Define a measure space.

State the Lebesgue-Radon-Nykodim Theorem.

What is the space $L^p(X,\mathcal{F},\mu)$?
Let $\mathcal{E}$ be a collection of subsets of $X$. Let $\mathcal{F}$ be the union of all $\sigma$-algebras generated by some sequence in $\mathcal{E}$; that is,

$$\mathcal{F} = \bigcup \{ \sigma((A_n)_{n=1}^{\infty}) : \forall n, A_n \in \mathcal{E} \}.$$

Show that $\mathcal{F} = \sigma(\mathcal{E})$. (Hint: is $\mathcal{F}$ a $\sigma$-algebra?)

Solution.

First note that if $A \in \mathcal{F}$, then there exists a sequence $(A_n)_n$ in $\mathcal{E}$ such that $A \in \sigma((A_n)_n) \subset \sigma(\mathcal{E})$. Thus, $\mathcal{F} \subset \sigma(\mathcal{E})$.

So it suffices to show that $\mathcal{F}$ is a $\sigma$-algebra.

$\emptyset \in \mathcal{F}$ is immediate.

If $A \in \mathcal{F}$ then there exists a sequence $(A_n)_n$ in $\mathcal{E}$ such that $A \in \sigma((A_n)_n)$. Thus, $A^c \in \sigma((A_n)_n) \subset \mathcal{F}$. So $\mathcal{F}$ is closed under complements.

Let $(A_n)_n$ be a sequence in $\mathcal{F}$. So for every $n$ there is a sequence $(E^1_k)_k$ in $\mathcal{E}$ such that $A_n \in \sigma((E^1_k)_k)$. Consider the family $(E^1_k)_{n,k}$. This is a sequence in $\mathcal{E}$. Let $\mathcal{G} =$
\[ \sigma((E^n_k)_{n,k}). \] Then by definition, \( G \subset F \). Also, for every \( n \), \( A_n \in \sigma((E^n_k)_k) \subset G \). Since \( G \) is a \( \sigma \)-algebra, we have that \( \bigcup_n A_n \in G \subset F \). Since this holds for any sequence \((A_n)_n\) in \( F \) we have show that \( F \) is closed under countable unions, and so \( F \) is a \( \sigma \)-algebra.
Let \((X, \mathcal{F}, \mu)\) be a measure space and let \(f \geq 0\) be a non-negative measurable function such that \(\int_X f \, d\mu < \infty\).

(a) \(\{12\}\)

Show that \(x : f(x) > 0\) is \(\sigma\)-finite.

(b) \(\{13\}\)

Show that for every \(\varepsilon > 0\) there exists \(A \in \mathcal{F}\) such that \(\mu(A) < \infty\) and

\[
\int_A f \, d\mu > \int_X f \, d\mu - \varepsilon.
\]

Solution.

(a) For every \(n > 0\) let \(A_n = \{ f > n^{-1} \}\). So \(\{ f > 0 \} = \bigcup_n A_n\). And it suffices to show that \(\mu(A_n) < \infty\) for all \(n\).

Indeed, for any \(n\), if \(x \in A_n\) then \(nf(x) > 1\). So,

\[
\mu(A_n) \leq n \cdot \int_{A_n} f \, d\mu \leq n \cdot \int f \, d\mu < \infty.
\]
(b) Since by (a) the set \( \{ f > 0 \} \) is \( \sigma \)-finite, we can write \( \{ f > 0 \} = \bigcup_n A_n \) where \((A_n)_n\) are pairwise disjoint measurable sets with \( \mu(A_n) < \infty \) for all \( n \). Thus,

\[
\int_X f \, d\mu = \int_{\{f > 0\}} f \, d\mu = \sum_n \int_{A_n} f \, d\mu.
\]

Write \( a_n := f_{A_n} \, d\mu \) which is a sequence of non-negative numbers. Since the sum \( \sum_n a_n \) converges, for any \( \varepsilon > 0 \) we may find \( N \) large enough so that \( \sum_{n > N} a_n < \varepsilon \).

