EXERCISES ON FOURIER SERIES

Problem 1. Let $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the Fourier series of $f(x)$. Find the Fourier series of the following functions:

(1) $g(x) = f(x+\alpha)$;
(2) $h(x) = e^{imx} f(x)$ (m an integer).

Problem 2. Let $\sum_{n=-\infty}^{\infty} |\alpha_n| < \infty$, $\sum_{n=-\infty}^{\infty} |\beta_n| < \infty$, and let $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$, $g(x) = \sum_{n=-\infty}^{\infty} \beta_n e^{inx}$.

(1) Show that the series $\sum_{n=-\infty}^{\infty} \alpha_{m-n}\beta_n$ converges for every integer m, and that
$$\sum_{n=-\infty}^{\infty} |\gamma_n| \le \sum_{n=-\infty}^{\infty} |\alpha_n| \cdot \sum_{n=-\infty}^{\infty} |\beta_n|,$$
where $\gamma_m = \sum_{n=-\infty}^{\infty} \alpha_{m-n}\beta_n$.
(2) Let $h(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$. Show that $h(x) = f(x)g(x)$. Show also that
$$\|h\|_\infty \le \sum_{n=-\infty}^{\infty} |\alpha_n| \cdot \sum_{n=-\infty}^{\infty} |\beta_n|.$$

Problem 3. Use the Fourier series of the function $f(x) = \cos ax$ on the interval $[-\pi, \pi]$, where a is not an integer, to show that
$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right),$$
$$\cot a\pi = \frac{1}{a\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right).$$

Problem 4. Find the Fourier series of the function
$$f(x) = \begin{cases} 2 + \dfrac{x}{2\pi}, & -\pi \le x < 0 \\ 2, & 0 \le x \le \pi \end{cases}.$$
and draw the graph of the sum of the Fourier series on the interval $[-3\pi, 3\pi]$.

Problem 5. Find the Fourier series on the interval $[-l, l]$ for the function
$$f(x) = \begin{cases} 0, & -l \le x \le -b \\ 1, & -b < x < b, \\ 0, & b \le x \le l \end{cases}.$$
On which subintervals $[\alpha, \beta] \subseteq [-l, l]$ does the series converge uniformly?

Problem 6. Assume that $f(x)$ is $k-1$ times continuously differentiable on $[-\pi, \pi]$ with $f^{(j)}(-\pi) = f^{(j)}(\pi)$, $j = 0, \ldots, k-1$, and k times piecewise continuously differentiable. Show that the Fourier coefficients c_n of $f(x)$ satisfy $\lim_{n\to\infty} n^k c_n = 0$.

Problem 7. Find the Fourier series of the following functions:

(1) $f(x) = 9\cos(x) + 7\sin(2x) + 11\cos(3x)$, $x \in [-\pi, \pi]$.

(2) $f(x) = \begin{cases} \sin(x) & 0 < x \le \pi \\ \cos(x) & -\pi \le x \le 0 \end{cases}$.

(3) $f(x) = |x^3|$, $x \in [-\pi, \pi]$.

Problem 8. Find the complex Fourier series of the following functions:

(1) $f(x) = \sin x/2$, $x \in [-\pi, \pi]$.

(2) $f(x) = \pi - x^2$, $x \in [-\pi, \pi]$.

(3) $f(x) = \begin{cases} e^{ix} & 0 < x < \pi \\ e^{-ix} & -\pi \le x \le 0 \end{cases}$.

Problem 9. Find the Fourier series of the function $f(x) = \sin(px/2)$, $p \neq 0$, $x \in [-\pi, \pi]$, and use Parseval's identity to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(1-4n^2)^2} = \frac{\pi^2}{64}$.

Problem 10. Find the Fourier series of

$$f(x) = \begin{cases} h^2, & h \le x \le \pi \\ 0, & -\pi \le x \le h \end{cases}$$

($h \neq 0$), and use it to compute $\displaystyle\sum_{n=1}^{\infty} \frac{(1-(-1)^n \cos(2n))}{n^2}$.

Problem 11. (1) Show that for all $0 < r < 1$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1-r^2}{1-2r\cos x + r^2}.$$

(2) Let $P_r(x) = \dfrac{1-r^2}{1-2r\cos x + r^2}$ ($P_r(x)$ is called the Poisson kernel). Let $f(x)$ be a piecewise continuous function on $[-\pi, \pi]$ with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)P_r(t)\,dt = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx}$$

where the series converges absolutely and uniformly in x.

(3) Let $f(x)$ be a continuous periodic function on $(-\infty, \infty)$ with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. Show that

$$\lim_{r\to 1-} \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = f(x)$$

uniformly in x.

(Hint: show that $\left\{\dfrac{1}{2\pi}P_r(x)\right\}_{r\to 1-}$ is a periodic approximate identity.)

(Remark: this problem shows one way (there are many others) to "confront" the fact that the Fourier series of a continuous periodic function fails to converge — we "improve" the convergence by adding the factors $r^{|n|}$.)

Problem 12. (1) A finite sum of the form $\sum_{k=M}^{N} c_k e^{ikx}$ (where M and N are integers, $M < N$) is called a trigonometric polynomial. Let $f(x)$ be a continuous periodic function on $(-\infty, \infty)$, show that there is a sequence of trigonometric polynomials $\{p_n(x)\}_{n=1}^{\infty}$ that converges unifomly to $f(x)$ on $(-\infty, \infty)$. (Hint: use the previous problem.)

(2) Use the previous item to give a direct proof (without using Chebyshev polynomials) that the normalized trigonometric / exponential system in $L^2_{\text{pc}}[-\pi, \pi]$ is closed.

Problem 13. (1) Let $\{Q_n(t)\}_{n=1}^{\infty}$ be a periodic approximate identity, i.,e., a sequence of piecewise continuous functions on the interval $(-\infty, \infty)$ that are periodic with period 2π and that satisfy the following conditions:

(a) $Q_n(t) \geq 0$ for all t.

(b) For every δ, $0 < \delta < \pi$, $Q_n(t) \underset{n\to\infty}{\longrightarrow} 0$ uniformly in t on $[-\pi, -\delta] \cup [\delta, \pi]$.

(c) $\int_{-\pi}^{\pi} Q_n(t)\, dt = 1$.

Show that for any piecewise continuous function $f(x)$ on $(-\infty, +\infty)$ that is periodic with period 2π

$$\int_{-\pi}^{\pi} f(x-t) Q_n(t)\, dt \underset{n\to\infty}{\longrightarrow} \frac{f(x-0) + f(x+0)}{2}$$

for all x.

(2) Let $f(x)$ be a piecewise continuous periodic function on $(-\infty, \infty)$ with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. Show that

$$\lim_{r\to 1-} \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = \frac{f(x-0) + f(x+0)}{2}$$

for all x.