SOME KEY FACTS AND FORMULAE ABOUT ORTHOGONAL POLYNOMIALS

For ρ a strictly positive piecewise continuous function on (a, b) with $\int_a^b \rho(x)\,dx < \infty$ (improper Riemann integral), we consider the inner product

$$\langle f, g\rangle_\rho = \int_a^b f(x)\overline{g(x)}\rho(x)\,dx$$

on the space of piecewise continuous functions on $[a, b]$ (with the usual identification of two functions that agree except at finitely many points); we denote this space by $L^2_{\rho,\mathrm{pc}}[a, b]$. We also consider the cases of semiinfinite and infinite intervals $(-\infty, b]$, $[a, +\infty)$, $(-\infty, \infty)$; in these cases the space consists of piecewise continuous functions on the interval with

$$\int_{-\infty}^b |f(x)|^2\rho(x)\,dx < \infty,$$

correspondingly in the other cases; and we assume further that

$$\int_{-\infty}^b |x|^k\rho(x)\,dx < +\infty,$$

correspondingly in the other cases, so that all the polynomials belong to the space. [A function on an interval is called piecewise continuous if it is piecewise continuous on any finite closed subinterval.]

There always exists a sequence of orthogonal polynomials $\{\psi_n(x)\}_{n=0}^\infty$ that is uniquely determined up to scaling; it can be obtained by applying the Gramm-Schmidt orthogonalization process to the sequence of monomials $\{x^n\}_{n=0}^\infty$ [so in particular ψ_n have real coefficients].

Notice that if the interval is symmetric ($a = -b$) and the weight is even ($\rho(-x) = \rho(x)$) then even and odd functions are orthogonal with respect to $\langle\cdot,\cdot\rangle_\rho$, and therefore orthogonal polynomials are even for n even and odd for n odd, $\psi_n(-x) = (-1)^n\psi_n(x)$.

EXAMPLES:

(1) $\rho(x) = 1$ on $[-1, 1]$ (Legendre polynomials): using integration by parts,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

is orthogonal to x^m for $m < n$, so that it is a sequence of orthogonal polynomials. We can also calculate $\|P_n\|^2$ — using n integrations by parts it reduces to calculating $I_n = \int_{-1}^1 (1-x^2)^n\,dx$ for which one obtains (writing $(1-x^2)^n dx = (1-x^2)^{n-1}dx + x\frac{1}{2n}\,d(1-x^2)^{n-1}$ and integrating the second summand by parts) the equation $I_n = I_{n-1} - \frac{1}{2n}I_n$. When the dust settles, $\|P_n\|^2 = \frac{2}{2n+1}$. Leading coefficient: $\frac{(2n)!}{2^n(n!)^2}$. First several polynomials: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

(2) $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$ (Chebyshev polynomials of the 1st kind):

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n} (1-x^2)^{n-1/2}$$

is a polynomial of degree n, and it is orthogonal to x^m for $m < n$ and $\|T_n\|^2 = \pi/2$ for $n > 0$, $\|T_0\|^2 = \pi$ — the proofs are similar to (1). Leading coefficient: 2^{n-1}. First several polynomials: $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$.

The trigonometric formula:

$$T_n(\cos\theta) = \cos n\theta$$

[$\cos n\theta$ is a polynomial of degree n in $\cos\theta$ by de Moivre formula, the orthogonality follows from the othogonality of the cosines — and then the equation follows by uniqueness and noticing that the norms coincide and the leading coefficients are both positive].

(3) $\rho(x) = \sqrt{1-x^2}$ on $[-1, 1]$ (Chebyshev polynomials of the 2nd kind): similar to (2),

$$U_n(X) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!} \frac{1}{\sqrt{1-x^2}} \frac{d^n}{dx^n} (1-x^2)^{n+1/2},$$

$|U_n\|^2 = \pi/2$. Leading coefficient: 2^n. First several polynomials: $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$.

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

We have that $\frac{d}{dx} T_n(x) = n U_n(x)$.

(4) $\rho(x) = e^{-x^2}$ on $(-\infty, +\infty)$ (Hermite polynomials):

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n},$$

$\|H_n\|^2 = 2^n n! \sqrt{\pi}$ [details are again similar — integrations by parts — where the crucial thing now is that any polynomial times e^{-x^2} tends to 0 at $\pm\infty$; the calculation of the norm $\|H_n\|^2$ reduces in the end to calculation of the integral $\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$]. Leading coefficient: 2^n. First several polynomials: $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$.

In general normalized orthogonal polynomials satisfy a three term recurrence relation

$$x\psi_n(x) = b_n \psi_{n-1}(x) + a_n \psi_n(x) + b_{n+1} \psi_{n+1}(x),$$

where the coefficients are easy to find by comparing the leading coefficients.

EXAMPLES:

(1) $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$,
(2) $2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$,
(3) same for $U_n(x)$,
(4) $2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$.

Notice that the recurrence relation gives an easy way to calculate the polynomials recursively starting from the first two ($n = 0$ and $n = 1$); for a practical calculation of successive orthogonal polynomials this is usually the easiest method.

For a finite interval, normalized orthogonal polynomials form a closed orthonormal system in $L^2_{\rho,\mathrm{pc}}[a,b]$; this follows from the Weierstrass' Theorem on uniform polynomial approximation of continuous functions on a finite interval, since

- any piecewise continuous function can be approximated in $\|\cdot\|_{2,\rho}$ by continuous functions,
- on a finite interval, convergence in the supremum norm implies convegences in L^2_ρ.

[For an infinite interval, normalized orthogonal polynomials may or may not form a closed orthonormal system, depending on the weight; Hermite polynomials do — we will discuss it later on in the course.]

The fact that the normalized trigonometric system (real or complex) is closed follows immediately from the fact that $\{T_n\}$ is closed (this shows that the cosines are a closed system on $[0,\pi]$, hence for even functions on $[-\pi,\pi]$) and that $\{U_n\}$ is closed (this shows that the sines are a closed system on $[0,\pi]$, hence for odd functions on $[-\pi,\pi]$).