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משך הבחינה: 3 שעות

אין חומר עזר

בהצלחה!

חלק א: שאלת חובה: הגדרות ומשפטים

חלק זה שווה 25 נקודות

(א) | 9 נקי׳ |

הגדירו מרחב מידה

Define a measure space.

(ב) | 8 נקי׳ |

נסחו את משפט Lebesgue-Radon-Nykodim

State the Lebesgue-Radon-Nykodim Theorem.

(ג) | 8 נקי׳ |

מהו המרחב $L^p(X, \mathcal{F}, \mu)$?

What is the space $L^p(X, \mathcal{F}, \mu)$?

חלק ב: הוראות:

בחלק זה עליכם לבחור 3 שאלות מתוך 4 (שלוש בלבד)

כתבו בתחילת המחברת באופן ברור על אלו שאלות עניתם

כתבו באופן ברור את תשובותיכן, ונמקו כל תשובה באופן מתמטי

שאלה 1: | 25 נקי׳ |

נניח $\mathcal{E}$ אוסף של תתי קבוצות של X.

נגדיר את $\mathcal{F}$ להיות האיחוד של כל ה-σ-אלגבראות הנוצרות מסדרה כלשהי ב-$\mathcal{E}$; כלומר:

$$\mathcal{F} = \bigcup \{\sigma((A_n)_{n=1}^{\infty}) \ : \ \forall \ n \ , \ A_n \in \mathcal{E}\} \, .$$

הראו ש- $\mathcal{F} = \sigma(\mathcal{E})$. רמז: האם $\mathcal{F}$ σ-אלגברה?

Let $\mathcal{E}$ be a collection of subsets of X. Let $\mathcal{F}$ be the union of all σ-algebras generated by some sequence in $\mathcal{E}$; that is,

$$\mathcal{F} = \bigcup \{\sigma((A_n)_{n=1}^{\infty}) \ : \ \forall \ n \ , \ A_n \in \mathcal{E}\} \, .$$

Show that $\mathcal{F} = \sigma(\mathcal{E})$. (Hint: is $\mathcal{F}$ a σ-algebra?)

Solution.

First note that if $A \in \mathcal{F}$, then there exists a sequence $(A_n)_n$ in $\mathcal{E}$ such that $A \in \sigma((A_n)_n) \subset \sigma(\mathcal{E})$. Thus, $\mathcal{F} \subset \sigma(\mathcal{E})$.

So it suffices to show that $\mathcal{F}$ is a σ-algebra.

$\emptyset \in \mathcal{F}$ is immediate.

If $A \in \mathcal{F}$ then there exists a sequence $(A_n)_n$ in $\mathcal{E}$ such that $A \in \sigma((A_n)_n)$. Thus, $A^c \in \sigma((A_n)_n) \subset \mathcal{F}$. So $\mathcal{F}$ is closed under complements.

Let $(A_n)_n$ be a sequence in $\mathcal{F}$. So for every n there is a sequence $(E_k^n)_k$ in $\mathcal{E}$ such that $A_n \in \sigma((E_k^n)_k)$. Consider the family $(E_k^n)_{n,k}$. This is a sequence in $\mathcal{E}$. Let $\mathcal{G} =$

$\sigma((E_k^n)_{n,k})$. Then by definition, $\mathcal{G} \subset \mathcal{F}$. Also, for every n, $A_n \in \sigma((E_k^n)_k) \subset \mathcal{G}$. Since $\mathcal{G}$ is a σ-algebra, we have that $\bigcup_n A_n \in \mathcal{G} \subset \mathcal{F}$. Since this holds for any sequence $(A_n)_n$ in $\mathcal{F}$ we have show that $\mathcal{F}$ is closed under countable unions, and so $\mathcal{F}$ is a σ-algebra.

שאלה 2: נניח $(X,\mathcal{F},\mu)$ מרחב מידה ו-$f \geq 0$ פונקציה מדידה אי-שלילית כך ש-$\int_X f d\mu < \infty$.

(א) | 12 נקי |

הראו ש-$\{x \ : \ f(x) > 0\}$ היא σ-סופית.

(ב) | 13 נקי |

הראו שלכל $\varepsilon > 0$ קיימת $A \in \mathcal{F}$ כך ש-$\mu(A) < \infty$ וכן

$$\int_A f d\mu > \int_X f d\mu - \varepsilon.$$

Let $(X,\mathcal{F},\mu)$ be a measure space and let $f \geq 0$ be a non-negative measurable function such that $\int_X f d\mu < \infty$.

(a) | 12 |

Show that $\{x \ : \ f(x) > 0\}$ is σ-finite.

(b) | 13 |

Show that for every $\varepsilon > 0$ there exists $A \in \mathcal{F}$ such that $\mu(A) < \infty$ and

$$\int_A f d\mu > \int_X f d\mu - \varepsilon.$$

Solution.

