**Exercise 1.** Show that the event that there exists an infinite component is translation invariant.

*Solution to Exercise 1.* If $\varphi \in \text{Aut}(G)$ then $\varphi$ maps infinite connected subsets to infinite connected subsets. So if $\omega$ is a subgraph containing an infinite component, then $\varphi \omega$ also contains an infinite component. Also, if $\omega$ contains only finite components, then $\varphi \omega$ contains only finite components.

Let $A$ be the event that there exists an infinite component. Then the above is just $\omega \in A \iff \varphi \omega \in A$, which implies $A = \varphi A$.

This holds for all $\varphi \in \text{Aut}(G)$ so $A$ is translation invariant. □

**Exercise 2.** Let $G$ be an infinite transitive graph, and let $E \subseteq E(G), |E| < \infty$ be some finite subset. Then, there exists $\varphi \in \text{Aut}(G)$ such that $\varphi E \cap E = \emptyset$.

*Solution to Exercise 2.* Fix some vertex $x \in G$. Let $r = \max \{\text{dist}(e,x) : e \in E\}$. Let $R > 3r$ and choose a vertex $y \in G$ such that $\text{dist}(x,y) > R$. Let $\varphi \in \text{Aut}(G)$ be such that $\varphi(x) = y$.

Then, since $\varphi$ is a graph automorphism, it preserves distances. So for any edge $e$ such that $\text{dist}(e,x) \leq r$, we have that $\text{dist}(\varphi(e), y) \leq r$ and so $\text{dist}(\varphi(e), x) > R - r > r$. Thus, for any $e \in E$ we have that $\varphi(e) \notin E$. That is, $\varphi E \cap E = \emptyset$. □

**Exercise 3.** Show that $\{x \leftrightarrow \infty\}$ is an increasing event.

Show that $\{x \leftrightarrow y\}$ is an increasing event.

Show that $A$ is increasing if and only if $A^c$ is decreasing.

Show that the union of increasing events is increasing.

Show that the intersection of increasing events is increasing.
Show that \{x \text{ is an isolated vertex} \} is a decreasing event.

Give an example of an event that is not increasing or decreasing.

**Solution to Exercise 3.** If \( \omega \leq \eta \) and \( \omega \) is such that \( \omega \in \{x \leftrightarrow \infty\} \), then the infinite component of \( x \) in \( \omega \) is open in \( \eta \), so \( \eta \) also contains an infinite component for \( x \).

In general, if \( \omega \leq \eta \), then for every \( z \), the component of \( z \) in \( \omega \) is contained in the component of \( z \) in \( \eta \). So if \( x \leftrightarrow y \) in \( \omega \) then \( x \leftrightarrow y \) in \( \eta \).

Let \( A \) be an increasing event, and let \( B \) be a decreasing event. Let \( \omega \leq \eta \). If \( \eta \in A^c \), then \( \eta \not\in A \), so it cannot be that \( \omega \in A \), which implies that \( \omega \in A^c \). If \( \omega \in B^c \) then \( \omega \not\in B \) so \( \eta \not\in B \) (because \( B \) is decreasing) and so \( \eta \in B^c \). Since this is true for all \( \omega \leq \eta \), we get that \( A^c \) is decreasing and \( B^c \) is increasing.

Suppose that \((A_n)_n\) are increasing events. Let \( A = \bigcup_n A_n \). Suppose that \( \omega \in A \), and that \( \eta \geq \omega \). Then, there exists \( n \) such that \( \omega \in A_n \), and since \( A_n \) is increasing, also \( \eta \in A_n \). So \( \eta \in A \). Thus, \( A \) is increasing.

Let \( B = \bigcap_n A_n \). If \( \eta \geq \omega \) and \( \omega \in B \) then \( \omega \in A_n \) for all \( n \). Since \( A_n \) are all increasing, \( \eta \in A_n \) for all \( n \). So \( \eta \in B \).

The event that \( x \) is an isolated vertex is the event that \( x \not\leftrightarrow y \) for all \( y \sim x \). So the intersection of decreasing events. That is, the event that \( x \) is an isolated vertex is the complement of the union of increasing events, and so a decreasing event.

Consider the event \( A = \{x \leftrightarrow \infty, \deg(x) = 1\} \). Then opening edges adjacent to \( x \) ruins the event, however, closing edges may disconnect \( x \) from infinity, so \( A \) is neither increasing nor decreasing. \( \square \)

**Exercise 4.** Let \( G \) be a graph. A function \( f : \{0, 1\}^{E(G)} \to \mathbb{R} \) is increasing if \( \omega \leq \eta \) implies \( f(\omega) \leq f(\eta) \).

Show that for an event \( A \), \( 1_A \) is increasing if and only if \( A \) is an increasing event.

**Solution to Exercise 4.** Let \( f = 1_A \).
Assume that $A$ is increasing. For any $\omega \leq \eta$, if $\omega \not\in A$ then $f(\omega) = 0 \leq f(\eta)$. If $\omega \in A$ then since $A$ is increasing $\eta \in A$ and so $f(\omega) = 1 = f(\eta)$. Since this holds for all $\omega \leq \eta$, we get that $f$ is increasing.

Now assume that $f$ is increasing. Let $\omega \leq \eta$, and assume that $\omega \in A$. So $1 = f(\omega) \leq f(\eta)$ which implies that $f(\eta) = 1$ and so $\eta \in A$. Since this holds for all $\omega \leq \eta$, we get that $A$ is increasing. \hfill \square

**Exercise 5.** Show that $p_c(Z) = 1$.

**Solution to Exercise 5.** Let $p < 1$. It suffices to show that $\Theta_Z(p) = 0$.

First we investigate the event $\{0 \leftrightarrow \infty\}$. Let $A_n$ be the event that both edges $\{n, n + 1\}$ and $\{-n, -(n + 1)\}$ are closed. So $\mathbb{P}_p[A_n] = (1 - p)^2$. Since for different $n$ these edges are different, we have that $(A_n)_n$ are independent, and also $(A^c_n)_n$ are independent. Thus,

$$
\mathbb{P}_p[\bigcap_n A^c_n] = \lim_{N \to \infty} \mathbb{P}_p[\bigcap_{n=1}^N A^c_n] = \lim_{N \to \infty} \prod_{n=1}^N [1 - (1 - p)^2] = 0
$$

because for $p < 1$ we have $1 - (1 - p)^2 < 1$. Thus,

$$
\mathbb{P}_p[\exists n : A_n] = 1.
$$

That is, $\mathbb{P}_p$-a.s. there exists $n$ such that both $\{n, n + 1\}$ and $\{-n, -(n + 1)\}$ are closed. This implies that $\mathbb{P}_p$-a.s. $C(0) \subset [-n, n]$ and so finite. Thus, $\mathbb{P}_p[0 \leftrightarrow \infty] = 0$.

Now, there was nothing special about the vertex 0 in this argument. One could replace 0 with any other vertex. So, we conclude that for any $x \in \mathbb{Z}$, $\mathbb{P}_p[x \leftrightarrow \infty] = 0$. Summing over all $x$ we have,

$$
\Theta_Z(p) = \mathbb{P}_p[\exists x : x \leftrightarrow \infty] \leq \sum_x \mathbb{P}_p[x \leftrightarrow \infty] = 0.
$$

\hfill \square