Problem 12, Practice Assignment No. 5

Consider the matrix equation

$$x_1 A + x_2 A^2 + \cdots + x_{2014} A^{2014} = 0_{2\times 2} \tag{1}$$

with the (scalar) unknowns $x_1, x_2, \ldots, x_{2014}$. Since a 2×2 matrix has 4 entries, this is in fact a linear system of 4 equations in 2014 unknowns; furthermore, this is a homogeneous linear system. Since $2014 > 4$, it follows that this system has a non-trivial solution, i.e., there is a solution $x_1, x_2, \ldots, x_{2014}$ of (1) with not all the xs equal to zero.

Let j, $1 \le j \le 2014$, be the largest index so that $x_j \neq 0$. We then have

$$x_1 A + x_2 A^2 + \cdots + x_{j-1} A^{j-1} + x_j A^j = 0_{2\times 2},$$

and

$$A^j = -\frac{x_1}{x_j} A - \frac{x_2}{x_j} A^2 - \cdots - \frac{x_{j-1}}{x_j} A^{j-1}.$$

We can multiply both sides by the matrix A^{2014-j}; we obtain

$$A^{2014} = -\frac{x_1}{x_j} A^{1+2014-j} - \frac{x_2}{x_j} A^{2+2014-j} - \cdots - \frac{x_{j-1}}{x_j} A^{2013},$$

meaning that A^{2014} belongs to the linear span of $A, A^2, \ldots, A^{2013}$, as required.