Percolation - 201.2.0101

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Home Exam Spring 2013

Due date: August 5, 2013

By this date it should be in my box at BGU.

It is a good idea to send me an email when you have submitted your work.

Instructions

• You may use any resources you find such as lecture notes, books, the internet. If you use solutions of exercises found in such, you should reference them in your solution. (For example, “in my solution I use a technique from: G. Orwell. Animal Farm. Chapter 5.”.) If you use such resources, you should not copy a solution, but rather understand it and write it up in your own words.

• Work in pairs is permitted, but only in pairs, not more. Do not write up the solutions together, but rather after understanding the solution, each of you should write them up on your own.

• You may use any theorems proved in class, but any other propositions you wish to use - you should prove.

• Please write clearly. State your claims and proofs in a mathematical format, with precise notation and correct formalism.

• Make sure your ID no. appears on your solutions.

• Each exercise is worth 20 points, although the different parts of a specific exercise are not all equal. Extra points will be awarded for original ideas and proofs.

• By taking this exam, you agree to these terms, and declare that you will conform to them.
Exercise 1. Regarding the number of infinite components in percolation:

(A) Let $G$ be an infinite connected graph (not necessarily transitive). Let $p \in [0, 1]$. Show that for $p$-percolation on $G \times \mathbb{Z}$ the number of infinite components is constant a.s. and is either 0, 1 or $\infty$.

(Recall that the graph $G \times \mathbb{Z}$ has vertex set $V(G) \times \mathbb{Z}$ and edges given by $(x,k) \sim (y,k)$ if and only if $x \sim y$ and $(x,k) \sim (x,k+1)$.)

(B) Give an example of a connected bounded degree graph $G$ such that the number of infinite components in $p$-percolation on $G$ is not a constant (that is, show that for some $k$, $0 < P_p[N = k] < 1$, where $N$ is the random variable that is the number of infinite components).


(A) An auxiliary claim: Show that if $\Gamma$ is a finite connected graph, and $x, y, z \in \Gamma$ three distinct vertices, then there exists a vertex $v \in \Gamma$ such that there exist three edge disjoint paths in $\Gamma$, $\alpha : x \to v$, $\beta : y \to v$ and $\gamma : z \to v$ (it may be that $v \in \{x, y, z\}$ in which case one may choose an empty path).

[Hint: A spanning tree of $\Gamma$ is a connected subgraph $T$ of $\Gamma$ such that $T$ is a tree and $V(T) = V(\Gamma)$. Every finite connected graph has at least one spanning tree.]

(B) Consider $p$-bond percolation on $G$, with $p < p_c$, if $x, y, z$ are three vertices (not necessarily distinct) such that $x, y, z$ are all in the same component of $p$-bond percolation on $G$, then there exists a vertex $v \in G$ such that $\{v \leftrightarrow x\} \circ \{v \leftrightarrow y\} \circ \{v \leftrightarrow z\}$.

(C) Use the above to prove that

$$\mathbb{E}_p[|C|^2] \leq (\mathbb{E}_p[|C|])^3.$$ 

Exercise 3. Give an example of an infinite connected graph $G$ with bounded degrees and a vertex $o \in G$ such that $p_c(G) = 1$ and such that $G$ has exponential
volume growth; i.e. there exists a constant $c > 1$ and $o \in G$ such that for all integers $r > 0$, $|B(o, r)| \geq c^r$.

Show this for both site and bond percolation.

**Exercise 4.** Let $G$ be an infinite connected simple graph (no self loops or multiple edges) and $o \in G$ some vertex. Suppose that $G$ is $d$-regular, $d \geq 2$.

In this exercise, it will be useful to consider the following set: Let $C_{n,b}$ be the set of all connected subsets of $G$ that contain $o$ and have exactly $n$ vertices and boundary size $b$; that is all connected subsets $S$ such that $o \in S$, $|S| = n$ and for

$$\partial S = \{y \notin S : y \sim S\}$$

$|\partial S| = b$.

(A) Show that if $C_{n,b} \neq \emptyset$ then $b \leq dn$. Show that if $n = 1$ then $b = d$ and if $n \geq 2$ then $b \leq (d - 1)n$.

(B) Show that for any $p \in (0, 1)$,

$$\sum_{n,b} |C_{n,b}| p^n (1-p)^b \leq 1.$$

(C) Let $A_n$ be the set of connected subsets of $G$ that contain $o$ and have exactly $n$ vertices. Show that

$$|A_n| \leq \left(\frac{d^d}{(d-1)^{d+1}}\right)^n.$$

**Exercise 5.** Let $d > 1$. In this exercise we will show in steps that in site percolation on $\mathbb{Z}^d$, for $p_c = p_c(\mathbb{Z}^d)$, there exist a constant $c = c(d) > 0$ such that

$$\mathbb{P}_{p_c}[0 \leftrightarrow \partial_r(0)] \geq c r^{(1-d)/2}.$$

That is, the probability to be connected to distance $r$ does not decay exponentially (as in the sub-critical case).

