EXERCISES ON ORTHOGONAL POLYNOMIALS

Problem 1. (1) Prove that the Legendre polynomials satisfy the differential equation:

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}P_n(x)\right)+n(n+1)P_n(x)=0.$$

(2) Prove that all the roots of $P_n(x)$ are distinct and are contained in the interval $(-1,1)$. (Hint: notice that all the derivatives of $(1-x^2)^n$ up to order $n-1$ vanish at ± 1; now apply Rolle's Theorem successively to $(1-x^2)^n$ and its derivatives to see that $\frac{d^j}{dx^j}(1-x^2)^n$, $j \le n$, has j distinct zeroes in $(-1,1)$.)

(3) Find a polynomial P of degree at most 4 such that $\int_{-1}^{1}|x^5-P(x)|^2\,dx$ is minimal.

Problem 2. (1) Show that the Chebyshev polynomials of the first kind satisfy the differential equation:

$$(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0.$$

(2) Show that the Chebyshev polynomials of the second kind satisfy the differential equation:

$$(1-x^2)U_n''(x)-3xU_n'(x)+n(n+2)U_n(x)=0.$$

(3) Show that $T_n(T_m(x))=T_{nm}(x)$.

(4) Show that $T_n'(x)=nU_{n-1}(x)$.

(5) Find a polynomial P of degree at most 4 such that $\int_{-1}^{1}|x^5-P(x)|^2\,\frac{dx}{\sqrt{1-x^2}}$ is minimal.

(6) Find a polynomial P of degree at most 4 such that $\int_{-1}^{1}|x^5-P(x)|^2\,\sqrt{1-x^2}\,dx$ is minimal.

Problem 3. (1) Show that the Hermite poloynomials satisfy the differential equation:

$$\left(e^{-x^2/2}H_n'(x)\right)'+ne^{-x^2/2}H_n(x)=0.$$

(2) Prove that all the roots of $H_n(x)$ are distinct.

(3) Show that $H_n'(x)=2xH_n(x)-H_{n+1}(x)$.

(4) Find a polynomial P of degree at most 4 such that $\int_{-\infty}^{\infty}|x^5-P(x)|^2\,e^{-x^2/2}\,dx$ is minimal.

Problem 4. (1) Show that $L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right)$ is a polynomial of degree n with leading coefficient $1/n!$; these are called Laguerre polynomials.

(2) Show that the Laguerre polynomials are orthogonal polynomials with respect to the weight $\rho(x) = e^{-x}$ on $[0, +\infty)$, and that $\|L_n\|_{2,\rho} = 1$.
(3) Show that the Laguerre polynomials satisfy the three term recurrent relation:
$$xL_n(x) = -(n+1)L_{n+1}(x) + (2n+1)L_n(x) - nL_{n-1}(x).$$
(4) Show that the Laguerre polynomials satisfy the differential equation:
$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0.$$
(5) Show that all the roots of $L_n(x)$ are distinct and are contained in the interval $(0, +\infty)$.
(6) Find a polynomial P of degree at most 4 such that that $\displaystyle\int_0^\infty |x^5 - P(x)|^2\, e^{-x}\, dx$ is minimal.

Problem 5. (1) Show that the Chebyshev polynomials of the first kind are orthogonal in the space of piecewise continuous functions on the interval $[-1, 1]$ with the inner product
$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\frac{dx}{\sqrt{1-x^2}} + \int_{-1}^{1} f'(x)\overline{g'(x)}\,\sqrt{1-x^2}\,dx,$$
and calculate their norms with respect to this inner product. (Hint: use Problem 2.4.)
(2) Show that there does not exist a piecewise continuous function f on $[-1, 1]$ such that
$$\lim_{n\to\infty} n^{-\alpha}\left(\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} + n\int_{-1}^{1} f'(x)U_{n-1}\,\sqrt{1-x^2}\,dx\right) = c,$$
where $\alpha \le -1/2$ and c is a nonzero complex number. (Hint: use Bessel's inequality.)
(3) Find a polynomial P of degree at most 4 such that
$$\int_{-1}^{1} |x^5 - P(x)|^2\frac{dx}{\sqrt{1-x^2}} + \int_{-1}^{1} |5x^4 - P'(x)|^2\,\sqrt{1-x^2}\,dx$$
is minimal.