(a) For every $n > 0$ let $A_n = \{f > n^{-1}\}$. So $\{f > 0\} = \bigcup_n A_n$. And it suffices to show that $\mu(A_n) < \infty$ for all n.

Indeed, for any n, if $x \in A_n$ then $nf(x) > 1$. So,

$$\mu(A_n) \leq n \cdot \int_{A_n} f d\mu \leq n \cdot \int f d\mu < \infty.$$

(b) Since by (a) the set $\{f > 0\}$ is σ-finite, we can write $\{f > 0\} = \biguplus_n A_n$ where $(A_n)_n$ are pairwise disjoint measurable sets with $\mu(A_n) < \infty$ for all n. Thus,

$$\int_X f d\mu = \int_{\{f>0\}} f d\mu = \sum_n \int_{A_n} f d\mu.$$

Write $a_n := \int_{A_n} f d\mu$ which is a sequence of non-negative numbers. Since the sum $\sum_n a_n$ converges, for any $\varepsilon > 0$ we may find N large enough so that $\sum_{n>N} a_n < \varepsilon$. Set

$$A := \biguplus_{n=1}^{N} A_n.$$

So

$$\mu(A) = \sum_{n=1}^{N} \mu(A_n) < \infty.$$

Also,

$$A^c \cap \{f > 0\} = \{f > 0\} \setminus A = \biguplus_{n>N} A_n,$$

so

$$\int_{A^c} f d\mu = \int_{A^c \cap \{f>0\}} f d\mu = \sum_{n>N} \int_{A_n} f d\mu = \sum_{n>N} a_n < \varepsilon.$$

Thus,

$$\int_X f d\mu = \int_A f d\mu + \int_{A^c} f d\mu < \int_A f d\mu + \varepsilon.$$

שאלה 3: | 25 נקי׳ |

נתון מרחב מידה $(X, \mathcal{F}, \mu)$. נניח $(f_n)_n$ סדרה של פונקציות מדידות אי-שליליות.

נניח f פונקציה מדידה כך שהסדרה $(f_n)_n$ מתכנסת במידה ל-f

הראו

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of non-negative measurable functions, and let f be a measurable function such that $(f_n)_n$ converges to f in **measure**.

Show that

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Solution.

Let $(f_{n_k})_k$ be a subsequence such that

$$\lim_{k\to\infty} \int f_{n_k} d\mu = \liminf_n \int f_n d\mu.$$

So we want to show that $\int f d\mu \leq \lim_{k\to\infty} \int f_{n_k} d\mu$. Since $(f_{n_k})_k$ is a subsequence, we have that for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \mu \{|f_{n_k} - f| > \varepsilon\} \leq \limsup_n \mu \{|f_n - f| > \varepsilon\} = 0.$$

So $(f_{n_k})_k$ converges in measure to f.

Let $g_k := f_{n_k}$ for all k, which converge in measure to f. By a theorem in class we now have that there is a further subsequence $(g_{k_j})_j$ such that $\lim_{j\to\infty} g_{k_j} = f$ a.e. Since these are all non-negative functions, Fatou's Lemma tells us that

$$\int f d\mu \leq \liminf_j \int g_{k_j} d\mu.$$

However, the sequence $(\int g_{k_j} d\mu)_j$ is a subsequence of the converging sequence $(\int f_{n_k} d\mu)_k$ which converges to $\liminf_n \int f_n d\mu$. So the limit is

$$\int f d\mu \leq \liminf_j \int g_{k_j} d\mu \leq \lim_{j\to\infty} \int g_{k_j} d\mu = \lim_{k\to\infty} \int f_{n_k} d\mu = \liminf_n \int f_n d\mu.$$

שאלה 4: נתון מרחב מידה σ-סופי $(X, \mathcal{F}, \mu)$. נניח $\mathcal{G} \subset \mathcal{F}$ תת-σ-אלגברה של $\mathcal{F}$.
נסמן: $\nu = \mu|_{\mathcal{G}}$. נניח $f \in L^1(X, \mathcal{F}, \mu)$.

(א) | 15 נקי |

הראו שקיימת פונקציה $g \in L^1(X, \mathcal{G}, \nu)$ כך שלכל $A \in \mathcal{G}$

$$\int_A f d\mu = \int_A g d\nu.$$

(ב) | 10 נקי |

הראו שאם קיימות שתי פונקציות $g, g' \in L^1(X, \mathcal{G}, \nu)$ המקיימות שלכל $A \in \mathcal{G}$

$$\int_A g d\nu = \int_A f d\mu = \int_A g' d\nu,$$

אז מתקיים: $g = g'$ ν-כמעט-בכל-מקום.

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $\nu = \mu|_{\mathcal{G}}$.