Set

\[
A := \bigcup_{n=1}^N A_n.
\]

So

\[
\mu(A) = \sum_{n=1}^N \mu(A_n) < \infty.
\]

Also,

\[
A^c \cap \{ f > 0 \} = \{ f > 0 \} \setminus A = \bigcup_{n > N} A_n,
\]

so

\[
\int_{A^c} f \, d\mu = \int_{A^c \cap \{ f > 0 \}} f \, d\mu = \sum_{n > N} \int_{A_n} f \, d\mu = \sum_{n > N} a_n < \varepsilon.
\]

Thus,

\[
\int_X f \, d\mu = \int_A f \, d\mu + \int_{A^c} f \, d\mu < \int_A f \, d\mu + \varepsilon.
\]

Let \((X, \mathcal{F}, \mu)\) be a measure space. Let \((f_n)_n\) be a sequence of non-negative measurable functions, and let \( f \) be a measurable function such that \((f_n)_n\) converges to \( f \) in measure.
Show that
\[ \int f d\mu \leq \liminf_{n \to \infty} \int f_n d\mu. \]

**Solution.**

Let \((f_n)_k\) be a subsequence such that
\[ \lim_{k \to \infty} \int f_n d\mu = \liminf_n \int f_n d\mu. \]

So we want to show that \( \int f d\mu \leq \lim_{k \to \infty} \int f_n d\mu \). Since \((f_n)_k\) is a subsequence, we have that for all \( \varepsilon > 0 \),
\[ \lim_{k \to \infty} \mu \{|f_n - f| > \varepsilon\} \leq \limsup_n \mu \{|f_n - f| > \varepsilon\} = 0. \]

So \((f_n)_k\) converges in measure to \( f \).

Let \( g_k := f_n \) for all \( k \), which converge in measure to \( f \). By a theorem in class we now have that there is a further subsequence \((g_k)_j\) such that \( \lim_{j \to \infty} g_k = f \) a.e. Since these are all non-negative functions, Fatou’s Lemma tells us that
\[ \int f d\mu \leq \liminf_j \int g_k d\mu. \]

However, the sequence \((\int g_k d\mu)_j\) is a subsequence of the converging sequence \((\int f_n d\mu)_k\) which converges to \( \liminf_n \int f_n d\mu \). So the limit is
\[ \int f d\mu \leq \liminf_j \int g_k d\mu \leq \lim_{j \to \infty} \int g_k d\mu = \lim_{k \to \infty} \int f_n d\mu = \liminf_n \int f_n d\mu. \]
Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Let $\nu = \mu|_{\mathcal{G}}$.

(a) \quad 15 \quad Suppose that $f \in L^1(X, \mathcal{F}, \mu)$. Show that there exists $g \in L^1(X, \mathcal{G}, \nu)$ such that for every $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A g d\nu.$$  

(b) \quad 10 \quad Suppose that for $f \in L^1(X, \mathcal{F}, \mu)$ there are two such functions $g, g' \in L^1(X, \mathcal{G}, \nu)$ such that for all $A \in \mathcal{G}$,

$$\int_A g d\nu = \int_A f d\mu = \int_A g' d\nu.$$  

Show that $g = g' \nu$-a.e.
Solution.

(a) Because \( f \in L^1 \), we know that \( |f| < \infty \) \( \mu \)-a.e., so we may assume that \( |f| < \infty \).

