Exercise 1. Let $G$ be a graph, and let $\eta \in \{0, 1\}^{E(G)}$ and $E \subset E(G)$. Let $X : \{0, 1\}^{E(G)} \to \mathbb{R}$ be a random variable. Show that $X_{\eta,E}$ is measurable with respect to $\mathcal{T}_E$ (that is, $X_{\eta,E}$ does not depend on the edges in $E$).

Solution to Exercise 1. Suppose that $\omega, \omega' \in \{0, 1\}^{E(G)}$ are such that $\omega(e) = \omega'(e)$ for all $e \notin E(G)$. Then, $\omega_{\eta,E} = \omega'_{\eta,E}$. So $X_{\eta,E}(\omega) = X_{\eta,E}(\omega')$. Thus, $X \in \sigma(\omega(e) : e \notin E) = \mathcal{T}_E$. \qed

Exercise 2. For percolation on $G$, let $X,Y$ be decreasing random variables. Let $Z$ be an increasing random variable. Show that for any $p$,

- $\mathbb{E}_p[XZ] \leq \mathbb{E}_p[X] \cdot \mathbb{E}_p[Z]$,
- $\mathbb{E}_p[XY] \geq \mathbb{E}_p[X] \cdot \mathbb{E}_p[Y]$.

Solution to Exercise 2. Note that if $X,Y$ are decreasing then $-X,-Y$ are increasing. So

$$\mathbb{E}_p[XY] = \mathbb{E}_p[(-X)(-Y)] \geq \mathbb{E}_p[-X] \cdot \mathbb{E}_p[-Y] = \mathbb{E}_p[X] \cdot \mathbb{E}_p[Y],$$

and

$$\mathbb{E}_p[XZ] = -\mathbb{E}_p[(-X)Z] \leq -\mathbb{E}_p[-X] \mathbb{E}_p[Z] = \mathbb{E}_p[X] \cdot \mathbb{E}_p[Z].$$

\qed

Exercise 3. Let $H$ be a subgraph of $G$. Show that $p_c(H) \geq p_c(G)$.

Solution to Exercise 3. For $p$-percolation on $G$ let $C_H(x)$ be the component of $x$ when the percolation is restricted to $H$. \qed
Let \( p > p_c(H) \). So for \( x \in H \),

\[
\theta_{G,x}(p) = P_p[|C(x)| = \infty] \geq P_p[|C_H(x)| = \infty] = \theta_{H,x}(p) > 0.
\]

So \( p > p_c(G) \).

This holds for all \( p > p_c(H) \) which completes the proof. \( \square \)

**Exercise 4.** Show that any finite cut-set must contain a minimal cut-set.

**Solution to Exercise 4.** Let \( \Pi \) be a cut-set of size \( n \). Let \( \Pi_n = \Pi \). If there exists \( e \in \Pi_n \) such that \( \Pi_n \setminus \{e\} \) is a cut-set, let \( \Pi_{n-1} = \Pi_n \setminus \{e\} \). Continue this way, until we have some cut-set \( \Pi_m, 1 \leq m \leq n \), with the property that for any \( e \in \Pi_m \), the set \( \Pi_m \setminus \{e\} \) is not a cut-set. That is, \( \Pi_m \) is a minimal cut-set, contained in \( \Pi_n = \Pi \). \( \square \)

**Exercise 5.** Give an example of a transitive graph and some \( p \in (0,1) \) for which the number of infinite components in percolation is a.s. \( \infty \).

**Solution to Exercise 5.** Let \( G = \mathbb{T}_3 \) the 3-regular tree. Let \( o \in G \) be some root vertex. Let \( x, y, z \) be the descendant of \( o \). Let \( T_x, T_y, T_z \) be the sub-trees composed of the descendants of \( x, y, z \) respectively. So each of \( T_x, T_y, T_z \) is a rooted binary tree. Thus, the component of the root in percolation on \( T_x, T_y, T_z \) is a Galton-Watson process with offspring distribution \( \text{Bin}(2,p) \). If \( p > \frac{1}{2} \) then there exists \( \theta > 0 \) such that

\[
P_p^{T_x}[x \leftrightarrow \infty] = P_p^{T_y}[y \leftrightarrow \infty] = P_p^{T_z}[z \leftrightarrow \infty] = \theta > 0.
\]

Since the edge sets \( \{\{o, x\}, \{o, y\}, \{o, z\}\}, E(T_x), E(T_y), E(T_z) \) are mutually disjoint, we have that

\[
P_p[o \not\leftrightarrow x, o \not\leftrightarrow y, o \not\leftrightarrow z, x \leftrightarrow \infty, y \leftrightarrow \infty, z \leftrightarrow \infty] = (1 - p)^3\theta^3 > 0.
\]

But this event implies that there are at least 3 infinite components, because \( x \) can connect to \( y \) only through \( o \), and similarly for the other pairs from \( \{x, y, z\} \). By
the 0, 1, $\infty$ law this implies that the number of infinite components must $\mathbb{P}_p$-a.s. be infinite. \qed