(a) | 15 |

Suppose that $f \in L^1(X, \mathcal{F}, \mu)$. Show that there exists $g \in L^1(X, \mathcal{G}, \nu)$ such that for every $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A g d\nu.$$

(b) | 10 |

Suppose that for $f \in L^1(X, \mathcal{F}, \mu)$ there are two such functions $g, g' \in L^1(X, \mathcal{G}, \nu)$ such that for all $A \in \mathcal{G}$,

$$\int_A g d\nu = \int_A f d\mu = \int_A g' d\nu.$$

Show that $g = g'$ ν-a.e.

Solution.

(a) Because $f \in L^1$, we know that $|f| < \infty$ μ-a.e., so we may assume that $|f| < \infty$. First assume that f is positive and μ is finite. In this case, consider the function

$$\rho(A) := \int_A f d\mu$$

defined for all $A \in \mathcal{G}$. First of all, we showed in class that this defines a finite positive measure on $(X, \mathcal{G})$. Moreover, if $\nu(A) = 0$ for some $A \in \mathcal{G}$, since $\nu = \mu|_{\mathcal{G}}$ we have that $\mu(A) = 0$, and so $\rho(A) = \int_A f d\mu = 0$. Since this holds for all $A \in \mathcal{G}$, the signed measure ρ is absolutely continuous with respect to the measure ν. Since μ is finite, so is ν. Thus, by the Radon-Nykodim Theorem there exists a positive integrable $g = \frac{d\rho}{d\nu} \in L^1(X, \mathcal{G}, \nu)$ such that $d\rho = g d\nu$; that is, for all $A \in \mathcal{G}$,

$$\int_A f d\mu = \rho(A) = \int_A d\rho = \int_A g d\nu.$$

Now, if μ is only σ-finite, then write $X = \biguplus_n X_n$ with $\mu(X_n) < \infty$. Consider $\nu_n(A) := \mu(A \cap X_n)$ for all $A \in \mathcal{G}$. So $\nu = \sum_n \nu_n$. Define $\rho_n(A) := \int_{A \cap X_n} f d\mu$. Since

$$\sum_{j=1}^{n} f \mathbf{1}_{A \cap X_j} \nearrow f \mathbf{1}_A,$$

by monotone convergence we get that

$$\rho(A) := \int_A f d\mu = \sum_n \int_{A \cap X_n} f d\mu = \sum_n \rho_n(A).$$

Also, as above, if $\nu_n(A) = 0$ then $\mu(A \cap X_n) = 0$ and so $\rho_n(A) = 0$. So $\rho_n << \nu_n$. Since ν_n is finite, $g_n := \frac{d\rho_n}{d\nu_n}$ exists and is in $L^1(X, \mathcal{G}, \nu_n)$. Specifically, g_n is $\mathcal{G}$-measurable. Also, since $\rho_n(A) = 0$ for $A \cap X_n = \emptyset$, we have that g_n can be chosen such that it is supported on X_n. Define $g = \sum_n g_n$. Since $(X_n)_n$ are disjoint and so g_n have disjoint support, we get that g is always finite and well defined. Also, since $g_n = g_n \mathbf{1}_{X_n}$, by monotone convergence again

$$\int_A g d\nu = \int \sum_n g_n \mathbf{1}_{A \cap X_n} d\nu = \sum_n \int_{A \cap X_n} g_n d\nu_n = \sum_n \rho_n(A) = \rho(A).$$

Specifically,

$$\int_X g d\nu = \rho(X) = \int_X f d\mu < \infty,$$

so $g \in L^1(X, \mathcal{G}, \nu)$.

Now, for the case that f is a general (not necessarily positive) function in L^1. Write $f = (f_1 - f_2) + i(f_3 - f_4)$ for $f_j \in L^1$ positive. By the previous case, there exist real-valued functions $g_j \in L^1(X, \mathcal{G}, \nu)$ such that for any $A \in \mathcal{G}$ and $j = 1, 2, 3, 4$ we have

$$\int_A f_j d\mu = \int_A g_j d\nu.$$

By linearity of the integral we get that for all $A \in \mathcal{G}$,

$$\int_A f d\mu = \int_A f_1 d\mu - \int_A f_2 d\mu + i \cdot \int_A f_3 d\mu - i \cdot \int_A f_4 d\mu = \int_A (g_1 - g_2) + i(g_3 - g_4) d\nu.$$

So we may choose $g = g_1 - g_2 + i(g_3 - g_4)$ which is a function in $L^1(X, \mathcal{G}, \nu)$.

(b) Suppose that g, g' are as in the question. Then for all $A \in \mathcal{G}$,

$$\int_A g d\nu = \int_A g' d\nu.$$

We have shown in class that this implies that $g = g'$ ν-a.e.