PRACTICE HOMEWORK NO. 10: EIGENVALUES AND DIAGONALIZATION

Problem 1. Define $T\colon F_3[t] \to F_3[t]$ by

$$T(at^2 + bt + c) = (-4a - 6b - 2c)t^2 + (5a + 8b + 5c)t + (-4a - 7b - 6c).$$

Is T diagonalizable for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$? If yes, find a diagonalizing basis and the diagonal matrix representing T in this basis.

(Hint: If a rational number p/q, $p, q \in \mathbb{Z}$, is a root of a polynomial with integer coefficients, then p divides the constant term and q divides the coefficient of the highest power of t.)

Problem 2. Define $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by

$$T\left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right) = \begin{bmatrix} z & y \\ x & w \end{bmatrix}.$$

Show that T is diagonalizable and compute T^n $(n \in \mathbb{N})$ and $(I + T)^4$.

Problem 3. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Find the eigenvalues of A^4, $4A^3 - 2A + I$.

Problem 4.

(1) Let $A = \left[\begin{smallmatrix} 5 & -12 \\ 2 & -5 \end{smallmatrix}\right] \in \mathbb{C}^{2\times 2}$. Find all matrices $B \in \mathbb{C}^{2\times 2}$ so that $B^2 = A$.

(2) Let $A = \left[\begin{smallmatrix} -3 & 0 \\ -3 & 0 \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$. Find all matrices $B \in \mathbb{R}^{2\times 2}$ so that $-B^2 + 2B = A$.

Problem 5. Let F be a field containing the field $\mathbb{Q}$, and define $T\colon F^3 \to F^3$ by

$$T((x, y, z)) = (ax + ay + z, 2y + 2z, z)$$

$(a \in F)$. Find all the values of a so that T is not diagonalizable. Does the answer depend on the field F?

Problem 6.

(1) Let $A \in \mathbb{R}^{3\times 3}$ with the characteristic polynomial $f = t^3 - 2t^2 + 2$. Is A diagonalizable over $\mathbb{Q}$? over $\mathbb{R}$? over $\mathbb{C}$?

(2) Let $A \in \mathbb{R}^{2\times 2}$ with the characteristic polynomial $f = t^2 - 1$ and with eigenvector $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Compute $A^{10}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

Problem 7. Which among the following matrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

represent the same linear transformation on $\mathbb{R}^3$ with respect to different bases?

Problem 8. Let A be an upper triangular matrix having n distinct elements on the diagonal. Prove that A is diagonalizable.

Problem 9. Let $A \in F^{n\times n}$ be such that the sum of the elements of A in every row equals 1. Show that 1 is an eigenvalue of A.

Problem 10. Let $T\colon V \to V$ be a linear transformation on a vector space V of dimension n over a field F, and assume that the characteristic polynomial f of T splits over F into linear factors. Show that $\det T$ equals $(-1)^n$ times the product of the eigenvalues of T, where each eigenvalue is repeated as many times as its multiplicity.

Problem 11. We define the *trace* of a matrix $A \in F^{n\times n}$ to be the sum of its diagonal elements.

(1) Show that the trace of A is minus the coefficient of t^{n-1} in the characteristic polynomial of A.
(2) We define the trace of a linear transformation T on a finite dimensional vector space to be the trace of a matrix representing T with respect to some basis; show that the trace of T is well defined.
(3) Assume that the characteristic polynomial f of T splits over F into linear factors. Show that the trace of T equals the sum of the eigenvalues of T, where each eigenvalue is repeated as many times as its multiplicity.

Problem 12. Let $B \in F^{n\times n}$ be a diagonalizable matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_s$ of multiplicity $k_1, \dots, k_s$ respectively. Let $\mathcal{C} = \{A \in F^{n\times n} \colon BA = AB\}$. Show that $\mathcal{C}$ is a vector subspace of $F^{n\times n}$ of dimension $k_1^2 + \cdots + k_s^2$.

Problem 13. Let $T\colon V \to V$ be a linear transformation on a finite dimensional vector space V over F.

(1) Let v be an eigenvector of T belonging to the eigenvalue λ, and let $n \in \mathbb{N}$. Show that v is an eigenvector of T^n belonging to the eigenvalue λ^n.
(2) Let v be an eigenvector of T belonging to the eigenvalue λ, and let $g \in F[t]$. Show that v is an eigenvector of $g(T)$ belonging to the eigenvalue $g(\lambda)$.
(3) Assume that $p \in F[t]$ splits into linear factors over F and that $p(\lambda) \neq 0$ for each eigenvalue λ of T. Show that $p(T)$ is invertible.
(4) Assume that F is algebraically closed and let $g \in F[t]$. Show that the eigenvalues of $g(T)$ are of the form $g(\lambda)$ where λ is an eigenvalue of T.

Problem 14. Let $B \in F^{n\times n}$ and define $L_B\colon F^{n\times n} \to F^{n\times n}$ by $L_B(A) = BA$. Show that:

(1) λ is an eigenvalue of L_B if and only if λ is an eigenvalue of B.
(2) Let λ be an eigenvalue of B having k linearly independent eigenvectors; then λ is an eigenvalue of L_B having kn linearly independent eigenvectors.
(3) L_B is diagonalizable if and only if B is diagonalizable.