PRACTICE HOMEWORK NO. 9: DETERMINANTS

Problem 1. Compute the determinants of the following matrices:

$$\begin{bmatrix} 1 & 2 & -9 & 2 \\ 1/4 & 3/4 & 9/2 & 5/4 \\ -3 & 0 & 27 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \in \mathbb{Q}^{4\times 4}, \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{F}_2^{4\times 4},$$

$$\begin{bmatrix} 1 & i & -1 & i \\ 0 & 1 & 0 & i \\ 1+i & 2+i & 3+i & 4+i \\ i & 2i & 3i & 4i \end{bmatrix} \in \mathbb{C}^{4\times 4}, \qquad \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix} \in F^{4\times 4} \ (a \in F, F \text{ a field}).$$

Problem 2. Check that the following matrices over $\mathbb{R}$ are invertible by verifying that $\det A \neq 0$, and then compute A^{-1} using the classical adjoint formula $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$:

$$\begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \ (\theta \in \mathbb{R}).$$

Problem 3. Solve the following two systems of linear equations over $\mathbb{Q}$ using Cramer's rule:

$$x + y + z = 11,\ 2x - 6y - z = 0,\ 3x + 4y + 2z = 0;$$
$$3x - 2y = 4,\ 3y - 2z = 6,\ 3z - 2x = -1.$$

Problem 4.

(1) A matrix $A \in F^{n\times n}$ is called skew-symmetric (or anti-symmetric) if $A^\top = -A$. Show that if A is a skew-symmetric matrix of odd order n over a field F that does not contain the field $\mathbb{F}_2$ of two elements, then $\det A = 0$.
(2) Is the statement true for matrices of even order n? (Prove, or give a counterexample.)
(3) Is the stement true for matrices of odd order over the field $\mathbb{F}_2$? (Prove, or give a counterexample.)

Problem 5. A matrix $A \in F^{n\times n}$ is called orthogonal if $A^\top A = I$. Show that if A is an orthogonal matrix, then $\det A = \pm 1$. For every n, given an example of an orthogonal matrix $A \in \mathbb{R}^{n\times n}$ so that $\det A = -1$.

Problem 5. A matrix $A \in \mathbb{C}^{n\times n}$ is called unitary if $A^* A = I$ (where A^* denotes the conjugate transpose of a complex matrix). Show that if A is a unitary matrix, then $|\det A| = 1$.

Problem 6. Prove the following statements, or give a counterexample:

(1) If A and B are row equivalent matrices, then $\det A = \det B$.
(2) If A and B are row equivalent matrices and $\det A = 0$, then $\det B = 0$.
(3) If A and B are row equivalent matrices and $\det A \neq 0$, then $\det B \neq 0$.
(4) $\det(A + B) = \det A + \det B$.
(5) If a square matrix A satisfying $A^2 - A + I = 0$, then $\det A \neq 0$.

Problem 6. Let $B \in F^{n\times n}$ and define a function T_B on $F^{n\times n}$ by $T_B(A) = AB - BA$. Show that T_B is a linear transformation, and that $\det T_B = 0$.

Problem 7.

(1) Let $A \in F^{n\times n}$, $A \neq 0$. An $r \times r$ submatrix of A is any $r \times r$ matrix obtained by deleting $n - r$ rows and $n - r$ columns of A. The *determinant rank* of A is the largest integer r, $1 \leq r \leq n$, such that there is some $r \times r$ submatrix of A with a nonzero determinant. Show that the determinant rank of A equals the rank of A.
(2) Let $A \in F^{n\times n}$ be a matrix of rank $n - 1$. Show that $\operatorname{adj} A$ has rank 1, and that every nonzero column of $\operatorname{adj} A$ is a basis of $\ker A$.

Problem 8. Let $B \in F^{n\times n}$ and define functions L_B and R_B on $F^{n\times n}$ by $L_B(A) = BA$ and $R_B(A) = AB$. Show that L_B and R_B are linear transformations, and that $\det L_B = (\det B)^n$, $\det R_B = (\det B)^n$.

Problem 9. Denote by $V(\alpha_1, \alpha_2 \ldots, \alpha_n)$ ($\alpha_1, \alpha_2 \ldots, \alpha_n \in F$, F a field) the determinant of the so called *Vandermonde matrix*

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{bmatrix}.$$

(1) Show that $V(t, \alpha_2 \ldots, \alpha_n)$ is a polynomial of degree at most $n - 1$, with the coefficient of t^{n-1} equal to $V(\alpha_2 \ldots, \alpha_n)$, and with roots $\alpha_2 \ldots, \alpha_n$.
(2) Assume that $\alpha_2 \ldots, \alpha_n$ are distinct. Show that

$$V(t, \alpha_2 \ldots, \alpha_n) = V(\alpha_2 \ldots, \alpha_n)(t - \alpha_2) \cdots (t - \alpha_n).$$

(3) Show that

$$V(\alpha_1, \alpha_2 \ldots, \alpha_n) = (\alpha_2 - \alpha_1) \cdots (\alpha_2 - \alpha_n)$$
$$(\alpha_3 - \alpha_2) \cdots (\alpha_n - \alpha_2) \cdots (\alpha_n - \alpha_{n-1})$$

(that is, for each $i = 1, 2, \ldots, n-1$, we take a product $(\alpha_{i+1} - \alpha_i) \cdots (\alpha_n - \alpha_i)$, and then multiply all of these together). (**Hint**: prove the statement by induction on n using the previous item.).