Exercise 1. Let $A$ be an event. We say that an edge $e$ is pivotal in $\omega$ for $A$ if for $\omega \in A, \omega + \delta_e \notin A$ or $\omega \notin A, \omega + \delta_e \in A$. Here

$$(\omega + \delta_e)(e') = 1_{\{e' \neq e\}}\omega(e') + 1_{\{e' = e\}}(1 - \omega(e))$$

flips the value of $\omega$ at $e$.

That is, $e$ is pivotal in $\omega$ for $A$ if flipping the state of the edge $e$ changes whether $\omega$ is in $A$.

- Show that

$$\{\omega : e \text{ is pivotal in } \omega \text{ for } A\} = A_{1,e} \Delta A_{0,e},$$

where $A_{j,e} = \{\omega : \omega_{j,e} \in A\}$.

- Show that if $A$ is increasing

$$\mathbb{E}[\partial_e 1_A] = \mathbb{P}[\{\omega : e \text{ is pivotal in } \omega \text{ for } A\}].$$

Solution to Exercise 1. Fix an edge $e$. For an event $F$ and $j \in \{0, 1\}$ let

$$F_j = \{\omega : \omega_{j,e} \in F\}.$$

Let $I = \{\omega : e \text{ is pivotal in } \omega \text{ for } A\}$.

First, note that since $\omega = \omega_{0,e}$ or $\omega = \omega_{1,e}$, we have that $\omega \in I$ if and only if either $\omega_{1,e} \in A, \omega_{0,e} \notin A$ or $\omega_{0,e} \in A, \omega_{1,e} \in A$. So

$$I = (A_1 \setminus A_0) \cup (A_0 \setminus A_1) = A_1 \Delta A_0.$$

Now, if $A$ is increasing then $A_0 \subset A_1$ so $I = A_1 \setminus A_0$. Also,

$$(1_A)_{j,e} = 1_{\{\omega_{j,e} \in A\}} = 1_{A_j}.$$
so
\[ \partial_e 1_A = 1_{A_1} - 1_{A_0}. \]
Thus,
\[ \mathbb{E}[\partial_e 1_A] = \mathbb{P}[A_1] - \mathbb{P}[A_0] = \mathbb{P}[I]. \]

Exercise 2. Let \( G \) be a graph. Show that there exists a graph \( H \) such that site percolation on \( H \) is equivalent to bond percolation on \( G \). Specifically,
\[ p_c^s(H) = p_c^b(G). \]

Solution to Exercise 2. Define the graph \( H \) as follows. The vertices of \( H \) are the edges of \( G \): \( V(H) = E(G) \). The edges of \( H \) are defined by \( e \sim e' \) if and only if as edges in \( G \), \( e \) and \( e' \) share a vertex. \( \square \)