Exercise 1. Show that $\text{dist}_G$ defines a metric on $G$.

(Recall that a metric is a function that satisfies:

- $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ iff $x = y$.
- $\rho(x, y) = \rho(y, x)$.
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. )

Exercise 2. Prove that connectivity is an equivalence relation in any graph.

Exercise 3. Let $G = \langle S \rangle$ be a finitely generated group with symmetric generating set $S$. Show that the Cayley graph of $G$ is transitive. For every $x, y \in G$ give an example of an automorphism $\varphi_{x,y}$ that maps $x \mapsto y$.

Solution to Exercise 3. For $x, y \in G$ let $g = yx^{-1}$, so $gx = y$. Define $\varphi(a) = ga$ for all $a \in G$. This is obviously a bijection that maps $\varphi(x) = y$.

For any $a, b \in G$ we have that $a = bs$ if and only if $\varphi(a) = ga = gbs = \varphi(b)s$. So $a \sim b$ if and only if $\varphi(a) \sim \varphi(b)$. Hence $\varphi$ is a (graph-) automorphism of $G$. \qed

Exercise 4. Let $\omega$ be $p$-bond-percolation on a graph $G$.

Show that the event that $|C(x)| = \infty$ is measurable.

Show that the event that there exists an infinite component in $\omega$ is measurable.

Show that this event is a tail event.

Solution to Exercise 4. First, note that the event that there exists an infinite component is the event that there exists $x$ such that $|C(x)| = \infty$. So it suffices to show that for every $x$, the event that $x \leftrightarrow \infty$ is measurable.
Fix $x \in G$ and for every $r > 0$ let $B(x, r)$ be the ball of radius $r$ around $x$ (in the graph metric). Let $A_r = G \setminus B(x, r)$.

If for some $r$, $x \not\leftrightarrow A_r$ then $C(x) \subset B(x, r)$ and so $C(x)$ is finite. If $C(x)$ is finite, then for $r = \max \{\text{dist}(y, x) : y \leftrightarrow x\}$ we have that $C(x) \subset B(x, r)$. So $x \not\leftrightarrow A_{r+1}$. That is, $\{x \leftrightarrow \infty\} = \bigcap_r \{x \leftrightarrow A_r\}$. So it suffices to show that for any $r > 0$, $\{x \leftrightarrow A_r\}$ is measurable.

Fix $r > 0$. The event $\{x \leftrightarrow A_r\}$ depends only on edges at distance at most $r$ from $x$. So $\{x \leftrightarrow A_r\}$ is measurable.

Now, the event that there exists an infinite component is unchanged if we change a finite number of edges: indeed, suppose $\omega$ is such that there exists an infinite component. Then changing one edge will either have no effect, or split the infinite component into two, where at least on has to still be infinite. On the other hand, if $\omega$ is such that there is no infinite component, then changing on edge can either do nothing or connect two finite components, the resulting component still being finite.

So the event that there exists an infinite component is a tail event. $\square$