2) Show that $(g\phi)' = g'\phi + g\phi'$. (Of course, the derivatives of $g\phi$ and of ϕ are taken in the sense of distributions.)

(3) Show that $(g\phi)^{(k)} = \sum_{j=0}^{k} \dfrac{k!}{j!(k-j)!} g^{(j)}\phi^{(k-j)}$.

Problem 2. Let

$$f(x) = \begin{cases} x^2, & x < 1, \\ x^2 + 2x, & 1 \le x < 2, \\ 2x, & x \ge 2. \end{cases}$$

Find the distributional derivative f'_{dist} in two ways:

(1) using the general rule for finding the distributional derivative of a piecewise continuously differentiable function,

(2) writing $f(x) = x^2 H(2-x) + 2xH(x-1)$, where

$$H(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0, \end{cases}$$

is the Heaviside function (so that $H'_{\text{dist}} = \delta$), and using Problem 1.

Problem 3. (1) Show that

$$x^j \delta^{(k)}(x) = \begin{cases} (-1)^j k!/(k-j)!\delta^{(k-j)}(x), & j \le k, \\ 0, & j > k. \end{cases}$$

(2) Show that for any C^∞ function g on $(-\infty, \infty)$,

$$g(x)\delta^{(k)}(x) = \sum_{j=0}^{k} (-1)^j \frac{k!}{j!(k-j)!} g^{(j)}(0)\delta^{(k-j)}(x).$$

Problem 4. Define

$$f_k(x) = \begin{cases} k, & k^{-1} < x < 2k^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that

(1) $f_k(x) \longrightarrow_{k\to\infty} 0$ pointwise (for all x), but $f_k(x) \longrightarrow_{k\to\infty} \delta(x)$ in the sense of distrubitions (weakly).

(2) $f_k(x)^2 \longrightarrow_{k\to\infty} 0$ pointwise (for all x), but $f_k(x)^2$ does not converge in the sense of distrubitions (weakly).

Problem 5. Let $\{x_k\}_{k=1}^{\infty}$ be any sequence of real numbers with $\lim_{k\to\infty} |x_k| = \infty$. Show that $\lim_{k\to\infty} \delta(x - x_k) = 0$.

Problem 6. Show that

$$\frac{\delta(x+h)-\delta(x)}{h} \longrightarrow_{h\to 0} \delta'(x).$$

Problem 7. Show that $k^N \sin kx \longrightarrow_{k\to\infty} 0$ for any integer N.

Problem 8.

(1) Compute the real Fourier series of the square wave

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$$

(2) Continue f periodically to all of $(-\infty, \infty)$, and find its distributional derivative.

(3) Use the previous two items to show that

$$\sum_{m=1}^{\infty} \cos(2m-1)x = \frac{\pi}{2}\left(\sum_{k=-\infty}^{\infty} \delta(x-2k\pi) - \sum_{k=-\infty}^{\infty} \delta(x-(2k+1)\pi)\right).$$