NOTES ON APPROXIMATE IDENTITIES

Theorem 1. *Let $\{Q_n(t)\}_{n=1}^{\infty}$ be a sequence of piecewise continuous functions on the interval $[-l,l]$ that satisfies the following conditions:*

(1) $Q_n(t) \geq 0$ *for all* $t \in [-l,l]$.
(2) *For every* δ, $0<\delta<l$, $Q_n(t) \underset{n\to\infty}{\longrightarrow} 0$ *uniformly in* t *on* $[-l,-\delta]\cup[\delta,l]$.
(3) $\int_{-l}^{l} Q_n(t)\,dt = 1$.

Then for any continuous function $f(x)$ *on* $(-\infty,+\infty)$,

$$\int_{-l}^{l} f(x-t)Q_n(t)\,dt \underset{n\to\infty}{\longrightarrow} f(x)$$

uniformly on every finite interval $[a,b]$.

Proof. By Property 3 we have that

$$f(x) = f(x)\cdot\int_{-l}^{l} Q_n(t)\,dt = \int_{-l}^{l} f(x)Q_n(t)\,dt.$$

Therefore

$$\begin{aligned}&\left|\int_{-l}^{l} f(x-t)Q_n(t)\,dt - f(x)\right| \\ &\qquad = \left|\int_{-l}^{l} f(x-t)Q_n(t)\,dt - \int_{-l}^{l} f(x)Q_n(t)\,dt\right| \\ &\qquad\qquad \leq \int_{-l}^{l} |f(x-t)-f(x)|Q_n(t)\,dt,\end{aligned}$$

where at the last stage we used Property 1 to get rid of the absolute value sign on $Q_n(t)$.

Since $f(x)$ is continuous on the closed interval $[a-l,b]$, it is bounded there and uniformly continuous. Denote $M=\max_{[a-l,b]}|f(x)|$. Given $\epsilon>0$, we find $\delta>0$ such that

$$|f(x)-f(y)| < \frac{\epsilon}{2} \tag{♡}$$

for all $x,y\in[a-l,b]$ with $|x-y|<\delta$. We then use Property 2 to find n_0 such that for all $n>n_0$ and all $t\in[-l,-\delta]\cup[\delta,l]$,

$$Q_n(t) < \frac{\epsilon}{8lM}. \tag{♠}$$

For $x\in[a,b]$ we can now write

$$\begin{aligned}&\left|\int_{-l}^{l} f(x-t)Q_n(t)\,dt - f(x)\right| \leq \int_{-l}^{l} |f(x-t)-f(x)|Q_n(t)\,dt \\ &\quad = \left(\int_{-l}^{-\delta} + \int_{\delta}^{l}\right)|f(x-t)-f(x)|Q_n(t)\,dt + \int_{-\delta}^{\delta}|f(x-t)-f(x)|Q_n(t)\,dt.\end{aligned}$$

In the first integral, we can bound $|f(x-t)-f(x)| \le |f(x-t)|+|f(x)|$ from above by $2M$ and use (♠) to bound $Q_n(t)$; in the second integral, we can use (♡) to bound $|f(x-t)-f(x)|$ (since $|(x-t)-x| = |t| < \delta$). Therefore

$$\left|\int_{-l}^{l} f(x-t)Q_n(t)\,dt - f(x)\right|$$
$$\le \left(\int_{-l}^{-\delta} + \int_{\delta}^{l}\right) 2M \cdot \frac{\epsilon}{8lM}\,dt + \int_{-\delta}^{\delta} \frac{\epsilon}{2} Q_n(t)\,dt.$$

Replacing both integrals by integrals over $[-l,l]$ will only increase the expression — for the second integral we use here again Property 1, so this is

$$\le \int_{-l}^{l} \frac{\epsilon}{4l}\,dt + \int_{-l}^{l} \frac{\epsilon}{2} Q_n(t)\,dt = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

where in the end we used again Property 3.

We have thus shown that for every $\epsilon > 0$ there exists n_0 such that for all $x \in [a,b]$ and all $n > n_0$,

$$\left|\int_{-l}^{l} f(x-t)Q_n(t)\,dt - f(x)\right| < \epsilon.$$

Hence

$$\int_{-l}^{l} f(x-t)Q_n(t)\,dt \underset{n\to\infty}{\longrightarrow} f(x)$$

uniformly in x on $[a,b]$. □

A sequence of functions $\{Q_n(t)\}$ satisfying Properties 1–3 or a variation thereof is called a (positive) approximate identity (or a Dirac sequence).

Property 2 can be replaced by: for every $\delta > 0$, $\left(\int_{-l}^{-\delta} + \int_{\delta}^{l}\right) Q_n(t)\,dt \underset{n\to\infty}{\longrightarrow} 0.$

Here are two variations of Theorem 1. The first one deals with periodic functions and the second one deals with functions vanishing outside of a finite interval.

Theorem 2. *Let $\{Q_n(t)\}_{n=1}^{\infty}$ be a sequence of piecewise continuous functions on the interval $(-\infty,\infty)$ that are periodic with period 2π and that satisfy the following conditions:*

(1) *$Q_n(t) \ge 0$ for all t.*
(2) *For every δ, $0 < \delta < \pi$, $Q_n(t) \underset{n\to\infty}{\longrightarrow} 0$ uniformly in t on $[-\pi,-\delta] \cup [\delta,\pi]$.*
(3) *$\int_{-\pi}^{\pi} Q_n(t)\,dt = 1$.*

Then for any continuous function $f(x)$ on $(-\infty,+\infty)$ that is periodic with period 2π

$$\int_{-\pi}^{\pi} f(x-t)Q_n(t)\,dt \underset{n\to\infty}{\longrightarrow} f(x)$$

uniformly on $(-\infty,\infty)$.

Theorem 3. *Let $\{Q_n(t)\}_{n=1}^{\infty}$ be a sequence of piecewise continuous functions on the interval $(-\infty,\infty)$ that satisfies the following conditions:*

(1) *$Q_n(t) \ge 0$ for all t.*
(2) *For every $\delta > 0$, $Q_n(t) \underset{n\to\infty}{\longrightarrow} 0$ uniformly in t on $(-\infty,-\delta] \cup [\delta,\infty)$.*

(3) $\int_{-\infty}^{\infty} Q_n(t)\,dt = 1$.

Then for any continuous function $f(x)$ *on* $(-\infty, +\infty)$ *that vanishes outside of a finite interval,*

$$\int_{-\infty}^{\infty} f(x-t)Q_n(t)\,dt \underset{n\to\infty}{\longrightarrow} f(x)$$

uniformly on $(-\infty, \infty)$*.*