1. We say that a collection $(\Omega(x))_{x \in \mathbb{Z}^d}$ is a $(p, \Delta)$-almost independent percolation if:
For every $x$, $\Omega(x)$ is a Bernoulli random variable, with $E[\Omega(x)] = P[\Omega(x) = 1] \leq p$ (that is, the probability that $x$ is open is at most $p$).

For every two subsets $A, B \subset \mathbb{Z}^d$ such that $\text{dist}(A, B) > \Delta$ we have that $(\Omega(x))_{x \in A}$ is independent of $(\Omega(x))_{x \in B}$. (Here the distance is the graph distance in $\mathbb{Z}^d$.)

Note that in a $(p, \Delta)$-almost independent percolation, it is not necessarily true that vertices close to one another are independent.

Show that if $S$ is a finite connected subset of $\mathbb{Z}^d$ containing 0, then in a $(p, \Delta)$-almost independent percolation $\Omega$,

$$P[S \text{ is open in } \Omega] \leq p^{|S|/V},$$

where $C(z, r) := \{z : ||z||_\infty \leq r\}$ and $V = |C(0, \Delta)|$ is the size of a $L^\infty$-ball of radius $\Delta$ in $\mathbb{Z}^d$.

(2) Show that for any $d, \Delta$ there exist $p, c_1, c_2 > 0$ such that if $\Omega$ is a $(p, \Delta)$-almost independent percolation on $\mathbb{Z}^d$, then in $\Omega$:

$$P[0 \leftrightarrow \partial_r(0) \text{ in } \Omega] \leq P[|C_\Omega(0)| \geq r] \leq c_1 e^{-c_2 r}.$$

(It may be useful to use part (C) of Exercise 4.)

(3) Consider $p$-site percolation on $\mathbb{Z}^d$ (totally independent case). Use the previous items to show that there exist $q, c_1, c_2 > 0$ such that the following holds. If there exists $r > 0$ such that

$$P_p[C(0, r) \leftrightarrow \mathbb{Z}^d \backslash C(0, 3r)] \leq q$$

then for all $R > 0$,

$$P_p[C(0, R) \leftrightarrow \mathbb{Z}^d \backslash C(0, 3R)] \leq c_1 e^{-c_2 R/r}.$$

(Hint: Define an appropriate almost independent percolation. Note that a tessellation of $\mathbb{Z}^d$ by $L^\infty$-balls of radius $r$ has an isomorphic graph structure to that of $\mathbb{Z}^d$.)
(4) Consider $p$-site percolation on $\mathbb{Z}^d$. Use the BK inequality to show that there exist $q, c_1, c_2 > 0$ such that if there exists $r > 0$ such that
\[ P_p[0 \leftrightarrow \partial_r(0)] \leq qr^{(1-d)/2} \]
then for all $R > 0$
\[ P_p[0 \leftrightarrow \partial_R(0)] \leq c_1 e^{-c_2 R / r} . \]

(5) Conclude that for $p = p_c$ there exists $q > 0$ such that for all $r > 0$,
\[ P_{p_c}[0 \leftrightarrow \partial_r(0)] \geq qr^{(1-d)/2} . \]
**Bonus Exercise**

- This exercise is with 100 points solved. That means if you solve it, you get 100 and do not need to solve the other exercises.
- Better to show me a solution first, just in case...
- At the moment I do not know how to solve it, so it may (or may not, who knows?) be difficult.
- You are not allowed to solve it for high dimensions using the method known as “lace expansion”.

**Bonus Exercise.** A random walk on $\mathbb{Z}^d$ is a sequence of vertices in $\mathbb{Z}^d$, say $(Z_n)_n$ such that for all $n$,

$$
P[Z_{n+1} = y \mid Z_n = x, Z_{n-1}, \ldots, Z_0] = \Pr[Z_{n+1} = y \mid Z_n = x] = \frac{1}{2d} 1_{(y \sim x)}.
$$

For a set $S \subset \mathbb{Z}^d$ we define

$$
T_S = \inf \{ n \geq 1 : Z_n \in S \}.
$$

(Note that we don’t count time $n = 0$.)

Given a subset $S \subset \mathbb{Z}^d$ we define the **index** of $S$ to be the number

$$
I(S) := \mathbb{E}[T_S \mid Z_0 = 0],
$$

where $\mathbb{E}$ is expectation with respect to the random walk measure. ($I(S)$ may be infinite.)

Consider $\Omega_p$, $p$-percolation on $\mathbb{Z}^d$. For every $p \in (0, 1)$ define a random subset: If $\theta(p) > 0$ then let $S_p$ be the (unique) infinite component in $\Omega_p$. If $\theta(p) = 0$ let $S_p = \emptyset$.

Prove the following for some $d \geq 3$.

- Show that if $p > p_c(\mathbb{Z}^d)$ then $\Pr_p$-a.s. $I(S_p)$ is finite.
- Prove or provide a counter-example: If $\theta(p) > 0$ then $\Pr_p$-a.s. $I(S_p)$ is finite.
- Prove that $\mathbb{E}_{p_c}[I(S_{p_c})] = \infty$. 