First assume that \( f \) is positive and \( \mu \) is finite. In this case, consider the function

\[
\rho(A) := \int_A f \, d\mu
\]

defined for all \( A \in \mathcal{G} \). First of all, we showed in class that this defines a finite positive measure on \((X, \mathcal{G})\). Moreover, if \( \nu(A) = 0 \) for some \( A \in \mathcal{G} \), since \( \nu = \mu|_{\mathcal{G}} \) we have that \( \mu(A) = 0 \), and so \( \rho(A) = \int_A f \, d\mu = 0 \). Since this holds for all \( A \in \mathcal{G} \), the signed measure \( \rho \) is absolutely continuous with respect to the measure \( \nu \). Since \( \mu \) is finite, so is \( \nu \). Thus, by the Radon-Nykodim Theorem there exists a positive integrable \( g = \frac{d\mu}{d\nu} \in L^1(X, \mathcal{G}, \nu) \) such that \( d\rho = g \, d\nu \); that is, for all \( A \in \mathcal{G} \),

\[
\int_A f \, d\mu = \rho(A) = \int_A d\rho = \int_A g \, d\nu.
\]

Now, if \( \mu \) is only \( \sigma \)-finite, then write \( X = \biguplus_n X_n \) with \( \mu(X_n) < \infty \). Consider \( \nu_n(A) := \mu(A \cap X_n) \) for all \( A \in \mathcal{G} \). So \( \nu = \sum_n \nu_n \). Define \( \rho_n(A) := \int_{A \cap X_n} f \, d\mu \).

Since

\[
\sum_{j=1}^n f \mathbf{1}_{A \cap X_j} \nearrow f \mathbf{1}_A,
\]

by monotone convergence we get that

\[
\rho(A) := \int_A f \, d\mu = \sum_n \int_{A \cap X_n} f \, d\mu = \sum_n \rho_n(A).
\]

Also, as above, if \( \nu_n(A) = 0 \) then \( \mu(A \cap X_n) = 0 \) and so \( \rho_n(A) = 0 \). So \( \rho_n << \nu_n \).

Since \( \nu_n \) is finite, \( g_n := \frac{d\mu_n}{d\nu_n} \) exists and is in \( L^1(X, \mathcal{G}, \nu_n) \). Specifically, \( g_n \) is \( \mathcal{G} \)-measurable. Also, since \( \rho_n(A) = 0 \) for \( A \cap X_n = \emptyset \), we have that \( g_n \) can be chosen
such that it is supported on $X_n$. Define $g = \sum_n g_n$. Since $(X_n)_n$ are disjoint and so $g_n$ have disjoint support, we get that $g$ is always finite and well defined. Also, since $g_n = g_n 1_{X_n}$, by monotone convergence again

$$\int_A g d\nu = \int \sum_n g_n 1_{A \cap X_n} d\nu = \sum_n \int_{A \cap X_n} g_n d\nu_n = \sum_n \rho_n(A) = \rho(A).$$

Specifically,

$$\int_X g d\nu = \rho(X) = \int_X f d\mu < \infty,$$

so $g \in L^1(X, \mathcal{G}, \nu)$.

Now, for the case that $f$ is a general (not necessarily positive) function in $L^1$. Write $f = (f_1 - f_2) + i(f_3 - f_4)$ for $f_j \in L^1$ positive. By the previous case, there exist real-valued functions $g_j \in L^1(X, \mathcal{G}, \nu)$ such that for any $A \in \mathcal{G}$ and $j = 1, 2, 3, 4$ we have

$$\int_A f_j d\mu = \int_A g_j d\nu.$$

By linearity of the integral we get that for all $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A f_1 d\mu - \int_A f_2 d\mu + i \cdot \int_A f_3 d\mu - i \cdot \int_A f_4 d\mu = \int_A (g_1 - g_2) + i(g_3 - g_4) d\nu.$$

So we may choose $g = g_1 - g_2 + i(g_3 - g_4)$ which is a function in $L^1(X, \mathcal{G}, \nu)$.

(b) Suppose that $g, g'$ are as in the question. Then for all $A \in \mathcal{G}$,

$$\int_A g d\nu = \int_A g' d\nu.$$

We have shown in class that this implies that $g = g' \nu$-a.e.